Supplementary Appendices

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Linearized dynamics equations for the balance and steer of a bicycle: a benchmark and review

by

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Contents of this supplement

Appendices in the 28 page main body of the paper (not in this supplement) are

A) Definitions of the coefficients used in the equations of motion, and

B) A brief derivation of the governing equations.

These supplementary appendices (starting here on page 29) include:

1) A detailed history of bicycle dynamics studies with an expanded bibliography (all references from the main text and some more that we did not have space for, some with annotations).

2) A more detailed explanation of the verification of the linearized equations which was done with the aid of the numerical dynamics package SPACAR.

3) A more detailed explanation of the verification of the linearized equations which was done using the symbolic algebra package AutoSim.

4) A more detailed explanation of how lateral symmetry decouples lateral and forward motion and gives $\dot{v} = 0$ as one of the linearized equations of motion.

5) A reduced benchmark for testing less general bicycle simulations.

1. History of bicycle steer and dynamics studies

> *"Even now, after we've been building them for 100 years, it's very difficult to understand just why a bicycle works - it's even difficult to formulate it as a mathematical problem."* — Freeman Dyson interviewed by Stewart Brand in Wired News, February 1998.

This appendix builds on Hand (1988) and is the source of the brief literature review in the main body of the paper. We divide the literature on bicycle dynamics into three categories:

a) Qualitative explanations of stability and self-stability that do not use the differential equations of motion.

b) Dynamical analyses that use any of a number of simplifications precluding the study of hands-free self-stability.

c) Equations of motion describing a model that has, in principle, enough complexity to predict hands-free self-stability.

The historical discussion below is in chronological order within each of the three categories above.

(*a*) *Qualitative discussions of stability*

Basic features of balance by means of controlled steering are accessible without detailed equations, and are reasonably described in many papers. In contrast, the self-stability of a bicycle involves complex dynamic phenomena that seem to us to be beyond precise description without appeal to correct governing equations. Thus the qualitative discussions of self-stability below are necessarily less definitive.

1866 Lallement's velocipede-improvement U.S. Patent, which is on the addition of front-wheel pedals (as opposed to pushing the feet on the ground), includes a concise explanation of balancing by steering: "If the carriage is inclined to lean to the right, turn the wheel [to the right], which throws the carriage over to the left...". Within five years, the U. S. patent literature begins to show pictures of bicycles further improved with trail and an inclined steering axis. Whether or not these improvements conveyed genuine self-stability is not known.

1869 William Rankine, engineer and thermodynamics theorist, presents semi-quantitative observations on lean and steer of a velocipede. This seems to be the first description of 'countersteering' — briefly turning to the left to generate the rightward lean necessary for a steady rightward turn. The Wright brothers were later obsessed with this counter-steering aspect of bicycle control (see quote in, e.g., Åström *et al.* (2005)). Rankine discusses steer only by means of rider control and seems to have been unaware of the possibility of self-stability.

1896 Archibald Sharp, an engineering lecturer at what was to become Imperial College, publishes his book covering nearly all technical aspects of bicycle theory and practice, including sections on stability for which he earned an honorable mention in the 1898 Prix Fourneyron (see Bourlet below). Sharp also later authored the classic 11th edition Britannica (1910) entry on bicycles. In calculating the handlebar torque required to maintain a steady turn, Sharp's equation (6) is wrong, first by the typographical error of a sign change in the second parentheses, and second by neglecting the centrifugal force on the mass centre of the front assembly. Sharp also neglects precessional torque on the front wheel. However, Sharp explicitly recognizes the mechanical trail and implicitly recognizes the quantity we call S_A.

Sharp developed his equation to investigate no-hands riding. Sharp concludes, correctly in part (see Jones 1970 below), that the no-hands rider exercises control of steering through upper-body lean causing frame lean, leading to gyroscopic precession of the front wheel. A rider can thus control this precession and make corrective turns much like he or she would with direct handlebar torques. Archibald Sharp seemed unaware of the possibility of bicycle self-stability.

1896 Appell, in his dynamics textbook, summarizes Bourlet's analysis (see category (b) below) of balancing and steering a velocipede. Surprisingly, this master of the differential equations governing non-holonomic dynamics includes none in his discussion of bicycles.

The later 1890s are a period when numerous mathematical analyses are initiated, as described below. Appell mentions a few analyses both in later editions of his textbook (1899–1952), and in a monograph (1899) on the non-holonomic mechanics of rolling bodies.

1920 Grammel provides some discussion of gyroscopic moments in bicycling, but provides no equations of motion.

1929 Wallace's long technical paper on motorcycle design contains thoughtful qualitative discussions on his predictions about the handling characteristics of various motorcycle designs (pp. 177–184). He examines steer torque, including the contribution of toroidal tires and gyroscopic torques. Wallace's analysis of non-linear geometric effects (pp. 185–212) erroneously assumes no pitch of the rear frame due to steering.

1946 Maunsell quantitatively estimates the relative sizes of many of the potential effects that can cause an uncontrolled bicycle to turn into a fall. Although the paper does not use complex modelling, it clearly lays out and partially answers many questions about bicycle stability. Maunsell is candid about the difficulty of using full dynamics equations "I have not yet had time to follow out in full the long and involved calculations of [Carvallo's] paper... I hope to do so in the future." (Carvallo is discussed in section (c) below).

1970 David E. H. Jones's Physics Today article (re-printed in 2006) is perhaps the single best-known paper on bicycle stability. With simple experiments Jones showed that, for the bicycles he tried, both front-wheel spin momentum and positive mechanical trail were needed for self-stability. Jones also observed that a rider can easily balance almost any bicycle that was not self-stable by turning the handlebars appropriately. But when riding no-hands, Jones had difficulty stabilizing a bicycle whose front-wheel gyroscopic terms were cancelled by an added, counter-spinning wheel. And Jones was unable to master no-hands balance of a bicycle with negative trail. Jones's experimental observations indicate useful trends, but do not seem to represent precise parameter-space boundaries of what is or can be stable or controlled. On the theoretical side Jones wanted to counter the widely-quoted simple gyroscopic explanations of no-hands bicycle control presented for example in Sharp (1896) above and un-controlled no-hands bicycle stability presented, for example, in Klein & Sommerfeld (1910). His experiments with a variety of bicycles pointed to mechanical trail as another important factor in bicycle stability. Jones did no dynamical modelling, and focused only on trail's effect on steer torque as a function of lean. His thought was that the "static" torque would define the steering tendency for a leaned bicycle, and thereby explain self-stability. In effect Jones explored only the gravitational-potential part of one entry in the stiffness matrix, while ignoring the velocity-dependent centrifugal and gyroscopic terms and the effects of front assembly mass placed ahead of the steering axis. A variety of subsequent investigators have elaborated on Jones's non-dynamic potential-energy treatment.

1942–98 Various other qualitative discussions, none making use of already published governing dynamics equations, were authored by Arthur T. Jones (1942), Den Hartog (1948), Higbie (1974), Kirshner (1980), Le Hénaff (1987), and Cox (1998). Most of these papers, somewhat like David E.H. Jones (1970), describe one or another term in the dynamics equations (e.g., centripetal forces or gyroscopic terms) but overstate, we think, their singular role in bicycle stability.

1984 Foale's book comprehensively explores factors affecting motorcycle handling.

1988 Olsen & Papadopoulos' qualitative article discusses aspects of dynamic modelling based on the uncontrolled bicycle equations in Papadopoulos (1987).

1993 Patterson developed a series of dynamically based design rules for improving rider control authority.

1999 Cossalter presented an entire book with qualitative explanations of his decades of quantitative modelling work on motorcycle handling.

2004 Wilson's *Bicycling Science* includes a chapter by Papadopoulos which qualitatively discusses bicycle stability.

(*b*) *Simplified analyses that use dynamics*

Simplified dynamic models have appeared from the mid 1890s to the present day. These papers use one or more of the following three types of specializations:

i) Simplified geometry and/or mass distribution. In these models some collection of the following assumptions are made:

- inertia axes of rear frame are vertical/horizontal

- inertia axes of front frame are vertical/horizontal or aligned with steer axis
- no spin angular momentum of wheels
- point masses for the frames and/or wheels
- massless wheels
- massless front assembly
- vertical steer axis
- zero trail
- vanishing wheel radii

Such simplified models are generally incapable of self-stability, as one can deduce by analytical stability analysis of the more general model presented here, reduced to these special cases.

ii) No steer dynamics because steer is fully controlled by the rider. In these models balance is effected entirely as a result of rider-controlled steering angle, and the steer angle δ has no uncontrolled dynamics. For these models there is no need to derive the relatively less intuitive equation for steer dynamics. Appropriately controlled steer angle is indeed the only way to stabilize many simplified bicycles. Because velocipedes (primitive bicycles with vertical steer axis, no trail, and front-assembly essentially centred on the steer axis) were not self-stable, it is natural that all of the early mathematical analyses incorporated a controlled-steering assumption.

