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ON DYNAMICALLY EQUIVALENT FORCE SYSTEMS AND THEIR APPLICATION TO THE BALANCING OF A BROOM OR THE STABILITY OF A SHOE BOX

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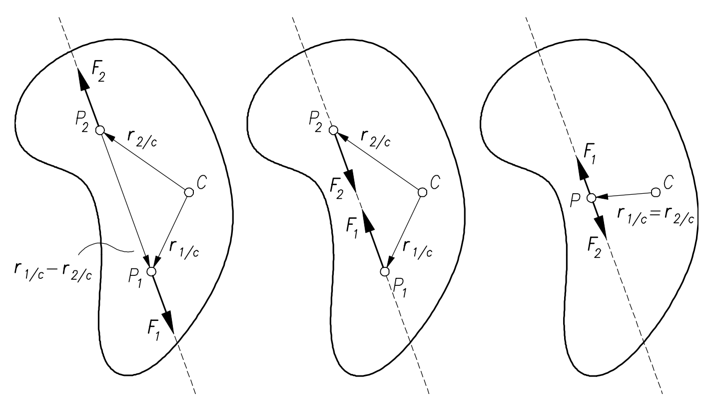
ABSTRACT

The stability of a rigid body on which two forces are in equilibrium can be assessed intuitively. In more complex cases this is no longer true. This paper presents a general method to assess the stability of complex force systems, based on the notion of dynamic equivalence. A resultant force is considered dynamically equivalent to a given system of forces acting on a rigid body if the contributions to the stability of the body of both force systems are equal. It is shown that the dynamically equivalent resultant force of two given constant forces applies at the intersection of its line of action and the circle put up by the application points of the given forces and the intersection of their lines of action. The determination of the combined center of mass can be considered as a special case of this theorem. Two examples are provided that illustrate the significance of the proposed method. The first example considers the suspension of a body, by springs only, that is statically balanced for rotation about a virtual stationary point. The second example treats the roll stability of a ship, where the metacentric height is determined in a natural way.

INTRODUCTION

The stability of a two-force system in equilibrium can be assessed intuitively at a single glance. Static equilibrium is achieved when the two forces are of equal magnitude, opposite sense, and have the same line of action. However, although all in equilibrium, the rigid bodies in Fig. 1 have different stability. Assuming constant forces (both in terms of magnitude and direction), it is readily seen (a more rigorous derivation will follow) that stable equilibrium results if the forces are pointing away from each other (Fig. 1a), whereas unstable equilibrium results if the forces are pointing towards each other (Fig. 1b). In the borderline case, the two points of application coincide, rendering the system in neutral equilibrium (Fig. 1c). Evidently, the point of application of the forces on their line of action is of vital importance to the *stability*, even though it does not affect the static equilibrium itself.

In an *n*-force system in equilibrium, the judgment of the stability is much less obvious. Figure 2 shows an example of a rigid body on which three forces are in equilibrium. One



(a) (b) (c)

Figure 1. Two-force system in equilibrium acting on a rigid body: (a) stable equilibrium, (b) unstable equilibrium, (c) neutral equilibrium.

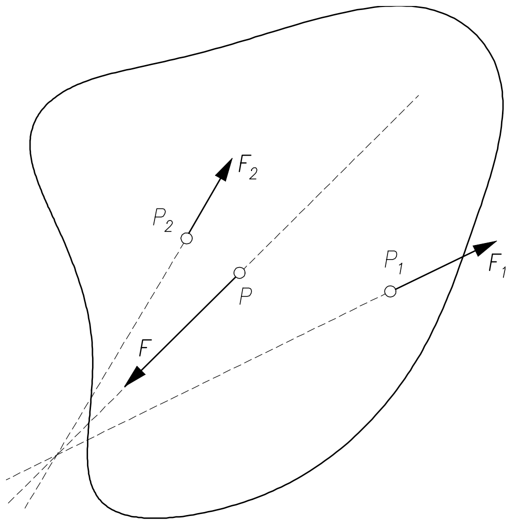


Figure 2   A three-force system acting on a rigid body. Static equilibrium determines the magnitude and line of action of the third force, but the stability is not readily assessed.

strategy to assess the stability of such systems would be to compose forces two by two until a two-force system is obtained, the stability of which can then be assessed as above. Clearly, the conventional procedure of force composition is not sufficient, since it does not yield the point of application of the resultant force. Since the conventional procedure is aimed at equilibrium (not at stability), it yields in fact the *statically equivalent force system*: an equivalent force for which the point of application on the line of action is not relevant. Therefore, in order to find the equivalent stability, a procedure is required to compose forces in a *dynamically equivalent* way, *i.e.* in such a way that the stability contribution of the resultant force is equal to the stability contribution of the two original forces. This implies that in addition to the magnitude and line of action of the resultant force, also the point of application is to be found.

This paper proposes a procedure for the determination of the dynamically equivalent resultant force system, for any given force system. The treatise will be limited to the planar case of rigid body motion. The study will not be limited to the judgment of stability of static equilibrium. It will be shown that the proposed procedure for the determination of dynamically equivalent forces is valid for the contribution of forces to any state of motion.

STATIC EQUIVALENCE

This section will use the Newton-Euler equations of motion to investigate the contribution of forces to the nominal state of motion of a rigid body. In matrix form, the equations of motion for a rigid body under the influence of n external forces read:

$$\begin{bmatrix} \Sigma \boldsymbol{F}_i \\ \Sigma(\boldsymbol{A}\boldsymbol{r}_{i/c})^T \boldsymbol{F}_i \end{bmatrix} = \begin{bmatrix} m\boldsymbol{E}_2 & 0 \\ 0 & I_c \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{r}}_c \\ \ddot{\varphi} \end{bmatrix} \tag{1}$$

where m is the mass of the rigid body, $\boldsymbol{E}_2$ is the 2x2 identity matrix, I_c is the mass moment of inertia about the center of mass C, $\ddot{\boldsymbol{r}}_c$ is the acceleration of the center of mass, $\ddot{\varphi}$ is the rotational acceleration of the rigid body, $\boldsymbol{F}_i$ are external forces acting on the body, while the summation runs from i=1 through n, where n is the number of forces. The vector notation is as follows. The vector $\boldsymbol{r}_i$ is the position vector of the point of application P_i of the force $\boldsymbol{F}_i$ relative to the inertial reference frame, whereas $\boldsymbol{r}_{i/c}$ denotes a position vector relative to point C of the rigid body, both expressed in the global coordinate system. The subscript ^{T} denotes transposition, and the matrix $\boldsymbol{A}$, which reads:

$$\boldsymbol{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

is in fact the rotation matrix for $\pi/2$ radians and is used to effect the planar form of the vector multiplication $\boldsymbol{r}_{i/c} \times \boldsymbol{F}_i$. It is noted that the summation convention is not used, but individual variables are considered.