Note that controlled-steer-angle treatments cannot illuminate a bicycle's self stability because, in the small-angle regime, a bicycle with locked steering has no self stability. Many modern studies of controlled stability also reasonably use one or more of the mechanical simplifications as described in (i) above.

iii) Mathematically simplified models. To make the mathematics more tractable, or to illuminate controlling factors, some authors eliminate terms from the equations in an *ad hoc* fashion. A possible consequence of such mathematical, as opposed to mechanical, simplifications is that the resulting equations may not describe any particular physical model, so that mechanics based theorems (such as energy conservation) or intuitions may not apply.

A common geometric issue. Many of these simplified-dynamics analyses include some non-linear terms (e.g., $\sin\phi$ instead of ϕ). However, all purportedly non-linear simplified-bicycle treatments of which we are aware, starting with Bourlet (1894), do not actually write non-linear equations that correctly describe any mechanically simplified model of a bicycle. That is, the equations are not a special or limiting case of the equations of Whipple and his followers. In these treatments wheel base, trail, frame pitch, path curvature and other such quantities are treated as being independent of the lean angle, even for non-zero steer angle. That all these quantities do actually vary with lean angle for an ideal bicycle is demonstrated by considering a small leftward steer angle. As the lean angle goes to -90 degrees, with the bicycle almost lying on its left side, the front contact point moves forward around the front wheel approximately by 90 degrees, while the rear contact point moves backward around the rear wheel the same amount. This alters the wheel base length, the angle between ground traces of the two wheels, and the trail. Depending on the frame geometry, this lean also places the front contact well outside the rear frame's symmetry plane, and introduces substantial pitch of the rear frame about the rear axle, relative to the ground trace of the rear wheel. Even the simplest bicycle (with vertical steering axis, zero trail, and vanishing front-wheel radius) is subject to at least an alteration of the front-wheel track direction, due to the lean of a steered wheel. In particular for such a bicycle, the angle α that the front wheel track makes with the line connecting rear and front wheel contacts should obey $\cos\phi \tan\alpha = \tan\delta$ rather than the commonly used $\alpha = \delta$ (where δ is steer and ϕ is lean).

In some cases the authors may be making conscious approximations that are valid for modest lean angles, in some cases they are making mathematical models that are not intended to literally describe any simplification of a bicycle, and in some cases these seem to be errors. The resulting governing equations are sometimes correct descriptions of an inverted pendulum mounted on a controlled tricycle. Such a tricycle might be considered to be a simple model of a bicycle. But such a tricycle is not any limiting case of the Whipple bicycle.

1894–1899 Mathematician Carlo Bourlet devotes several papers and both editions of his encyclopedic bicycle treatise to the lateral balance of a steer-controlled velocipede (vertical steer axis and no trail). All inertias have vertical principal axes, and spin angular momentum of the wheels is included. The treatment is largely non-linear, but has the front-contact geometry issues described above. When linearized, his (1899b) final lean equation (29*bis*) lacks the gyroscopic moment from steer rate, but is otherwise correct.

Bourlet considers steering moves that can eliminate a lean, or follow a path. His final and most technical paper on bicycle dynamics (1899b) was awarded the Prix Fourneyron (submitted 1897, awarded 1898). Bourlet claims to have outlined the practical design factors leading to self-stability in another book dedicated to the design of bicycles (which we have not been able to find), but he does not address them analytically.

The **Prix Fourneyron** prize is offered biannually by the French Académie des Sciences (Gauja, 1917). In 1897, the Fourneyron mechanics challenge was "Give the theory of movement and discuss more particularly the conditions of stability of velocipedic devices" and was later amplified to include "whether in a straight line or a curve, on a flat plane or a slope." Boussinesq and Léauté were on the prize committee, and Appell was interested in the entries. Bourlet, Sharp and Carvallo submitted entries, as did others whose names and works are unfamiliar to us. Bourlet won the 1000-franc first place, Carvallo shared second (another 1000 francs) with Jacob and Sharp received honourable mention. Other than this prize, we know nothing more of Jabob's work. Both Bourlet and Carvallo published their entries, and Appell prominently cited these and other papers in more than one book. Shortly after the prize was awarded, Boussinesq published his own thorough analysis, and Léauté also published a note (which we have not seen). It seems that the dynamical analysis of bicycles is a French innovation. Bourlet (1894) may have started this, then the Prix announcement produced a singular peak of bicycle research activity.

1899 Physicist Joseph Boussinesq wrote two papers (1899a,b) (and four prior 'notes') on velocipede balance and control. These are similar in approach and content to prior work by Bourlet but slightly anticipating Bourlet's later more sophisticated dynamical modelling. Boussinesq neglects gyroscopic contributions expressed as E/R for each wheel, which Bourlet remarks correctly is a minor effect for lean dynamics. Boussinesq also notes that the system's centre of mass can usually be displaced sideways slightly by upper-body lean relative to the frame (This is the means by which an inverted double pendulum can be balanced by actuation of the connecting hinge. This effect is just as applicable to a bicycle that is not moving forward and is presumably essentially irrelevant: very few people can balance a non-moving bicycle by this means.). Self-stability was not addressed. The simplest point-mass bicycle model (vertical steer axis, no wheel mass, zero-radius wheels, no trail, no mass in the front assembly or equivalently mass balanced with respect to the steer axis, and controlled steer) seems to be due to Boussinesq.

1899 G.R.R. Routh (son of famous dynamicist E.J. Routh) considers steering strategies for lean stability and path following of a slightly more general model of a velocipede than was considered by Bourlet (1899b) and Boussinesq (1899a,b).

1910 Bouasse, in his dynamics textbook, reviews some geometric relations from Bourlet (1899b), and presents the model and analysis of Boussinesq (1899a,b).

1915 Bower investigates the stability of an uncontrolled velocipede via linearized equations that are missing terms (Hand, 1988). However, Bower's central result, that such a bicycle has no self-stability, happens to be correct. Comparable treatments without fully correct equations are also presented in Pearsall (1922, citing Bower), Lowell & McKell (1982, citing Pearsall), and Fajans (2000, citing Lowell & McKell).

1934 Loĭcjanskiĭ & Lur'e, in their textbook, study an uncontrolled velocipede. This is cited by Letov (1959), Neĭmark & Fufaev (1972), and revisited in Lobas (1978). We have not seen this book.

1948 Timoshenko & Young's well-known dynamics text presents the Boussinesq (simplest) bicycle analysis of Bouasse (1910).

1955 Haag independently derives bicycle equations of motion in his book, but simplifies by inconsistently ignoring various terms involving trail, spin momentum, front assembly mass, cross terms in the potential energy, etc. The resulting incorrect differential equations of a simplified bicycle model lead him to conclude (incorrectly) that bicycle self-stability is never possible.

1959 Letov gives correct linearized lean equations for a Boussinesq bicycle, attributing it to Loĭcjanskiĭ & Lur'e. Gyroscopic torques on the steering due to lean rate are incorporated in the dynamics of the steer controller, with reference to Grammel.

1967 Neĭmark & Fufaev, in their classic text (English translation,1972) on nonholonomic dynamics consider the full Whipple model (see section (c) below).

They then simplify to a velocipede model (vertical head, no trail, fore-aft balanced front steering). In the velocipede model the only contribution to self-correcting steer is gyroscopic precession due to lean, the basic mechanism for no-hands but controlled stability discussed in, e.g., Sharp (1896). However, Neĭmark & Fufaev also include linear viscous damping in the steering column. Without this damping, the steer angle is proportional to the integral of the lean angle. They mistakenly omit the mass from the second term in equation (2.65) (English edition p. 354), leaving a dimensionally incorrect quantity ν to propagate through to equations (2.67) and (2.68). However, the overall form of their differential equations is correct. Even for this simple model they find self-stability if there is sufficiently large steering friction, a result we trust despite the algebra error noted above.

1995 Getz & Marsden consider the possibility of following an arbitrary path without falling over, when not only the steering but also forward speed may be controlled. Their simplified non-linear Boussinesq model incorporates no wheel radius nor wheel inertias. Like some others before them (e.g. Bourlet 1894) this paper makes geometric assumptions that are equivalent to modelling a bicycle as an inverted pendulum mounted on a tricycle (see discussion above on a "a common geometric issue").

2005 One small part of the paper by Åström, Klein & Lennartsson treats a simplified bicycle model. The paper also describes decades of experiments on bicycle stability as well as the development of super-stable bicycles for teaching disabled children to ride (see also Richard Klein's web page, listed in the bibliography for this paper). Åström *et al.* is also discussed briefly in section (c) below.

The simplified model in Åström *et al.* is aimed at basic explanation of bicycle control and self-stability. We comment here only on the sections relevant to "Self-Stabilization" and not on the paper's focus, which concerns control.

In Åström *et al.* the reductions leading to the simple model come in two stages, mechanical and then mathematical. First Åström *et al.* assume that the wheels have no spin momentum and are thus essentially skates. They also assume that the front assembly has no mass or inertia. However, both non-zero head angle and non-zero trail are allowed and both point-mass and general-inertia rear-frame mass distributions are considered. Åström *et al.* then add further mathematical simplifications by neglecting non-zero trail contributions except in the static (non-derivative) terms. This eliminates the steer acceleration term in equation (14) therein (lean dynamics), and alters the steer rate term. In equation (9) (steer dynamics), where all torques arise only through trail, this eliminates the terms involving steer rate, steer acceleration, and lean acceleration.

Their reduced second order unforced (uncontrolled) steer equation implies that steer angle is proportional to lean angle (note the contrast with the integral feedback implicit in Neĭmark & Fufaev above). The resulting system is thus stabilized in the same way a skateboard is self-stable. In a skateboard mechanical coupling in the front "truck" enforces steer when there is lean, see Hubbard (1979) and pages 6 and 17 in Papadopoulos (1987). That bicycle lean and steer coupling might approximately reduce to the much simpler skateboard coupling is certainly an attractive idea.

However, the governing stability equation in Åström *et al.* , equation (15) appears to show the emergence of self-stability at high-enough speeds for quite arbitrary bicycle parameters. Examination of the full fourth order equations here (the pair of second order equations) applied to their simplified bicycle (without their additional mathematical simplifications) seems to show that stability is only obtained for special parameters. For example, the point-mass version is never stable. An extended-mass version can be stable, but only with a rather special mass distribution, as discussed in Papadopoulos (1987), on page 6 and figure 3 therein. Even for those parameter values in which their mechanical model can have self-stability it is not clear that having steer proportional to lean is an appropriate description of self-steer dynamics. So we have some doubt about the reduction of Åström *et al.* of even a simple class of bicycle models to second order skateboard-like equations. Limebeer and Sharp (2006) also question the conclusion of Åström *et al.* about the central role of trail in stability.

2006 Limebeer & Sharp present a large colourful historical review of various issues associated with bicycle and motorcycle handling, including anecdotes, simple models and complex models. One small part of Limebeer & Sharp includes an analysis of a Boussinesq-like simple bicycle. The non-linear lean equations therein implicitly assume a zero-radius front wheel. Also, in the first lean equation (4) the term $(\sigma - \dot{\phi}/v)$ was mistyped and should be $(\sigma - \dot{\psi}/v)$, which vanishes. Lean equation (5), and its linearization which is used for control analysis, are fully correct.

(*c*) *Equations of motion for a Whipple bicycle*

Here we discuss literature on linear equations of motion for more general bicycle models with uncontrolled steering. These are models that are similar to the Whipple model used in this paper. Papers in which e.g. toroidal wheels, tire-slip models, frame or rider elastic deformation, rider steering inputs or rider-controlled torso lean were difficult to remove from the analysis are generally not discussed. Non-linear treatments are not discussed systematically. The non-linear literature is further reviewed in Basu-Mandal *et al.* (2007).

1897–1900 Carvallo shared second prize in the Prix Fourneyron (see discussion of Bourlet in section (b) above), for a 186-page monograph on the dynamics of an uncontrolled monocycle (a single wheel surrounding a rider) and bicycle. Carvallo was already an accomplished applied mathematician and mechanician when he submitted this paper in 1897. As far as we know, this is the first genuine analysis of bicycle self-stability and slightly precedes Whipple. Although Carvallo's bicycle is slightly specialized, relative to Whipple, by neglecting the mass and moments of inertia of the front frame (in comparison to those of the front wheel), his equations for his model are correct. Carvallo identified the four standard eigenmodes, and presented equations for the upper (capsize) and lower (weave) limiting velocities for hands-free stability. Carvallo mentions the use of Grassman's geometric calculus, and stability calculations similar to Routh-Hurwitz. According to Carvallo, bicycle constructors of his time recommended that the steer axis be designed to pass under the front axle, half way between axle and ground, a feature approximately maintained in present day bicycle designs.

1899 Francis J. W. Whipple, apparently unaware of Carvallo, undertakes the second substantive analysis of the self-stability of a bicycle. Whipple was a Cambridge University undergraduate at the time, and was a Second Wrangler in the Tripos mathematics exam. Whipple later had a long career in mathematical meteorology. See Limebeer & Sharp (2006) for a short biography. Whipple's model is equivalent to the model presented here. His paper was awarded Honourable Mention for the prestigious Smith's Prize. Whipple first undertook the difficult task of a fully non-linear analysis, which was flawed by an incorrect expression of the front-wheel ground–contact vertical constraint. However, when linearized this error is irrelevant, and Whipple's linearized equations are correct, except for a few typographical errors. Whipple's results include scaling rules, the dynamic modes (nowadays known as weave and capsize), rider control inputs via torso lean, etc. Whipple also recognized that the exponential decay of lean and steer pertubations is not inconsistent with energy conservation. He cites Bourlet. Because of ambiguity in submission and publication times, Whipple is sometimes credited as the first to write equations of motion for a complex bicycle model, but it seems to us that Carvallo was actually first. Although Whipple had the same editor as Routh, neither cited the other.

Whipple and Carvallo laid solid foundations which have mostly been unused. Despite Carvallo being cited in two books by Appell, and both authors being cited by Klein & Sommerfeld (1910), and mentioned both in the 11th edition *Encyclopedia Britannica* (Gyroscope article), and in Grammel's 1920 gyroscope textbook, their achievements languished for decades. The only path by which Carvallo seemingly influenced posterity is via Noether (see Klein & Sommerfeld (1910), next in this list) who seems to follow his equations. Noether's analysis was expanded to the full Whipple model by Döhring (1953, 1955) Döhring (1955) was later translated into English by CALSPAN. Then, in turn, Döhring's equations were slightly misquoted by Singh & Goel (1971). As far as we know, no-one ever used Whipple's equations of motion.

1910 Klein & Sommerfeld's fourth volume on gyroscopes appears with an extensive chapter on bicycles written by Fritz Noether (brother of mathematician Emmy). These governing equations for a slightly simplified bicycle model (like Carvallo's), derived by Newton–Euler techniques used for other gyroscopic systems, are equivalent to those in Carvallo (1899) and are fully correct. While Noether claims to have compared his equations with Whipple as well as Carvallo, he erroneously states that Whipple used a Lagrangian derivation, and acknowledges neither Whipple's more general model nor his typographical errors. Noether's discussion of gyroscopic contributions and on degrees of freedom for holonomic and non-holonomic systems is clear and informative. Noether is keen to point out (incorrectly we think) that gyroscopic effects are necessary for self-stability, and that steering torques from the trailing front ground contact are not sufficient for stability. In effect Noether introduces, explains and dismisses the trail effects that were later a central interest of David E.H. Jones (1970).

1948 Kondo in Japan wrote reports on bicycles between 1948 and 1964. In discussion of a paper by Fu, Kondo says he wrote equations of motion for the meeting of JSME in November, 1948, unpublished (we have not seen this). Neither have we checked Kondo's later work that included tire models.

1949 Herfkens writes a report deriving equations for the Whipple model for the Dutch Institute for Bicycle Development (in Dutch). The linearized equations of motion are correct, except for some typos. On page 12, Eq. (28), $\cot(\beta)$ should be $\cot(\alpha)$. There is a missing term and misplaced brackets on page 13, and on page 14 the subscript of b_5 is missing in Eq. (34). The coefficients on page 15 agree with ours. Note that his steering angle β is our $\delta\cos(\lambda)$. Using Routh–Hurwitz stability criteria, he looked at the effect of some key parameters (namely trail and front-wheel inertia, and head angle) on the range of self-stability. He knew of Carvallo and Whipple but found them too analytical. Herfkens' report never seems to have been printed, distributed or cited. We only found it through a listing in the Delft card catalogue.

1951 Manning, in a technical report of the Road Research Laboratory in Britain, appears to provide correct non-linear configuration geometry, and a well-organized derivation of the linearized equations of motion for a full Whipple model. We have not yet checked the equations in detail, but the work shows great care. Manning acknowledges Carvallo's work but states "[it has] not yet been compared with the results in this note". He also writes "even if this work is merely a repetition of Carvallo's, it will be valuable to have the theory in a more accessible form, in a more up-to-date notation, and in English." That is exactly the sentiment of our present paper (but with respect to Whipple). Ironically, Manning's report is stamped "<u>RESTRICTED</u> Not for publication" and seems essentially unknown to the world. It is referenced by Roland (1973b) and the first edition of *Bicycling Science* by Whitt and Wilson (1974). Manning's paper is for sale by the Road Research Laboratory in Britain.