Suppose that $\Sigma\boldsymbol{F}_i$ includes $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ among other forces, and that $\Sigma(\boldsymbol{A}\boldsymbol{r}_{i/c})^T\boldsymbol{F}_i$ includes their moment contributions $(\boldsymbol{A}\boldsymbol{r}_{1/c})^T\boldsymbol{F}_1$ and $(\boldsymbol{A}\boldsymbol{r}_{2/c})^T\boldsymbol{F}_2$, among other terms. Then a single force $\boldsymbol{F}_r$ has the same contribution to the nominal state of motion, as represented by Eq. 1, under the following conditions:

$$\boldsymbol{F}_r = \boldsymbol{F}_1 + \boldsymbol{F}_2 \tag{3}$$

$$(\boldsymbol{A}\boldsymbol{r}_{r/c})^T \boldsymbol{F}_r = (\boldsymbol{A}\boldsymbol{r}_{1/c})^T \boldsymbol{F}_1 + (\boldsymbol{A}\boldsymbol{r}_{2/c})^T \boldsymbol{F}_2 \tag{4}$$

These equations form the basis for the well-known conditions for static force composition. The resulting transformed force systems, often called "equivalent force systems" (*e.g.* [1]), are in fact *statically* equivalent force systems. Moreover, the contribution of the force $\boldsymbol{F}_r$ thus found is equivalent to the contribution of the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ together, not only to the state of static equilibrium but also to any nominal state of motion. Furthermore, it can be shown that the conditions 3 and 4 are valid with respect to any point of the rigid body (*i.e.* point C need not be the center of mass).

DYNAMIC EQUIVALENCE

The vector equation Eq. 3 determines the magnitude and direction of the resultant force $\boldsymbol{F}_r$, whereas the line of action of $\boldsymbol{F}_r$ is determined by the scalar equation Eq. 4. However, Eqs. 3 and 4 do *not* determine the point of application of the force $\boldsymbol{F}_r$. This is not important for the contribution to the nominal state, but for the *stability* of the nominal state, the point of application of a force is essential. Therefore, it is important that *static* equivalence is well distinguished from *dynamic* equivalence. In this paper, a resultant force will be considered dynamically equivalent to a system of forces acting on a rigid body if the contributions to the *stability* of the body of the resultant force and the original system of forces are equal.

Stability essentially is a dynamic phenomenon. Therefore, small variations about the nominal state of the body will be considered to investigate the stability of the nominal state of a rigid body in the planar case. The equations of motion for any nominal state of motion were given in Eq. 1. Expansion of these (for first order variations), subtraction of the nominal state, and rearranging of terms, yields the equations for the variations about the nominal state:

$$\boldsymbol{M}\Delta\ddot{\boldsymbol{x}} + \boldsymbol{K}\Delta\boldsymbol{x} = 0 \tag{5}$$

where the mass matrix is

$$\boldsymbol{M} = \begin{bmatrix} m\boldsymbol{E}_2 & 0 \\ 0 & I_c \end{bmatrix} \tag{6}$$

and where the tangent stiffness matrix is

$$\boldsymbol{K} = \begin{bmatrix} -\left(\Sigma\boldsymbol{F}_i\right)_{,\boldsymbol{r}_c} & -\left(\Sigma\boldsymbol{F}_i\right)_{,\varphi} \\ -\Sigma\left((\boldsymbol{A}\boldsymbol{r}_{i/c})^T\boldsymbol{F}_i\right)_{,\boldsymbol{r}_c} & -\left(\Sigma(\boldsymbol{A}\boldsymbol{r}_{i/c})^T\boldsymbol{F}\right)_{,\varphi} \end{bmatrix} \tag{7}$$

and where $\boldsymbol{x} = [\boldsymbol{r}_c^T \quad \varphi]^T$. A subscript with comma is used to denote partial derivatives, *e.g.* $\boldsymbol{F}_{,\varphi} = \partial\boldsymbol{F}/\partial\varphi$. The terms in the matrix $\boldsymbol{K}$ show that the forces and their moment contributions must be differentiated with respect to the position and orientation of the rigid body, $\boldsymbol{r}_c$ and φ, respectively, implying that the *character* of these forces affects the result. In the following sections, two kinds of forces will be addressed: constant forces and central linear forces.

Constant Forces

For constant forces, *i.e.* forces due to a homogenous force field, hence with invariant magnitude and direction, most of the elements in the tangent stiffness matrix $\boldsymbol{K}$ vanish. The only remaining term is $-(\Sigma(\boldsymbol{A}\boldsymbol{r}_{i/c})^T\boldsymbol{F}_i)_{,\varphi}$, representing the change in moments due to a small rotation. To elaborate this term, a local coordinate system is fixed to the body at point C such that $\boldsymbol{r}_i = \boldsymbol{r}_c + \boldsymbol{r}_{i/c} = \boldsymbol{r}_c + \boldsymbol{R}\boldsymbol{r}'_{i/c}$, where

$$\boldsymbol{R} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \tag{8}$$

is the rotation matrix describing the transformation from the body fixed coordinates $\boldsymbol{r}'$ into the space fixed coordinates $\boldsymbol{r}$ as in

$$\boldsymbol{r} = \boldsymbol{R}\boldsymbol{r}' \tag{9}$$

The rotation matrix is an orthogonal matrix (*e.g.* [2]), which means that

$$\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{E}_2 \tag{10}$$

A direct result from this orthonormality is that the transposed of $\boldsymbol{R}$ equals the inverse, as in $\boldsymbol{R}^T = \boldsymbol{R}^{-1}$. Furthermore, if we differentiate the identities in Eq. 10 with respect to φ we find:

$$\boldsymbol{R}_{,\varphi}\boldsymbol{R}^T + \boldsymbol{R}\boldsymbol{R}^T_{,\varphi} = 0 \quad \text{or} \quad \boldsymbol{R}_{,\varphi}\boldsymbol{R}^T + (\boldsymbol{R}_{,\varphi}\boldsymbol{R}^T)^T = 0 \tag{11}$$

Indeed, the matrix $\boldsymbol{R}_{,\varphi}\boldsymbol{R}^T$ is a skew symmetric matrix. Moreover, the previously presented matrix $\boldsymbol{A}$ is in fact defined by:

$$\boldsymbol{A} = \boldsymbol{R}_{,\varphi}\boldsymbol{R}^T \tag{12}$$

Returning to the term $-(\Sigma(\boldsymbol{A}\boldsymbol{r}_{i/c})^T\boldsymbol{F}_i)_{,\varphi}$ of the stiffness matrix, we can now elaborate this term as follows:

$$\begin{aligned} &-\left(\Sigma\left(\boldsymbol{A}\boldsymbol{r}_{i/c}\right)^T\boldsymbol{F}_i\right)_{,\varphi} = -\left(\Sigma\left(\boldsymbol{A}\boldsymbol{R}\boldsymbol{r}'_{i/c}\right)^T\boldsymbol{F}_i\right)_{,\varphi} = \\ &-\Sigma\left(\boldsymbol{F}_i^T\left(\boldsymbol{A}\boldsymbol{R}_{,\varphi}\boldsymbol{r}'_{i/c}\right)\right) = -\Sigma\left(\boldsymbol{F}_i^T\left(\boldsymbol{A}\boldsymbol{R}_{,\varphi}\boldsymbol{R}^T\boldsymbol{r}_{i/c}\right)\right) = \\ &-\Sigma\left(\boldsymbol{F}_i^T\left(\boldsymbol{A}\boldsymbol{A}\boldsymbol{r}_{i/c}\right)\right) = \Sigma\boldsymbol{F}_i^T\boldsymbol{r}_{i/c} \end{aligned} \tag{13}$$

where the equality $\boldsymbol{A}\boldsymbol{A} = -\boldsymbol{E}_2$ is used. Consequently, the stiffness matrix of a set of constant forces $\boldsymbol{F}_i$ acting on a rigid body reduces to:

$$\boldsymbol{K} = \begin{bmatrix} 0 & 0 \\ 0 & \Sigma\boldsymbol{F}_i^T\boldsymbol{r}_{i/c} \end{bmatrix} \tag{14}$$

This expression shows that the contribution to the stability of a constant force is characterized by the scalar product of the force vector and the position vector of its point of application.

The stability of the two-constant-force systems in Fig. 1 can now be investigated more rigorously. Evaluation of the stiffness maxtrix yields for the $\Sigma\boldsymbol{F}_i^T\boldsymbol{r}_{i/c}$ term:

$$\Sigma\boldsymbol{F}_i^T\boldsymbol{r}_{i/c} = \boldsymbol{F}_1^T\boldsymbol{r}_{1/c} + \boldsymbol{F}_2^T\boldsymbol{r}_{2/c} = \boldsymbol{F}_1^T\left(\boldsymbol{r}_{1/c} - \boldsymbol{r}_{2/c}\right) = \pm Fd \tag{15}$$

where $F = |\boldsymbol{F}_1|$ and $d = \left|\left(\boldsymbol{r}_{1/c} - \boldsymbol{r}_{2/c}\right)\right|$, *i.e.* the distance between points P_1 and P_2. Thus, we find values of Fd, $-Fd$ and zero for the systems in Figs. 1a, 1b, and 1c, respectively. Since these terms are in fact the powers of the exponential solution to the linear differential equation Eq. 5, it can be concluded that the system in Figs. 1a, 1b, and 1c are stable, unstable, and neutrally stable, respectively, for small rotational disturbances. Note that all of these systems are neutrally stable for small displacements.

More interestingly, the tangent stiffness matrix $\boldsymbol{K}$ of Eq. 14 can be used to find the force $\boldsymbol{F}_r$ that is dynamically equivalent to two constant forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, by demanding that their contributions to the tangent stiffness matrix $\boldsymbol{K}$ must be equal. This notion leads to the following equation:

$$\boldsymbol{F}_r^T \boldsymbol{r}_{r/c} = \boldsymbol{F}_1^T \boldsymbol{r}_{1/c} + \boldsymbol{F}_2^T \boldsymbol{r}_{2/c} \tag{16}$$

Thus an equation of scalar products is found, which, together with the equation of force vectors (Eqs. 3) and the equation of vector products (Eq. 4), uniquely defines the application point of the resultant force $\boldsymbol{F}_r$ yielding the same stability contribution, when constant forces are assumed. The application point found in this manner will be called the *dynamically equivalent application point* (*DEP*) of the resultant force. Equation 16 will be called the *stability equation* for the case of constant forces. As is true for the force and moment equations (Eqs. 3 and 4, respectively), it can be shown that the stability equation is valid for any point *C* on the rigid body.

DEP of two constant parallel forces

To investigate the implications of the stability equation, the special case of two *parallel* constant forces is investigated first. Consider for example two gravity forces $\boldsymbol{F}_1 = [0 \quad -m_1 g]^T$ and $\boldsymbol{F}_2 = [0 \quad -m_2 g]^T$ acting on a rigid body, as in Fig. 3a. Substitution of these expressions into Eqs. 3, 4, and 16, yields after elaboration:

$$r_{rx/c} = \frac{m_1}{m_1 + m_2} r_{1x/c} + \frac{m_2}{m_1 + m_2} r_{2x/c} \tag{17}$$

$$r_{ry/c} = \frac{m_1}{m_1 + m_2} r_{1y/c} + \frac{m_2}{m_1 + m_2} r_{2y/c} \tag{18}$$

Thus, the application point $P_r = (r_{rx/c}, r_{ry/c})$ of the dynamically equivalent force is found to be located on the line connecting P_1 and P_2, in such a way that $P_1P_r / P_rP_2 = m_2 / m_1$ as shown in Fig. 3b. This result corresponds to the well-known procedure of finding the combined center of mass of two particles, and demonstrates that the above derivation indeed yields equivalent dynamics. In fact, this result can be identified as a particular case of the proposed procedure.

DEP of two constant forces

The more general case of two constant forces of arbitrary direction (Fig. 4a) is considered next. The line of action of the statically equivalent force is known, and Eq. 16 is used to find its point of application. Graphical inspection of this equation reveals a remarkable phenomenon. The *DEP* is located on the circle defined by the application points of the original forces and the intersection of their lines of action, as is shown in Fig. 4b. Assuming that this is true, it will be shown that Eq. 16 results. First, it is realized that if $\boldsymbol{F}_r = \boldsymbol{F}_1 + \boldsymbol{F}_2$, then the projection of $\boldsymbol{F}_r$ on any straight line is equal to the summation

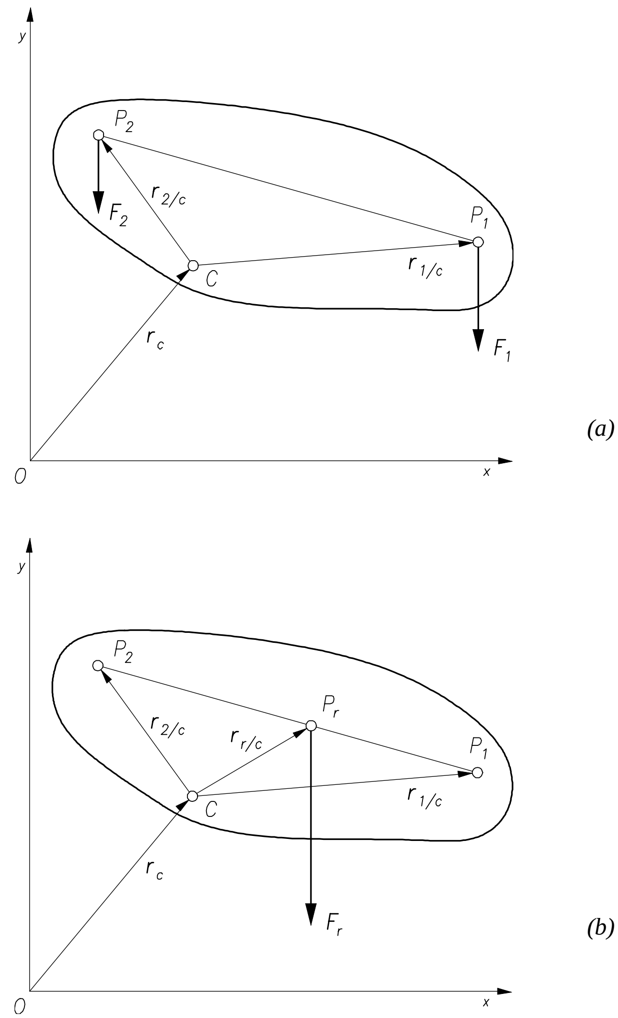


Figure 3. Two parallel constant forces acting on a rigid body: (a) given situation, (b) dynamic equivalent force. This example demonstrates the analogy with the determination of the combined center of mass of two particles.

of the projections of the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ on the same line. If the diameter of the circle through point *T* is selected, the following expression results for the projections of the forces on this line (Fig. 4c):

$$F_r D \cos\alpha_r = F_1 D \cos\alpha_1 + F_2 D \cos\alpha_2 \tag{19}$$

where $F_i = |\boldsymbol{F}_i|$, and where each term is multiplied by the diameter *D*. When *T* is selected as point *C*, the terms $D\cos\alpha_i$ are equal to $r_{i/T} = |\boldsymbol{r}_{i/T}|$. Furthermore, as the vectors $\boldsymbol{r}_{i/T}$ and $\boldsymbol{F}_i$ are collinear, the terms $F_i D \cos\alpha_i$ evolve into $\boldsymbol{F}_i^T \boldsymbol{r}_{i/T}$. Since $\boldsymbol{F}_r = \boldsymbol{F}_1 + \boldsymbol{F}_2$, the treatise is valid with respect to any point of the rigid body. Consequently, Eq. 19 is equivalent to

Eq. 16, which concludes the proof. Thus, the circle construction is a convenient way of finding the *DEP* of two given constant forces.