1953–1955 Ekkehard Döhring at the Technology University of Braunschweig, Germany, writes a Ph.D. thesis on the stability of a straight ahead running motorcycle. Döhring generalizes the model of Noether (Klein & Sommerfeld, 1910) to make the mass distributions as general as Whipple, whose work he seems not to have used. Döhring misdates Klein and Sommerfeld as 1890, the time when Klein and Sommerfeld started writing their multi-volume book. Döhring's equations agree with ours in detail (Hand, 1988).

Döhring's are the first perfectly correct equations of the Whipple model presented in the open literature (Whipple had small errors, Carvallo and Klein & Sommerfeld were slightly less general). Döhring also did some eigenvalue stability analysis and did experiments on a motor-scooter and two different motorcycles (1954) to validate his results. Döhring's 1955 paper was translated into English by CALSPAN but this translation is not published. Later citations of Döhring from Braunschweig follow CALSPAN and refer to him as Doehring, Brunswick.

Döhring's (1955) geometric equations (20) and (21) are not correct, but he did not use these for his final (correct) equations of motion.

Döhring mentions a "turn of the century" bicycle author named Galetti about whom we have no other information.

1963–1964 University of Wisconsin dissertations by Collins (1963) and Singh (1964) both involve multi-page equations employing chained parameter definitions. Collins relied on Wallace's (1929) problematic non-linear geometry, but this should not affect the correctness of his linearization. Although we did not compare Collins's equations in every detail, we noted a missing term and Psiaki (1979) found computational disagreement. Singh's subsequent conference and journal publications were based on Döhring's (1955) correct equations, rather than his own (see Singh and Goel (1971) below).

1966 Ge in Taiwan has a paper with a promising title. And Ge's other publications indicate expertise in rigid-body mechanics. But we have not seen the paper nor succeeded in contacting the person.

1967 Neĭmark & Fufaev, in their authoritative book on non-holonomic dynamics, present an exceptionally clear and thorough derivation of the equations of motion for a Whipple bicycle (we read only the 1972 English translation). Unfortunately, their treatment has several typographical errors, and also has a flaw in the potential energy: equation (2.30) which ignores downward pitch of the frame due to steering. This flaw was later corrected by Dikarev, Dikareva & Fufaev (1981) and independently by Hand (1988).

In 1970 there was a sudden increase in single-track vehicle research, perhaps because of the advent of digital computers and compact instrumentation, increased popularity of large motorcycles (and attendant accidents), and a surge in bicycle popularity. Most authors incorporated tire models which simplifies the equation formulation by avoiding having to implement kinematic constraints. But tire models add empirical parameters and complicate the resulting equations and their interpretation.

1970–1978 CALSPAN. One concentration of single-track research was at CALSPAN (then the Cornell Aeronautical Laboratory), funded by the U.S. government, Schwinn Bicycles and Harley-Davidson Motor Company. CALSPAN generated about 20 bicycle reports and papers. The CALSPAN program included hand calculations (involving linearized equations and algebraic performance indices for a somewhat simplified model), non-linear computer models (including high-order rider control inputs), and a comprehensive experimental program (including tire measurements and comparisons to experiments).

CALSPAN reports include: Rice & Roland (1970), Roland & Massing (1971), Roland & Lynch (1972), Rice & Roland (1972), Lynch & Roland (1972), Milliken (1972), Roland & Rice (1973), Roland & Kunkel (1973), Roland (1973a), Kunkel & Roland (1973), Roland (1973b), Anonymous (1973), Roland (1974), Rice (1974a), Davis & Cassidy (1974), Rice (1974b), Roland & Davis (1974), Rice (1974c), Davis (1975) Kunkel & Rice (1975), Anonymous (1975a), Rice *et al.* (1975), Anonymous (1975b), Kunkel (1975), Kunkel (1976), Rice (1976), Rice & Kunkel (1976), Rice (1978). Six of these reports are singled out below in their chronological places.

1970 Rice & Roland, in a CALSPAN report sponsored by the National Commission on Product Safety, included an appendix on non-linear equations (except linearized for small steer angles), where compliant, side-slipping tires avoid the need to apply lateral or vertical contact constraints. Rider lean relative to the frame is included. Thus the governing system includes all six velocities of a rigid body, plus the two extra degrees of freedom (steer and rider lean). The tabulated 8×8 first order system is forbiddingly complex, and terms such as wheel vertical force require a host of subsidiary equations to be defined. This report seems to contain the first use of the term 'mechanical trail' to describe the moment arm of the lateral front-contact forces about the steer axis.

1971 Roland & Massing, commissioned by the Schwinn bicycle company, write a CALSPAN report on the modelling and experimental validation of an uncontrolled bicycle. The mix of modelling, measuring, and testing is unusually thorough. After correcting an expression for tire slip, then linearizing and imposing constraints their equations agree with the equations here.

1971 Robin Sharp (unrelated to Archibald above) considers a model with tire slip, and front-assembly inertia tensor aligned with the steering axis. His partly non-linear model treats rear-frame pitch as zero, with a constant force acting upward on the front wheel. When he linearizes and takes the limit of infinite lateral tire stiffness, he introduces minor algebraic and typographical errors (see Hand 1988). Sharp does not base his equations on any prior work. This, Sharp's first of many bicycle and motorcycle dynamics papers, is the only paper we have mentioned so far that has had a lasting influence. This paper includes his original naming of the two major eigenmodes as 'weave' and 'capsize'. Most users of Sharp's equations include models for tire deformation.

1971 Singh & Goel say they use Döhring's (1955) equations (which are correct) and not Singh's (1964) equations (which are suspect). The equations of motion Singh & Goel present correspond well with Döhring's, except for two typographical errors in the first equation: in the first line, $V\ddot{\psi}$ should read $V\dot{\psi}$ and in the second line, $I_{13(I)}$ should read $I_{13(II)}$. Also they make use of the incorrect geometric relations (20) and (21) of Döhring (1955) (which are not used by Döhring himself). We were unable to reproduce Singh & Goel's reported eigenvalues.

1972 Roland & Lynch, commissioned by the Schwinn bicycle company, write a CALSPAN report on a rider control model for path tracking, bicycle tire testing, experimental tests to determine the effect of design parameters on the stability and manoeuvrability of the bicycle, and the development of computer graphics for display purposes. For the bicycle model the equations from Roland & Massing (1971) are used.

1972 In his Ph.D. thesis Weir explicitly compares his correct equations with the previous slightly incorrect and slightly specialized results of Sharp (1971).

Weir appears to be the first to perform such a check. Weir's thesis is widely cited.

1973 Eaton presents governing equations without derivation. He explains that he reconciled his own derivation with Sharp (1971) and Weir (1972), although using his own notation and somewhat embellishing the tire models.

1973 Roland (1973b) reports in the open literature, rare for CALSPAN, basically the same equations as in Roland & Massing (1971). Apparently few if any typos were corrected and some further typos seem to have been introduced.

1974 Rice (1974c) at CALSPAN uses simplified linearized analysis to develop steady-state and transient performance indices. He investigates the stiffness matrix (with rider lean included, statically equivalent to a lean moment), which requires only point-mass bicycle parameters. Much of the complication depends on tire parameters. As in Carvallo (1899) and Whipple (1899), formulae are given for capsize speed and for the low speed at which turning leaves the rear frame perfectly upright (when the displacement of the front contact and front centre of mass perfectly balance the lean moment of centrifugal force).

1975 Van Zytveld's Engineer's thesis on a robot bicycle controller develops equations that agree with ours, except for some incorrect terms involving 'rider lean' which drop out for the rigid rider assumption used in our Whipple model. According to van Zytveld, his advisor John Breakwell had developed independently equations of motion, without a rider-robot, that matched van Zytveld when simplified to remove rider lean (see also Breakwell 1982).

1976 Singh & Goel elaborate the Whipple model to allow deviations from left-right symmetry and incorporate more sophisticated tire models, leading to a very high order system of governing equations. The derivation appears to follow Sharp (1971) but we have not checked the results in detail.

1976 Rice writes a CALSPAN report on simplified dynamic stability analysis. He assumes all inertia tensors to have a vertical principal axis. This report explicitly identifies the frequently-occurring combination of terms which we call S_A.

1978 Weir & Zellner present Weir's equations but introduce a sign error in the mistaken belief they are making a correction, and commit several typographical errors (Hand 1988). Weir's thesis (1972), not Weir & Zellner, should be used for correct equations.

1978 Lobas (in translation misspelled as Gobas) extends the treatment by Neĭmark & Fufaev (1972) to add forward acceleration. When we set acceleration to zero, it appears that the static lean contribution to Lobas's steer equation is in error.