Stability of a three-force system in equilibrium

In the special case of a three-force system in equilibrium, the assessment of the stability now becomes straightforward and convenient by using the circle construction of Fig. 4. Figure 5 shows three cases of a rigid body which is in equilibrium under influence of the forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_r$. The body in Fig. 5a is in stable equilibrium, since replacing forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ by their dynamically equivalent resultant $\boldsymbol{F}_r$ (not shown) yields a system similar to Fig. 1a. Similarly, the system in Fig. 5b is in unstable equilibrium, whereas the system in Fig. 5c is in neutral equilibrium. Note that it can now be concluded that the body in Fig. 2 is in stable equilibrium, simply by constructing the circle.

Central linear forces

This section deals with forces generated by a central linear force field. One special type of central linear forces consists of forces generated by zero-free-length springs [3], which are of great benefit in the design of mechanisms in neutral equilibrium [4]. Due to the character of these forces, the tangent stiffness matrix $\boldsymbol{K}$ will contain more non-zero entries than in the case of constant forces. This section will derive the conditions for the dynamically equivalent force of two central linear forces, or, in particular, two zero-free-length springs.

The central linear force generated by a zero-free-length spring can be expressed as:

$$\boldsymbol{F}_i = k_i\left(\boldsymbol{a}_i - \boldsymbol{r}_i\right) = k_i\left(\boldsymbol{a}_i - \boldsymbol{r}_c - \boldsymbol{R}\boldsymbol{r}'_{i/c}\right) \tag{20}$$

where k_i is the spring stiffness; $\boldsymbol{a}_i$ is the position vector of the fixed end of the spring (the origin of the central linear force field), and $\boldsymbol{r}_i$ is the position vector of the moving end of the spring. The moment contribution of such a force with respect to an arbitrary point C of the body is:

$$\left(\boldsymbol{A}\boldsymbol{r}_{i/c}\right)^T \boldsymbol{F}_i = \boldsymbol{F}_i^T \boldsymbol{R}_{,\varphi}\, \boldsymbol{r}'_{i/c} = k_i\left(\boldsymbol{a}_i - \boldsymbol{r}_c - \boldsymbol{R}\boldsymbol{r}'_{i/c}\right)^T \boldsymbol{R}_{,\varphi}\, \boldsymbol{r}'_{i/c} \tag{21}$$

From Eqs. 20 and 21, the contributions due to this force to the elements of the stiffness matrix $\boldsymbol{K}$ (Eq. 5) can be derived:

$$-\left(\boldsymbol{F}_i\right)_{,\boldsymbol{r}_c} = k_i \boldsymbol{E}_2 \tag{22}$$

$$-\left(\boldsymbol{F}_i\right)_{,\varphi} = k_i \boldsymbol{R}_{,\varphi}\boldsymbol{r}'_{i/c} = k_i \boldsymbol{A}\boldsymbol{R}\boldsymbol{r}'_{i/c} = k_i \boldsymbol{A}\boldsymbol{r}_{i/c} \tag{23}$$

$$-\left(\left(\boldsymbol{A}\boldsymbol{r}_{i/c}\right)^T \boldsymbol{F}_i\right)^T_{,\boldsymbol{r}_c} = k_i\left(\boldsymbol{R}_{,\varphi}\boldsymbol{r}'_{i/c}\right)^T = k_i\left(\boldsymbol{A}\boldsymbol{R}\boldsymbol{r}'_{i/c}\right)^T = k_i\left(\boldsymbol{A}\boldsymbol{r}_{i/c}\right)^T \tag{24}$$

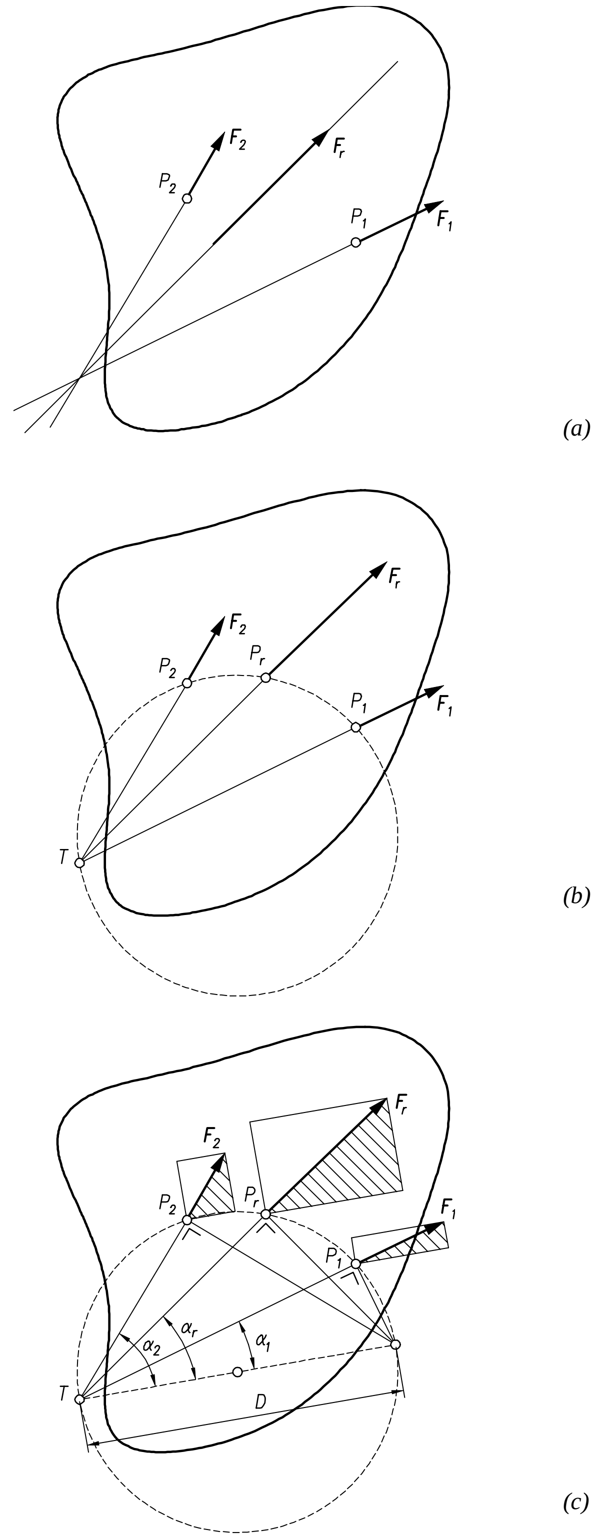


Figure 4 Two constant forces of arbitrary direction acting on a rigid body: (a) given situation and statically equivalent force that may apply anywhere on its line of action, (b) dynamically equivalent point of application, as determined by the proposed circle construction, (c) proof of the circle construction.

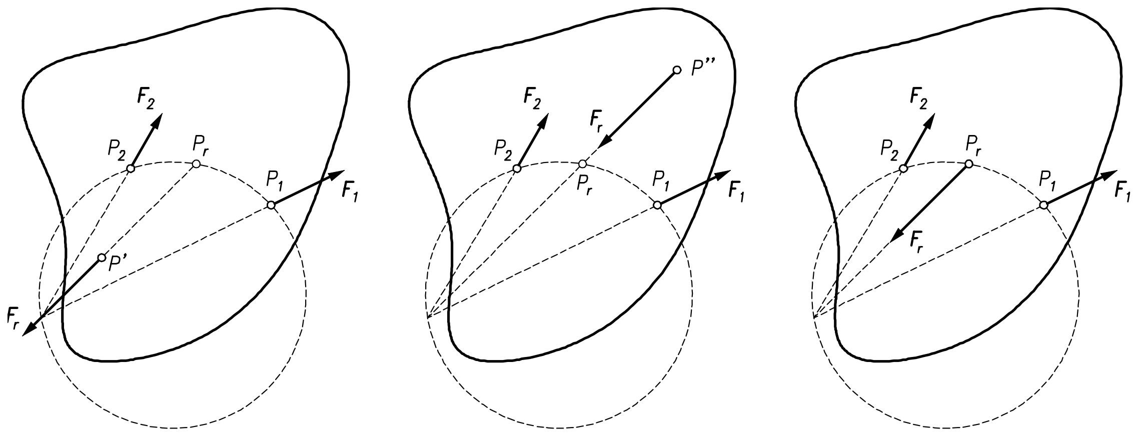


Figure 5. Systems of three constant forces in equilibrium acting on a rigid body: (a) stable equilibrium, (b) unstable equilibrium, (c) neutral equilibrium.

$$-\left(\left(\boldsymbol{A}\boldsymbol{r}_{i/c}\right)^T \boldsymbol{F}_i\right)_{,\varphi}^T = k_i\left(\boldsymbol{R}_{,\varphi}\boldsymbol{r}'_{i/c}\right)^T \boldsymbol{R}_{,\varphi}\boldsymbol{r}'_{i/c} - \boldsymbol{F}_i^T \boldsymbol{R}_{,\varphi\varphi}\boldsymbol{r}'_{i/c} =$$

$$k_i \boldsymbol{r}_{i/c}^T \boldsymbol{r}_{i/c} + \boldsymbol{F}_i^T \boldsymbol{r}_{i/c} = k_i \boldsymbol{a}_{i/c}^T \boldsymbol{r}_{i/c} \tag{25}$$

where the equalities $\boldsymbol{R}_{,\varphi} = \boldsymbol{A}\boldsymbol{R}$, $\boldsymbol{R}_{,\varphi}^T \boldsymbol{R}_{,\varphi} = \boldsymbol{E}_2$, $\left|\boldsymbol{r}'_{i/c}\right| = \left|\boldsymbol{r}_{i/c}\right|$, and $\boldsymbol{R}_{,\varphi\varphi} = -\boldsymbol{R}$, are used. Thus, the tangent stiffness matrix $\boldsymbol{K}$ for central linear forces evolves into:

$$\boldsymbol{K} = \begin{bmatrix} k_i \boldsymbol{E}_2 & k_i \boldsymbol{A}\boldsymbol{r}_{i/c} \\ k_i\left(\boldsymbol{A}\boldsymbol{r}_{i/c}\right)^T & k_i \boldsymbol{a}_{i/c}^T \boldsymbol{r}_{i/c} \end{bmatrix} \tag{26}$$

As compared to the constant-force stiffness matrix (Eq. 14), the following differences are apparent. Additional terms $k_i \boldsymbol{A}\boldsymbol{r}_{i/c}$ and $k_i\left(\boldsymbol{A}\boldsymbol{r}_{i/c}\right)^T$ are present as the off-diagonal elements. Consequently, the system is no longer indifferent with respect to arbitrary displacements. Pure translation is associated with a stiffness k_i due to the term $k_i \boldsymbol{E}_2$. Furthermore, it is remarkable that the lower right term is not replaced by a completely different term but is expanded with the term $k_i \boldsymbol{r}_{i/c}{}^T \boldsymbol{r}_{i/c}$ (compare Eq. 13 with Eq. 25), resulting in $k_i \boldsymbol{r}_{i/c}{}^T \boldsymbol{r}_{i/c} + \boldsymbol{F}_i^T \boldsymbol{r}_{i/c} = k_i \boldsymbol{a}_{i/c}{}^T \boldsymbol{r}_{i/c}$.

If now two central linear forces are to be replaced by a single equivalent one, the contribution to the stiffness matrix $\boldsymbol{K}$ due to the equivalent central linear force must be equal to the contribution due to the two original ones. Considering Eqs. 22 through 25 respectively, this leads to the following conditions for equal stability (stability equations):

$$k_r = k_1 + k_2 \tag{27}$$

$$k_r \boldsymbol{A}\boldsymbol{r}_{r/c} = k_1 \boldsymbol{A}\boldsymbol{r}_{1/c} + k_2 \boldsymbol{A}\boldsymbol{r}_{2/c} \tag{28}$$

$$k_r \boldsymbol{r}_{r/c}^T \boldsymbol{r}_{r/c} + \boldsymbol{F}_r^T \boldsymbol{r}_{r/c} =$$

$$k_1 \boldsymbol{r}_{1/c}^T \boldsymbol{r}_{1/c} + \boldsymbol{F}_1^T \boldsymbol{r}_{1/c} + k_2 \boldsymbol{r}_{2/c}^T \boldsymbol{r}_{2/c} + \boldsymbol{F}_2^T \boldsymbol{r}_{2/c} \tag{29}$$

Thus, when replacing two central linear forces by one, a total of seven equations are found: the vector equations Eq. 3 and Eq. 28, and the scalar equations Eq. 4, Eq. 27, and Eq. 29, which are to be solved for five unknowns (one scalar, k_r, and two vectors, $\boldsymbol{r}_r$ and $\boldsymbol{a}_r$). Consequently, no solutions are found in general. This leads to the conclusion that two central linear forces cannot generally be substituted by a single one in a dynamically equivalent manner. However, by imposing constraints on the system, solutions for at least two special cases are possible. The following section will give the first special case, the second one will be treated in the Examples section.

Special Case 1: Common Attachment

A first special case is possible when the zero-free-length springs are attached to the rigid body at the same point (Fig. 6). Then $\boldsymbol{r}_{1/c} = \boldsymbol{r}_{2/c}$ which, substituted in Eqs. 3 and 4, leads to $\boldsymbol{r}_{r/c} = \boldsymbol{r}_{1/c} = \boldsymbol{r}_{2/c}$. Together with Eq. 27, this immediately satisfies Eq. 28, while Eq. 29 now becomes:

$$\left(k_1 + k_2\right)\boldsymbol{r}_{r/c}^T \boldsymbol{r}_{r/c} + \boldsymbol{F}_r^T \boldsymbol{r}_{r/c} =$$

$$k_1 \boldsymbol{r}_{r/c}^T \boldsymbol{r}_{r/c} + \boldsymbol{F}_1^T \boldsymbol{r}_{r/c} + k_2 \boldsymbol{r}_{r/c}^T \boldsymbol{r}_{r/c} + \boldsymbol{F}_2^T \boldsymbol{r}_{r/c} \tag{30}$$

The $\boldsymbol{r}_{r/c}{}^T \boldsymbol{r}_{r/c}$ terms cancel out, which gives after rearranging:

$$\left(\left(k_1 + k_2\right)\boldsymbol{a}_r^T - k_1 \boldsymbol{a}_1^T - k_2 \boldsymbol{a}_2^T\right)\boldsymbol{r}_{r/c} = 0 \tag{31}$$

This should be valid for any $r_{r/c}$, and therefore, introducing the unit vector $\boldsymbol{e}_a$ and using the relations $\boldsymbol{a}_1 - \boldsymbol{a}_2 = a\boldsymbol{e}_a$ and $a_1 + a_2 = a$ (see Fig. 6), Eq. 25 evolves into:

$$\boldsymbol{a}_r = \frac{k_1}{k_1+k_2}\boldsymbol{a}_1 + \frac{k_2}{k_1+k_2}\boldsymbol{a}_2 \qquad (32)$$

From this result it is seen that $\boldsymbol{a}_r$ traces the line A_1A_2 as the stiffnesses k_1 and k_2 vary. It is also seen that $a_2 = ak_1/(k_1+k_2)$ and $a_1 = ak_2/(k_1+k_2)$. Thus, the relation $k_1a_1 = k_2a_2$ defines the location of point A_r on the line A_1A_2.