1979 Psiaki writes a dense Princeton undergraduate honors thesis on bicycle dynamics. Starting from a fully non-linear analysis based on Lagrange equations with non-holonomic constraints, he developed linearized equations for both an upright body and for a rigid bent body in hands-free turns. The equations of motion were complex and we have not checked them in detail, but his numerical results match ours to plotting accuracy suggesting, to us, correctness.

1981 Dikarev, Dikareva & Fufaev in equation (1.2) therein correct the errors in Neĭmark & Fufaev (1972). They write subtly about their "refinement" that "Note that in [Neimark and Fufaev] the expression for ϕ was obtained only to within first-order small terms... ". This should make their final equations correct, but we have not checked them in detail. This same error was corrected later independently by Hand (1988).

1985 Sharp presents a very comprehensive review of extended motorcycle dynamics equations, with an emphasis on capturing weave motions that seem to depend on tire and frame compliance. He has some errors in his description of the pre-1970 literature. Sharp (1985) identifies Sharp (1971) as 'confirmed', with which, but for minor errors, Hand (1988) agrees.

1987 Papadopoulos focused on achieving a compact notation and simple derivation of the equations of motion, using Hand's (1988) results as a check. The equations in the present paper are based on this Papadopoulos report.

1988 Hand's Cornell M.Sc. thesis compares a variety of publications and settles on a compact, transparent notation. Hand's thesis was advised by Papadopoulos and nominally by Ruina. Hand shows that several approaches, e.g. (Döhring 1955, Neĭmark & Fufaev 1972, Sharp 1971, and Weir 1972) all led to the same governing equations once errors were corrected. Hand, unaware of the work of Dikarev, Dikareva & Fufaev (1981), independently and similarly corrected Neĭmark & Fufaev (1972). Psiaki (2006, personal communication) also checked Hand's derivation which is similar to Neĭmark & Fufaev. Psiaki found terms missing from Hand's Lagrangian that fortunately have no effect on the equations of motion.

1988 Mears verified Weir's (1972) thesis and noted Weir's later (1978) errors. Mears also checked against Hand (1988).

The 1980s essentially mark an end to the development of sound equations for the Whipple bicycle model. Equations from Sharp, Weir or Eaton are widely cited as valid, even though explicit comparisons are rare. Subsequent research on motorcycle and bicycle dynamics tends to focus on elaborations necessary for modelling tire and frame deformations or on non-linear modelling.

1990 Franke, Suhr & Rieß derive non-linear equations of a bicycle, with neglect of some dynamic terms. This paper was the topic of an optimistic lead editorial in Nature by John Maddox (1990). We did not check the derivation. The authors did not find agreement between integration of their differential equations for small angles and the integration of the Papadopoulos (1987) equations (1990 — private communication). However, recently Lutz Aderhold (2005 — private communication) applied our benchmark bicycle parameters to an updated form of the Franke, Suhr & Rieß non-linear model and obtained agreement of eigenvalues in an approximately upright configuration, within plotting accuracy. Thus we expect that the well-conceived Franke *et al.* model is largely correct, but perhaps for details corrected by Aderhold.

1999 Lennartsson's PhD thesis has linear and non-linear numerical analysis of the Whipple bicycle model. In a 2006 personal communication Lennartsson said his 1999 numerical work agrees with the equations presented in this paper.

2004 Meijaard in preparing for this publication makes an independent derivation of the linearized equations of motion that agrees with the equations here.

2004 Schwab, Meijaard & Papadopoulos write a draft of the present paper and present it at a conference. The present paper uses a slightly different notation, and uses more carefully selected benchmark parameters. The present paper subsumes Schwab *et al.* (2004).

2005 Åström, Klein & Lennartsson present a wide-ranging paper, part of which is discussed in section (b) above. Another discussion in the paper builds on Schwab *et al.* (2004) and Papadopoulos (1987) and presents some parameter studies based on them. Åström *et al.* also presents Lennartsson's (1999) simulations from a general purpose rigid-body dynamics code. In addition to some non-linear dynamics observations, they show agreement with the benchmark equations in Schwab *et al.* (2004), although not with enough precision to assure correctness. Recently Lennartsson (2006 — private communication) made a high-precision comparison for the current benchmark parameters, and found agreement out to 12 decimal places.

2006 Meijaard & Schwab extend the Whipple bicycle model with torus wheels and the effects of braking and accelerations caused by moments at the hubs of the rear and front wheel, by a road gradient, and by aerodynamic drag.

2006 Kooijman, Schwab & Meijaard (2007) measure dynamic responses on an instrumented bicycle and validate the Whipple model by comparing between the experimentally measured eigenvalues and the eigenvalues predicted by the formulas here. They find good agreement in the speed range for 2 to 6 m/s.

2006 Limebeer & Sharp, in part of a large historical review paper, present the equations of Schwab *et al.* (2004) (the equations that the present publication archives) and also use the AutoSim model of Schwab *et al.* (2004) .

Although many reports, theses, and papers have models at least almost as general as Whipple's model, and many of these are largely correct, as yet there is no consensus that any peer-reviewed paper in English has correct equations. Carvallo (1899) and later Klein & Sommerfeld (1910) presented correct equations for a somewhat simplified bicycle. Whipple (1899) treated the general bicycle, but has a few typographical errors. By our (possibly incomplete) evaluation, the first error-free publication of full explicit equations for the general Whipple bicycle, and the only journal publication with full correct explicit equations, is that by Döhring (1955) in German. Sharp (1971) has a restricted front-assembly inertia, and introduces an error when specialized to tire-free rolling constraints. Singh & Goel (1971) introduce errors when presenting Döhring's correct equations. Weir & Zellner (1978) introduce an error when publishing Weir's correct thesis equations. Dikarev *et al.* (1981)

give correct equations implicitly. On the other hand, the theses by Weir (1972), Eaton (1973), van Zytveld (1975, when 'rider lean" is neglected'), Hand (1988) and Mears (1988) have correct explicit equations. Previous "gray" literature reports by the present authors also have correct explicit equations.

Supplementary (expanded) bibliography

This bibliography is a superset of the bibliography in the main paper. It includes all of the main paper's references and many more. Length limitations prevented a longer reference list in the main paper, but we wanted researchers to have access to a single comprehensive bibliography, this one. Reprints of some of the harder-to-find references are available at Andy Ruina's bicycle www site.

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2. SPACAR model

The SPACAR model for the benchmark bicycle is sketched in figure 5 and the input file for the SPACAR program describing this model is presented in §2 a.

Because the SPACAR program is based on finite element methods (FEM), the input file shows an FEM structure. The SPACAR input file is roughly divided into four parts: element declaration and connectivity, nodal data, boundary conditions, and some additional data like masses, inertias, applied forces and simulation settings.

In the first section of input in §2 a the elements are declared: they are given a type, a unique element number followed by a list of node numbers and an initial rotation axis. These element statements implicitly define the associated nodes. The nodes are either translational or rotational. A `hinge` element allows large relative rotation between two rotational nodes. A `wheel` element allows rolling contact at the contact point node. A `pinbody` element generates a node within a rigid body by which another finite element can be connected. Within this finite element approach a rigid body can be defined in two ways: either as a deformable element with all deformation modes set to zero or as a body with one three-degree-of-freedom translational node and one three-degree-of-freedom rotational node.

In the second section of the input file the nodes, which are placed at the centres of mass of the rigid bodies, are given their reference-configuration coordinates. Translational nodes have three coordinates (x, y, z) in a global reference frame whereas rotational nodes are parameterized by four Euler parameters. These parameters are set to $(1, 0, 0, 0)$, the unit transformation, in the reference configuration.

The approach in establishing a bicycle model is to consider it in a reference configuration: upright, orientated along the x-axis, and with the rear contact at the origin. This configuration is used to define nodal positions and rigid body orientations. Relative to this reference configuration it is easy to set an initial lean or steer angle and set the rates as initial conditions. However, to do a simulation from an arbitrary configuration, the system must be driven there by specifying a path from the initial configuration to the desired initial state.

Any consistent set of units may be used. Here SI units are used.

In the third section the boundary conditions are set. The implicit definition is that all nodes are free and all elements are rigid. A node's position or orientation in space can be fixed by the `fix` command; otherwise it is free to move in space. An element can be allowed to 'deform'; e.g. a hinge element is allowed to rotate, by the `rlse` command. A non-zero prescribed 'deformation' mode is specified by `inpute`, e.g. the forward motion of the bicycle in this example. For generating linearized equations of motion the `line` command identifies a degree of freedom to be used. The `enhc` command ties a non-holonomic constraint to a configuration space coordinate so as to identify those configuration coordinates for which the time derivative is not a velocity degree of freedom.

In the last section mass and inertia are added to the nodes, one value for translational nodes and six values for rotational nodes (the terms in the upper triangle portion of the inertia matrix in the initial configuration). Finally applied (constant) forces are added and some initial conditions and simulation settings are made.