So, two zero-free-length springs, k_1 and k_2, each attached with one end to a first rigid body and with the other end to a second rigid body, can be composed into a single zero-free-length spring k_r in a dynamically equivalent way for any relative movement of the rigid bodies, under the following conditions: Firstly, k_r must equal k_1+k_2 (due to Eq. 27), secondly, the free ends of the springs must be attached to the same point of application P_r, so $\boldsymbol{r}_{r/c} = \boldsymbol{r}_{1/c} = \boldsymbol{r}_{2/c}$ (assumed earlier); and thirdly, the fixed end A_r of the dynamically equivalent zero-free-length spring must be located on the line connecting A_1 and A_2, so that $k_1a_1 = k_2a_2$ (resulting from Eq. 32). Inversely, these equations can be used to resolve a single spring into two springs, where it is noted that this does not give a unique solution.

Potential Forces

The treatise above can be generalized when the applied forces can be derived from a potential function, *i.e.* when they are conservative. This is especially useful when the stability is to be assessed in cases where the original forces and their points of attachment are not easy to identify. Examples are distributed loads, such as hydrostatic pressure.

In the case of a rigid body, only the potential of the external forces is to be concerned, so the equations of motion can be written as follows:

$$\begin{bmatrix} V_{,\boldsymbol{r}_c} \\ V_{,\varphi} \end{bmatrix} = \begin{bmatrix} m\boldsymbol{E}_2 & 0 \\ 0 & I_c \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{r}}_c \\ \ddot{\varphi} \end{bmatrix} \qquad (33)$$

where $V = V(\boldsymbol{r}_c, \varphi)$ is the potential energy of the body. The variations about the nominal state of motion can be found by extending Eq. 33 and subtracting the nominal state, which results in:

$$\begin{bmatrix} m\boldsymbol{E}_2 & 0 \\ 0 & I_c \end{bmatrix} \begin{bmatrix} \Delta\ddot{\boldsymbol{r}}_c \\ \Delta\ddot{\varphi} \end{bmatrix} + \begin{bmatrix} -V_{,\boldsymbol{r}_c\boldsymbol{r}_c} & -V_{,\boldsymbol{r}_c\varphi} \\ -V_{,\varphi\boldsymbol{r}_c} & -V_{,\varphi\varphi} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{r}_c \\ \Delta\varphi \end{bmatrix} = 0 \qquad (34)$$

This result is completely equivalent with Eq. 5. Depending on the situation, either of these may be more convenient.

EXAMPLES

This section will provide two examples. The first example is in fact the second special solution to finding a dynamic equivalent of two central linear forces. The second one demonstrates the convenient use of the potential using the example of the stability of a floating vessel.

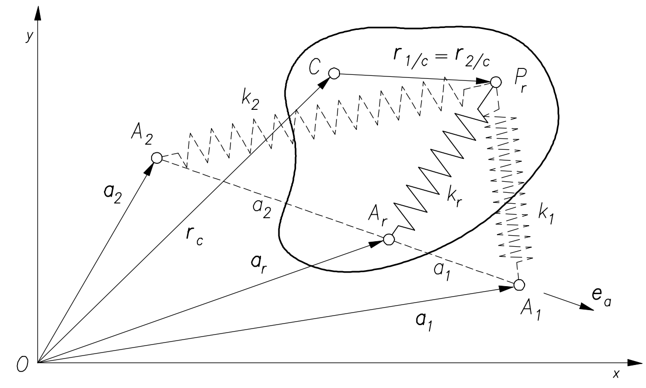


Figure 6. Two central linear forces, acting at the same point of a rigid body, can be composed into a single dynamically equivalent one.

The Balanced Broom

In addition to the common attachment, a second special case of a dynamical equivalent of two central linear forces is found when the motion is restricted to rotation (about a fixed point) only, and the two central linear forces are not replaced by a resultant central linear force but by a constant force. Under these distinct conditions, a solution can be found as follows. Due to the restriction to rotation, Eqs. 28 do not apply; and due to the replacement of two central linear forces with a constant force, dynamic equivalence is characterized by:

$$\boldsymbol{F}_r^T\boldsymbol{r}_{r/c} = k_1\boldsymbol{r}_{1/c}^T\boldsymbol{r}_{1/c} + \boldsymbol{F}_1^T\boldsymbol{r}_{1/c} + k_2\boldsymbol{r}_{2/c}^T\boldsymbol{r}_{2/c} + \boldsymbol{F}_2^T\boldsymbol{r}_{2/c} \qquad (35)$$

where the left side corresponds to the expression for constant forces. Now, together with the Eqs. 3 and 4, a total of four equations (one vector equation and two scalar equations) are found to solve for four unknowns (two vectors, $\boldsymbol{F}_r$ and $\boldsymbol{r}_{r/c}$).

The physical interpretation of this solution is still an open question, however an example confirming this phenomenon is present in the Balanced Broom (*a.k.a.* Floating Suspension [5]). This is a statically balanced mechanism consisting of one link with a mass at its end, suspended by two zero-free-length springs in such a way that a stationary pivot is obtained (Fig. 7). Static balance can be proved by using the potential. The total potential is the summation of the potentials of the springs and of the mass (with respect to *O*, see Fig. 7):

$$V = \tfrac{1}{2}k_1\left(\boldsymbol{a}_1 - \boldsymbol{r}_c - \boldsymbol{r}_{1/c}\right)^T\left(\boldsymbol{a}_1 - \boldsymbol{r}_c - \boldsymbol{r}_{1/c}\right) +$$

$$\tfrac{1}{2}k_2\left(\boldsymbol{a}_2 - \boldsymbol{r}_c - \boldsymbol{r}_{2/c}\right)^T\left(\boldsymbol{a}_2 - \boldsymbol{r}_c - \boldsymbol{r}_{2/c}\right) -$$

$$mg(\boldsymbol{r}_c + \boldsymbol{r}_{m/c})^T \boldsymbol{e}_z \tag{36}$$

where $\boldsymbol{r}_{i/c}$ runs from point C to the points where the moving spring ends attach to the link. The equilibrium position can be found from the equilibrium of forces:

$$V_{,\boldsymbol{r}_c} = -k_1(\boldsymbol{a}_{1/c} - \boldsymbol{r}_{1/c}) - k_2(\boldsymbol{a}_{2/c} - \boldsymbol{r}_{2/c}) + mg\boldsymbol{e}_z = 0 \tag{37}$$

$$\Rightarrow -k_1((1-p)a\boldsymbol{e}_z - r_1\boldsymbol{e}_r) - k_2(-pa\boldsymbol{e}_z + r_2\boldsymbol{e}_r) + mg\boldsymbol{e}_z = 0 \tag{38}$$

where the vector $(\boldsymbol{a}_i - \boldsymbol{r}_c) = \boldsymbol{a}_{i/c}$ is the vector from point C to the fixed spring attachment point A_i, and where r_1 and r_2 are defined according to $\boldsymbol{r}_{1/c} = r_1\boldsymbol{e}_r$ and $\boldsymbol{r}_{2/c} = -r_2\boldsymbol{e}_r$, respectively. This results in the following two conditions that are valid for any $\boldsymbol{e}_r$ or, equivalently, for any φ:

$$k_1 r_1 = k_2 r_2 \tag{39}$$

$$-k_1(1-p)a + k_2 pa + mg = 0 \tag{40}$$

In case of equal springs $k_1 = k_2 = k$, it is found from Eq. 39 that $r_1 = r_2 = r$, and from Eq. 40 that $p = \frac{1}{2} - mg/2ka$ or $z = a/2 - pa = mg/2k$. This expression is independent of the orientation of the link, which implies that the link has a stationary center of rotation, even though no physical joint is present. Thus, the link is restricted to rotation about a virtual pivot at C.