When the program is run, for each output time step, all system variables (coordinates, deformations, speeds, accelerations, nodal forces, element forces, etc.) are written to standard files which can later be read by other software for plotting or analysis. At every time step the numeric values of the coefficients of the SPACAR semi-analytic linearization are also written to standard files.

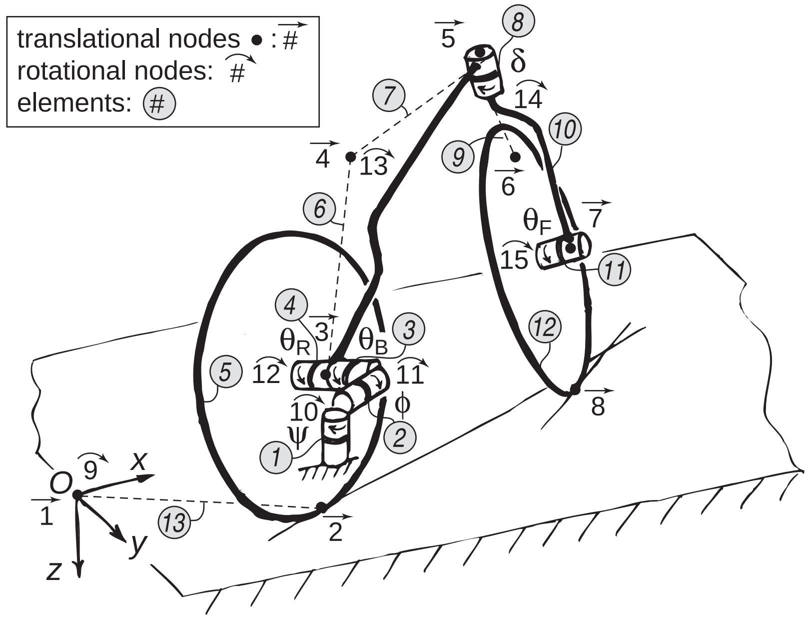

Figure 5. Sketch of the bicycle model for SPACAR input together with node numbers (straight arrows for translations $1\cdots8$, curved arrows for rotations $9\cdots15$) and element numbers encircled.

(*a*) *SPACAR Input file*

The sketch of this model is shown in figure 5.

```
% SPACAR input file for bicycle benchmark I
% SECTION 1, ELEMENT DECLARATION AND CONNECTIVITY:
% type number nodes     rotation axis vector
hinge    1     9 10      0 0 1  % yaw angle rear frame between node 9(ground) and 10
hinge    2    10 11      1 0 0  % lean angle rear frame between node 10 and 11
hinge    3    11 13      0 1 0  % pitch angle rear frame between node 11 and 13(frame)
hinge    4    13 12      0 1 0  % rear wheel rotation between 13(frame) and 12(wheel)
wheel    5     3 12  2   0 1 0  % rear wheel, cm nodes 3, 12, contact pnt 2
pinbody  6     4 13  3          % node 3(rear hub) in rigid body nodes 4, 13(frame)
pinbody  7     4 13  5          % node 5(head) in rigid body nodes 4, 13(frame)
hinge    8    13 14      0.32491969623291   0   1.0 % steering angle between 13 and 14
pinbody  9     5 14  6          % node 6(cm fork) in rigid body 5, 14(front frame)
pinbody 10     5 14  7          % node 7(front hub) in rigid body 5, 14(front frame)
hinge   11    14 15      0 1 0  % front wheel rotation between 14 and 15(wheel)
wheel   12     7 15  8   0 1 0  % front wheel, cm nodes 7, 15, contact pnt 8
pinbody 13     1  9  2          % node 2(rear contact pnt) in rigid body nodes 1, 9
% SECTION 2, NODAL DATA:
%  node  initial coordinates,      all rotational nodes are initialized:(1,0,0,0)
```

```
x   1    0    0  0     % fixed origin
x   2    0    0  0     % rear contact point
x   3    0    0 -0.3   % rear hub
x   4    0.3  0 -0.9   % cm rear frame + rigid rider
x   5    0.80757227339038 0 -0.9 % steering head
x   6    0.9  0 -0.7   % cm front fork + handle bars
x   7    1.02 0 -0.35  % front hub
x   8    1.02 0  0     % front contact point
% SECTION 3, BOUNDARY CONDITIONS:
% type number components
fix      1    1 2 3    % fix all(1,2,3) translations node 1(ground)
fix      9    1 2 3 4  % fix all(1,2,3,4)  rotations node 9(ground)
rlse     1    1        % release rotation(1) hinge 1: yaw
rlse     3    1        % release rotation(1) hinge 3: pitch
rlse    11    1        % release rotation(1) hinge 11: front wheel rotation
rlse    13    1 2 3    % release all relative displacements(1,2,3) in pinbody 13
inpute   4    1        % rotation(1) hinge 4 is prescribed motion for forward speed
line     2    1        % generate linearized eqns for rotation(1) hinge 2: lean
line     8    1        % generate linearized eqns for rotation(1) hinge 8: steering
% tie a a non-holonomic constraint to a configuration space coordinate
%type lmnt mode lmnt mode  (lmnt means element number)
enhc     5    4  13  1 % wheel 5 4=long slip tied to pinbody 13 1=x-disp node 2
enhc     5    5  13  2 % wheel 5 5=lat slip tied to pinbody 13 2=y-disp node 2
enhc    12    4   1  1 % wheel 12 4=long slip tied to hinge 1 1=yaw rear frame
enhc    12    5  11  1 % wheel 12 5=lat slip tied to hinge 11 1=front wheel rot
% SECTION 4, ADDITIONAL DATA: MASS, INERTIA, APPLIED FORCES, AND SIMULATION SETTINGS
%   node mass:(m) or mass moment of inertia:(Ixx,Ixy,Ixz,Iyy,Iyz,Izz)
mass  3   2.0                                 % mass    rear wheel
mass 12   0.0603  0  0        0.12 0  0.0603  % inertia rear wheel
mass  4  85.0                                 % mass    rear frame + rider
mass 13   9.2     0  2.4     11.0  0  2.8     % inertia rear frame + rider
mass  6   4.0                                 % mass    front frame + handle bars
mass 14   0.05892 0 -0.00756  0.06 0  0.00708 % inertia front frame + handle bars
mass  7   3.0                                 % mass    front wheel
mass 15   0.1405  0  0        0.28 0  0.1405  % inertia front wheel
%    node applied force vector (gravity used g = 9.81)
force  3  0 0  19.62   % gravity force rear wheel
force  4  0 0 833.85   % gravity force rear frame + rider
force  6  0 0  39.24   % gravity force front frame + handle bars
force  7  0 0  29.43   % gravity force front wheel
% initial conditions
ed  4 1 -3.333333333   % angular velocity in hinge 4(forward speed) set to -3.333333333
% simulation settings
epskin  1e-6           % set max constraint error for Newton-Raphson iteration
epsint  1e-5           % set max numerical integration error on coordinates
epsind  1e-5           % set max numerical integration error on speeds
timestep  100 2.0      % set number of output timesteps and simulation time
hmax  0.01             % set max step size numerical integration
end                    % end of run
eof                    % end of file
```

3. AutoSim model

The AutoSim 2.80 input file used for the bicycle model is listed below. The generalized coordinates and velocities are the same as those in the SPACAR model. Two massless intermediate reference frames have been introduced: a yawing frame describing the horizontal translation and yawing of the rear frame and a rolling frame describing the lean of the rear frame with respect to the yawing frame. These additional frames allow a better control over the choice of the generalized coordinates by the program. The holonomic constraint at the rear wheel is automatically satisfied. The holonomic constraint at the front wheel and the four non-holonomic constraints are explicitly defined in the input file. For more details on the syntax used see the AutoSim documentation.