Next, the rotation is investigated by differentiating the potential with respect to φ:

$$V_{,\varphi} = k_1(\boldsymbol{R}_{,\varphi}\boldsymbol{r}'_{1/c})^T(\boldsymbol{a}_1 - \boldsymbol{r}_c - \boldsymbol{R}\boldsymbol{r}'_{1/c}) + k_2(\boldsymbol{R}_{,\varphi}\boldsymbol{r}'_{2/c})^T(\boldsymbol{a}_2 - \boldsymbol{r}_c - \boldsymbol{R}\boldsymbol{r}'_{2/c}) - mg(\boldsymbol{R}_{,\varphi}\boldsymbol{r}'_{m/c})^T\boldsymbol{e}_z \tag{41}$$

This equation can be simplified because $(\boldsymbol{R}_{,\varphi}\boldsymbol{r}'_{i/c})^T\boldsymbol{R}\boldsymbol{r}'_{i/c} = \boldsymbol{r}'^{T}_{i/c}\boldsymbol{R}_{,\varphi}^T\boldsymbol{R}\boldsymbol{r}'_{i/c} = -\boldsymbol{r}'^{T}_{i/c}\boldsymbol{A}\boldsymbol{r}'_{i/c} = 0$. In addition, making use of Fig. 7, it is observed that $\boldsymbol{a}_1 - \boldsymbol{r}_c = (1-p)a\boldsymbol{e}_z$ and $\boldsymbol{a}_2 - \boldsymbol{r}_c = -pa\boldsymbol{e}_z$. Furthermore, if $\boldsymbol{e}_\varphi$ is used as unit vector perpendicular to $\boldsymbol{e}_r$, then $\boldsymbol{R}_{,\varphi}\boldsymbol{r}'_{1/c} = r_{1/c}\boldsymbol{e}_\varphi$ and $\boldsymbol{R}_{,\varphi}\boldsymbol{r}'_{2/c} = -r_{2/c}\boldsymbol{e}_\varphi$. Thus, Eq. 41 reduces to:

$$V_{,\varphi} = (k_1 r_{1/c}(1-p)a + k_2 r_{2/c}pa - mgr_m)\boldsymbol{e}_\varphi^T\boldsymbol{e}_z \tag{42}$$

where $r_m = |\boldsymbol{r}_{m/c}|$. Using Eq. 39, $k_1 r_{1/c} = k_2 r_{2/c} = kr$, it is seen that this expression equals zero for any orientation if:

$$kra - mgr_m = 0 \quad \text{or} \quad r_m / r = ka/mg \tag{43}$$

It is thus shown that the broom is statically balanced for rotation by two zero-free-length springs, while at point C a stationary center of rotation with no support force is obtained!

A solution to Eq. 35 is now possible when it is assumed that $\boldsymbol{F}_r$ is parallel to the line connecting the fixed ends of the springs and its point of application is located on the line connecting the moving ends of the springs. Hence, $\boldsymbol{F}_r = F_r\boldsymbol{e}_a$ and $\boldsymbol{r}_{r/c} = r_r\boldsymbol{e}_r$. Under these conditions, Eq. 35 becomes:

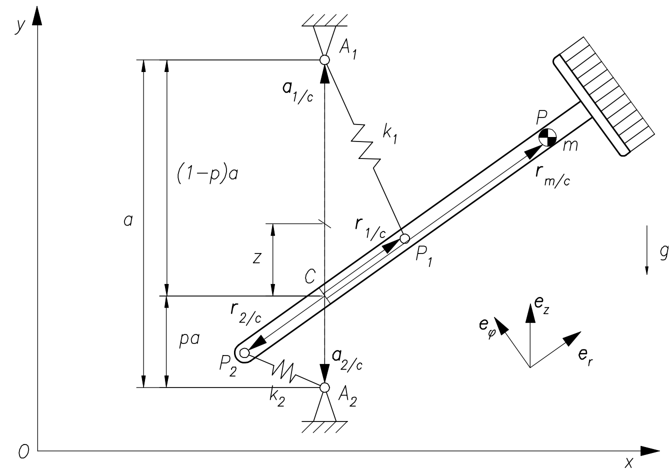


Figure 7. The Balanced Broom, demonstrating the fact that two central linear forces can be composed in a single dynamically equivalent constant force for a rotating body. The name for this mechanism is due to a confirmation experiment performed by Prof. Andy Ruina during a visit to our laboratory. An available broom was used by way of beam, while medical instrument covers out of latex served as approximate zero-free-length springs, with amazing result.

$$F_r r_r \boldsymbol{e}_a^T\boldsymbol{e}_r = k_1 a_1 r_1 \boldsymbol{e}_a^T\boldsymbol{e}_r + k_2 a_2 r_2 \boldsymbol{e}_a^T\boldsymbol{e}_r \tag{44}$$

A solution is found for any $\boldsymbol{e}_r$ and therefore for any angle φ when:

$$F_r r_r = k_1 a_1 r_1 + k_2 a_2 r_2 \tag{45}$$

This result confirms that one solution for the dynamically equivalent force of the two ideal spring forces is a constant force of magnitude F_r and directed along $\boldsymbol{e}_a$, acting on the lever at point P, located along the extension of P_1P_2 at a distance r_r from point C. In particular, this treatise proofs the neutral stability for rotation of the spring-suspended link when $\boldsymbol{e}_a$ is set vertical and a gravity force $\boldsymbol{F}_r = m\boldsymbol{g}$, equal and opposite to the resultant of the spring forces, is applied at point P, where $r_r = r_{m/c}$.

Roll stability of a ship

The metacenter of a ship is an example of a dynamically equivalent application point of a resultant force, *i.e.* the hydrostatic or buoyancy force (*e.g.* [6]). The position of the metacenter with respect to the center of gravity of the ship, the metacentric height, determines the stability of the roll motion of a ship. The motion is stable if the metacenter is above the center of gravity, in which case the metacentric height is taken positive.

If we draw the free body diagram of a ship of rectangular cross section in the equilibrium position (Fig. 8a), that is at zero roll angle, it is clear that the resulting hydrostatic force must act on the center line of the cross section of the ship. Yet, at a glance, it is not obvious were the point of application of this resultant is located in order to be dynamically equivalent with the hydrostatic forces. A first, incorrect guess would be the centre of gravity of the displaced water volume, also known as the centre of buoyancy. Determining the *DEP* for the hydrostatic forces is not so easy since, for a rotated ship, these forces change both in direction as well as in magnitude. Therefore a direct analysis as presented in Eq. 5 is rather cumbersome. A much easier approach is making use of the potential function for this conservative force field and subsequently take derivatives as proposed in Eqs. 33 and 34 to obtain the dynamically equivalent force system.