(a) *AutoSim Input file*

```
;;;;  This is the file fietsap2.lsp, with the benchmark1 model.
;; Set up preliminaries:
(reset)
(si)
(add-gravity :direction [nz] :gees g)
(set-names g "Acceleration of gravity" )
(set-defaults g 9.81) ; this value is used in the benchmark.
;; The name of the model is set to the string "fiets".
(setsym *multibody-system-name* "fiets")
;; Introduce a massless moving reference frame. This frame has x and y
;; translational degrees of freedoms and a yaw rotational degree of freedom.
( add-body yawframe :name "moving yawing reference frame"
  :parent n :translate (x y) :body-rotation-axes z
  :parent-rotation-axis z :reference-axis x :mass 0
  :inertia-matrix 0 )
;; Introduce another massless moving reference frame. This frame has a rolling
;; (rotational about a longitudinal axis) degree of freedom.
( add-body rollframe :name "moving rolling reference frame" :parent yawframe
  :body-rotation-axes (x) :parent-rotation-axis x :reference-axis y :mass 0
  :inertia-matrix 0 )
;; Add the rear frame of the bicycle. The rear frame has a pitching (rotation
;; about the local lateral y-axis of the frame) degree of freedom.
( add-body rear :name "rear frame" :parent rollframe
  :joint-coordinates (0 0 "-Rrw") :body-rotation-axes y
  :parent-rotation-axis y :reference-axis z :cm-coordinates (bb 0 "Rrw-hh")
  :mass Mr :inertia-matrix ((Irxx 0 Irxz) (0 Iryy 0) (Irxz 0 Irzz)) )
( set-names
  Rrw "Rear wheel radius"
  bb "Longitudinal distance to the c.o.m. of the rear frame"
  hh "Height of the centre of mass of the rear frame"
  Mr "Mass of the rear frame"
  Irxx "Longitudinal moment of inertia of the rear frame"
  Irxz "Minus product of inertia of the rear frame"
  Iryy "Transversal moment of inertia of the rear frame"
  Irzz "Vertical moment of inertia of the rear frame" )
( set-defaults Rrw 0.30 bb 0.3 hh 0.9
  Mr 85.0 Irxx 9.2 Irxz 2.4 Iryy 11.0 Irzz 2.8 )
```

```
;; Add the rear wheel of the vehicle. This body rotates
;; about the y axis of its physical parent, the rear frame.
( add-body rw :name "rear wheel" :parent rear :body-rotation-axes y
  :parent-rotation-axis y :reference-axis z :joint-coordinates (0 0 0)
  :mass Mrw :inertia-matrix (irwx irwy irwx) )
( set-names
  Mrw "mass of the rear wheel"
  irwx "rear wheel in-plane moment of inertia"
  irwy "rear wheel axial moment of inertia" )
(set-defaults Mrw 2.0 irwx 0.0603 irwy 0.12)
;; Now we proceed with the front frame.
;; Define the steering and reference axes of the front frame.
;; Add in the front frame: define some points.
( add-point head :name "steering head point B" :body n
  :coordinates (xcohead 0 zcohead) )
( add-point frontcmpoint :name "c.o.m. of the front frame" :body n
  :coordinates (xfcm 0 zfcm) )
( set-names
  epsilon "steering head angle"
  xcohead "x coordinate of the steering head point B"
  zcohead "z coordinate of the steering head point B"
  xfcm "x coordinate of the c.o.m. of the front frame"
  zfcm "z coordinate of the c.o.m. of the front frame" )
( set-defaults epsilon 0.314159265358979316
  xcohead 1.10 zcohead 0.0 xfcm 0.90 zfcm -0.70 )
( add-body front :name "front frame" :parent rear :body-rotation-axes z
  :parent-rotation-axis "sin(epsilon)*[rearx]+cos(epsilon)*[rearz]"
  :reference-axis "cos(epsilon)*[rearx]-sin(epsilon)*[rearz]"
  :joint-coordinates head :cm-coordinates frontcmpoint :mass Mf
  :inertia-matrix ((Ifxx 0 Ifxz) (0 Ifyy 0) (Ifxz 0 Ifzz))
  :inertia-matrix-coordinate-system n )
( set-names
  Mf "Mass of the front frame assembly"
  Ifxx "Longitudinal moment of inertia of the front frame"
  Ifxz "Minus product of inertia of the front frame"
  Ifyy "Transversal moment of inertia of the front frame"
  Ifzz "Vertical moment of inertia of the front frame" )
( set-defaults Mf 4.0
  Ifxx 0.05892 Ifxz -0.00756 Ifyy 0.06 Ifzz 0.00708 )
;; Add in the front wheel:
( add-point fw_centre :name "Front wheel centre point" :body n
  :coordinates (ll 0 "-Rfw") )
( add-body fw :name "front wheel" :parent front :body-rotation-axes y
  :parent-rotation-axis y :reference-axis "[nz]"
  :joint-coordinates fw_centre :mass Mfw :inertia-matrix (ifwx ifwy ifwx) )
( set-names
  ll "Wheel base"
  Rfw "Radius of the front wheel"
  Mfw "Mass of the front wheel"
  ifwx "In-plane moment of inertia of the front wheel"
  ifwy "Axial moment of inertia of the front wheel" )
(set-defaults ll 1.02 Rfw 0.35 Mfw 3.0 ifwx 0.1405 ifwy 0.28)
```

```
;; The system is complete, except for the contact constraints at the wheels.
;; The holonomic constraint at the rear wheel is automatically satisfied.
;; The rear wheel slip is zero.
( add-speed-constraint "dot(vel(yawframe0),[yawframex])+Rrw*(ru(rear)+ru(rw))"
  :u "tu(yawframe,1)" )
(add-speed-constraint "dot(vel(yawframe0),[yawframey])" :u "tu(yawframe,2)")
;; For the front wheel we have a holonomic constraint for the contact and two
;; non-holonomic slip constraints. The slip velocities are defined now.
(setsym singammafw "dot([fwy],[nz])")
(setsym cosgammafw "sqrt(1-@singammafw**2)")
(setsym fw_rad "([nz] - [fwy]*@singammafw)/@cosgammafw")
(setsym slipfw_long "dot(vel(fw0)+Rfw*cross(rot(fw),@fw_rad),[nx])")
;; No longitudinal slip on front wheel;
;; eliminate rotational velocity about the axis
(add-speed-constraint "@slipfw_long" :u "ru(fw)")
;; normal constraint; eliminate the pitch angle
(setsym slipfw_n "dot(vel(fw0)+Rfw*cross(rot(fw),@fw_rad),[nz])")
(add-speed-constraint "@slipfw_n" :u "ru(rear)")
(add-position-constraint "dot(pos(fw0),[nz])+Rfw*@cosgammafw" :q "rq(rear)")
;; No lateral slip on front wheel; eliminate yaw rate of the yawing frame
(setsym slipfw_lat "dot(vel(fw0)+Rfw*cross(rot(fw),@fw_rad),[ny])")
(add-speed-constraint "@slipfw_lat" :u "ru(yawframe)")
(dynamics)
(linear)
```

4. Decoupling of lateral and forward dynamics: $\dot{v} = 0$

Here we present in more detail why symmetry decouples lean and steer from forward motion in the linearized equations. As explained in §4 c, some configuration variables (ignorable coordinates) do not show up in the equations of motion and so are not of central interest. These include position $(x_{\mathrm{P}}, y_{\mathrm{P}})$ on the plane, the yaw ψ, and the net wheel rotations θ_{R} and θ_{F}. Of interest is the evolution of the right lean ϕ, the right steer δ, and backwards rear wheel rotation rate $\dot{\theta}_{\mathrm{R}}$. For conceptual and notational convenience define forward speed as $v = -r_{\mathrm{R}}\dot{\theta}_{\mathrm{R}}$ and use v instead of $\dot{\theta}_{\mathrm{R}}$ in the discussion below. First we establish the forward motion governing equation when there is no applied thrust.

Without writing explicit non-linear equations, we know they have this in-plane exact reference solution:

$$v(t) = v^*, \quad \phi(t) = 0 \quad \text{and} \quad \delta(t) = 0 \tag{B 1}$$

where v^* is an arbitrary constant.

The linearized equations are for small perturbations about this reference solution. For notational simplicity we take the lean and steer perturbations as merely ϕ and δ recognizing that we are discussing only infinitesimal values of these variables. For the forward motion take the perturbation to be $\hat{v}$.

For the argument below we depend only on the linearity of the equations, and not their detailed form. Take an arbitrary set of initial conditions to be $(\hat{v}_0, \phi_0, \delta_0)$. At some definite time later, say $t_d = 1\,\mathrm{s}$ for definiteness, the values of the speed lean and steer at t_d must be given by

$$\begin{bmatrix} \hat{v}_d \\ \phi_d \\ \delta_d \end{bmatrix} = \mathbf{A} \begin{bmatrix} \hat{v}_0 \\ \phi_0 \\ \delta_0 \end{bmatrix} \tag{B 2}$$

for any possible combination of $\hat{v}_0, \phi_0$, and δ_0. The matrix

$$\mathbf{A} = \begin{bmatrix} A_{vv} & A_{v\phi} & A_{v\delta} \\ A_{\phi v} & A_{\phi\phi} & A_{\phi\delta} \\ A_{\delta v} & A_{\delta\phi} & A_{\delta\delta} \end{bmatrix}$$

depends on which definite time t_d is chosen. Because the bicycle rolls on a flat horizontal isotropic plane and there is no time-dependent forcing, the coefficient matrix $\mathbf{A}$ is dependent on the time interval t_d but independent of the starting time.

Now consider an initial condition 1 where only the lean is disturbed:

$$\begin{bmatrix} \hat{v}_0^1 \\ \phi_0^1 \\ \delta_0^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

where we think of 1 as a small perturbation. This results in a perturbation a time t_d later of

$$\begin{bmatrix} \hat{v}_d^1 \\ \phi_d^1 \\ \delta_d^1 \end{bmatrix} = \begin{bmatrix} A_{v\phi} \\ A_{\phi\phi} \\ A_{\delta\phi} \end{bmatrix}$$

where the right side is the middle column of **A**. Now consider the opposite perturbation 2 with

$$\begin{bmatrix} \hat{v}_0^2 \\ \phi_0^2 \\ \delta_0^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

which results in a perturbation a time t_d later of

$$\begin{bmatrix} \hat{v}_d^2 \\ \phi_d^2 \\ \delta_d^2 \end{bmatrix} = \begin{bmatrix} -A_{v\phi} \\ -A_{\phi\phi} \\ -A_{\delta\phi} \end{bmatrix}$$

where the right side is the negative of the middle column of **A**; for a linear system negating the input negates the output.

Now we invoke lateral symmetry. If knocking a bicycle to the left causes it to speed up, knocking it to the right must cause it to speed up equally. So

$$v_d^2 = v_d^1 \quad \Rightarrow \quad A_{v\phi} = -A_{v\phi} \quad \Rightarrow \quad A_{v\phi} = 0.$$

Now we can similarly apply a rightwards perturbation to just the steer. On the one hand linearity requires a negative steer to have the negative effect on forward speed. On the other hand, lateral symmetry requires that a rightwards steer perturbation have an equal effect as a leftwards perturbation. Thus, by the same reasoning as for lean we get $A_{v\delta} = 0$.

Next, consider perturbations to just the forward speed $\hat{v}$. By symmetry these can cause neither left or right lean or steer. So $A_{\phi v} = A_{\delta v} = 0$. Thus symmetry reduces the matrix **A** to having zeros off the diagonal in both the first row and the first column.

Finally, we know the steady upright solution is an exact non-linear solution for any v^*. Assuming that the full non-linear equations have unique solutions for any given initial conditions, a perturbation in v^* merely leads to a new constant speed solution at the perturbed v^*. Thus, $\hat{v}_d = \hat{v}_0$ and $A_{vv} = 1$.

Altogether this means that the linearized equations giving the perturbed values of the state at time t_d in terms of the initial perturbation are necessarily of the form of equation (B 2) with

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_{\phi\phi} & A_{\phi\delta} \\ 0 & A_{\delta\phi} & A_{\delta\delta} \end{bmatrix}.$$

This form must hold for any t_d. Thus perturbations in lean ϕ and steer δ never have influence on the forward speed v and *vice versa*, perturbations in speed have no influence on lean and steer. Similarly, lean and steer rates $(\dot{\phi}, \dot{\delta})$ are also decoupled from forward motion. Further, because $\hat{v}_d = \hat{v}_0$ for all time, $\hat{v}$ is a constant so

$$\dot{v} = 0. \tag{B 3}$$

This is the first of the three linearized equations of motion.

Similar arguments show that forward forcing does not cause lean or steer and that lateral forcing does not cause changes in speed (to first order). Thus a bicycle which is forced to go at exactly constant speed in a full non-linear analysis has the same linearized lean and steer governing equations as for the bicycle that is free in forward motion. Such is confirmed by SPACAR numerical analysis where

1. For sufficiently small deviations from upright, both constant energy and constant speed give the same solutions (to about 9 digits) and

2. Both constant speed and constant energy give the same values for the numerical coefficients in the linearized equations. These are also the same as the values presented in the body of the paper here for our ad hoc linearization (to about 14 digits). These two comparisons were also performed by Lennartsson (2006, personal communication).

5. A simplified benchmark model

In a second benchmark various simplifications are made to permit comparison with less complete models. The design parameters are according to table 1 but with the following changes. Both wheels are planar, $I_{yy} = 2I_{xx}$, and identical with: $m_{\mathrm{R}} = m_{\mathrm{F}} = 3$ kg, $r_{\mathrm{R}} = r_{\mathrm{F}} = 0.35$ m, and $(I_{\mathrm{R}xx}, I_{\mathrm{R}yy}) = (I_{\mathrm{F}xx}, I_{\mathrm{F}yy}) = (0.14, 0.28)$ kgm^2. The mass of the rear frame and body assembly B is $m_{\mathrm{B}} = 85$ kg located at $(x_{\mathrm{B}}, z_{\mathrm{B}}) = (0.3, -0.9)$ m, whereas the mass moment of inertia is zero, $\mathbf{I}_{\mathrm{B}} = \mathbf{0}$. The front frame H has neither mass, $m_{\mathrm{H}} = 0$, nor inertia moments, $\mathbf{I}_{\mathrm{H}} = \mathbf{0}$. Substitution of these values of design parameters for the simplified benchmark bicycle in the expressions from appendix A results in the following values for the entries in the mass matrix from (A 20),

$$\mathbf{M} = \begin{bmatrix} 69.865 & 1.868\,727\,853\,976\,56 \\ 1.868\,727\,853\,976\,56 & 0.239\,079\,887\,561\,38 \end{bmatrix},$$

the entries in the constant stiffness matrix from (A 22) which are to be multiplied by gravity g,

$$\mathbf{K}_0 = \begin{bmatrix} -78.6 & -2.226\,580\,876\,684\,00 \\ -2.226\,580\,876\,684\,00 & -0.688\,051\,330\,245\,63 \end{bmatrix},$$

the coefficients of the stiffness matrix from (A 24) which are to be multiplied by the square of the forward speed v^2,

$$\mathbf{K}_2 = \begin{bmatrix} 0, & 74.779\,149\,614\,579\,71 \\ 0, & 2.306\,586\,620\,338\,71 \end{bmatrix},$$

and finally the coefficients of the "damping" matrix from (A 26) which are to be multiplied by the forward speed v,

$$\mathbf{C}_1 = \begin{bmatrix} 0 & , & 29.140\,558\,140\,953\,37 \\ -0.880\,193\,481\,747\,67, & & 1.150\,360\,143\,808\,13 \end{bmatrix}.$$

To facilitate comparison with equations or results derived using different methods, eigenvalues are presented. These eigenvalues in the forward speed range show the same structure as those from the full benchmark bicycle, see figure 3, but with slightly different values. The precise eigenvalues for the simplified bicycle benchmark at some forward speeds are presented in table 3.

Table 3. *Some characteristic values for the forward speed v and the eigenvalues λ from the linearized stability analysis for the simplified benchmark bicycle from §5. Fourteen digit results are presented for benchmark comparisons. (a) $v = 0$, weave speed v_w, capsize speed v_c and the speed with a double root v_d. In the forward speed range of $0 \leq v \leq 10$ m/s: (b) Complex (weave motion) eigenvalues λ_{weave}, and (c) Real eigenvalues λ.*

(a)

v [m/s]	λ [1/s]
$v = 0$	$\lambda_{s1} = \pm 3.321\,334\,354\,955\,67$
$v = 0$	$\lambda_{s2} = \pm 5.695\,461\,613\,073\,60$
$v_d = 0.804\,279\,462\,741\,01$	$\lambda_d = 4.043\,478\,683\,070\,60$
$v_w = 5.405\,811\,651\,738\,11$	$\lambda_w = 0 \pm 7.746\,411\,825\,301\,59\,i$
$v_c = 5.706\,991\,804\,685\,07$	0

(b)

v [m/s]	Re(λ_{weave}) [1/s]	Im(λ_{weave}) [1/s]
0	–	–
1	3.915 605 159 008 03	0.676 636 216 381 60
2	3.145 971 626 952 20	1.947 971 866 614 21
3	2.096 627 566 535 66	3.144 568 094 683 27
4	0.910 809 011 944 21	4.881 202 124 548 49
5	0.198 648 678 113 17	6.936 393 452 637 19
6	−0.245 683 866 155 55	8.903 125 360 683 31
7	−0.589 203 483 851 70	10.790 930 464 293 57
8	−0.883 875 624 871 00	12.628 966 109 587 14
9	−1.150 515 263 118 26	14.434 482 871 116 77
10	−1.399 313 952 184 76	16.217 648 368 548 84

(c)

v [m/s]	$\lambda_{capsize}$ [1/s]	$\lambda_{castering}$ [1/s]
0	−3.321 334 354 955 67	−5.695 461 613 073 60
1	−3.339 571 399 042 72	−6.577 674 865 894 17
2	−3.122 857 194 829 05	−7.341 157 952 916 98
3	−2.196 003 785 406 69	−8.255 359 188 427 08
4	−0.787 290 747 535 25	−9.378 471 064 036 38
5	−0.161 936 233 356 19	−10.665 540 857 474 20
6	0.039 380 255 445 46	−12.064 228 204 659 15
7	0.114 168 685 341 41	−13.538 013 346 083 71
8	0.143 031 193 913 90	−15.063 567 519 538 39
9	0.152 632 341 109 21	−16.625 925 337 159 89
10	0.153 494 106 064 82	−18.215 225 670 903 33