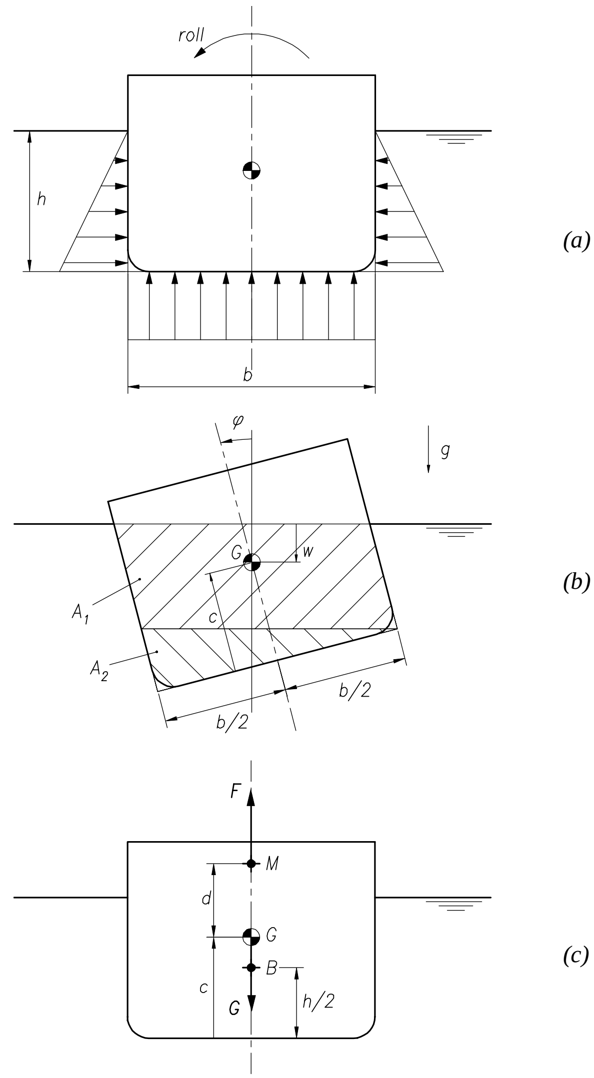


Figure 8. Ship with rectangular cross-section: (a) stability is not easily assessed based on the physical force system, (b) inclined vessel, (c) resulting dynamically equivalent force configuration reveals the state of stability at a glance.

Consider the ship in a displaced position (Fig. 8b). The potential function for the hydrostatic forces is minus the potential of the displaced water volume which is equal to the first moment of mass distribution with respect to the water line times the gravitational constant g. We can divide the immerged cross sectional area of the ship into two parts, a parallelogram A_1 and a triangle A_2 as given by

$$A_1 = \frac{b}{\cos\varphi}\left(c\cos\varphi + w - \frac{b}{2}\sin\varphi\right) \tag{46}$$

$$A_2 = \frac{b}{\cos\varphi}\left(\frac{b}{2}\sin\varphi\right) \tag{47}$$

where b is the width of the ship, c the distance of the center of gravity with respect to the bottom of the ship, and w the position of the center of gravity with respect to the waterline. With the distance of the centers of mass with respect to the waterline of the two parts, respectively, as given by

$$w_1 = \frac{1}{2}\left(c\cos\varphi + w - \frac{b}{2}\sin\varphi\right) \tag{48}$$

$$w_2 = c\cos\varphi + w + \frac{b}{6}\sin\varphi \tag{49}$$

the total potential of the hydrostatic forces now becomes

$$V = \rho g\ell\left(A_1 w_1 + A_2 w_2\right) \tag{50}$$

where ℓ is the length of the ship and ρ the density of water. The metacentric height of a ship is defined in the equilibrium position. In this position the roll angle will be zero due to symmetry. The submerged height h of the ship follows from the total weight of the ship G being equal to the buoyancy, as in

$$G = \rho g\ell\left(h + b\right) \tag{51}$$

Note that the buoyancy also can be found as the first partial derivative of the hydrostatic potential V with respect to vertical displacement w. With this height h, the displacement of the center of gravity w at equilibrium is now

$$w = h - c \tag{52}$$

The dynamically equivalent force system for the hydrostatic forces follows directly from the second order partial derivative of the hydrostatic potential V with respect to the roll angle φ (Eq. 34) resulting in

$$V_{,\varphi\varphi} = \rho g \ell (h+b)\left(\frac{1}{12}\left(\frac{b}{h}\right)^2 h + \frac{1}{2}h - c\right) \tag{53}$$

If we compare this to the result for a constant force, *i.e.* the term Fd in Eq. 15, we conclude that we can replace the resulting hydrostatic forces by a constant force

$$F = \rho g \ell (h+b) \tag{54}$$

acting at a distance d from the centre of gravity, where the distance d is in this case

$$d = \frac{1}{12}\left(\frac{b}{h}\right)^2 h + \frac{1}{2}h - c \tag{55}$$

This distance d is the so-called metacentric height of the ship. If we compare this to the result from literature (*e.g.* [6]) where the metacentric height GM is given by

$$GM = \frac{I_{oo}}{Vol} - BG \tag{56}$$

where Ioo is the area moment of inertia of the *horizontal* waterline area about the longitudinal axis of the ship, Vol is the displaced water volume, and BG the distance from the centre of buoyancy to the centre of gravity of the ship, we find

$$GM = \frac{1}{12}\frac{\ell b^3}{bh\ell} - \left(c - \frac{h}{2}\right) \tag{57}$$

This is in complete accordance with Eq. 49. Note that the *DEP* of the hydrostatic forces is not located at the center of buoyancy but at a distance $b^2/12h$ right above this point (Fig. 8c).

Stability of a shoe box

A clear example of the difference between the center of bouyancy B and the metacenter M is present in the case of a floating shoe box (Fig. 9a). We know from experience that this is a stable configuration. Yet, in most cases, the center of gravity G will be *above* the center of buoyancy B which gives the impression of an unstable system. Substituting the dimensions as presented in Fig. 9a into Eq. 55 we calculate

$$GM = \tfrac{3}{2}h \qquad \text{and} \qquad BM = 3h \tag{58}$$

Indeed, a very stable configuration! In conclusion, Fig. 9b shows the physical force system, while Fig. 9c shows the dynamically equivalent two-force system.

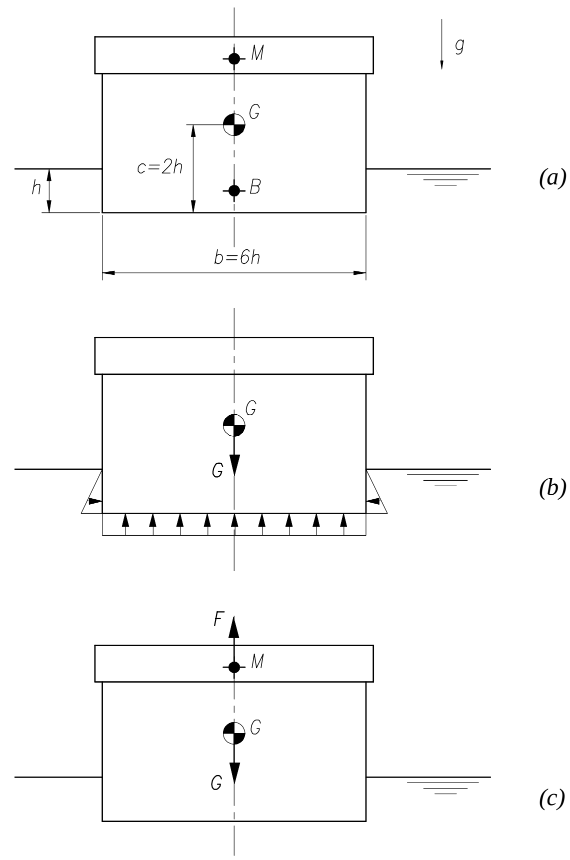


Figure 9. Shoe box: (a) given configuration of center of buoyance and center of gravity, (b) physical force system, (c) dynamically equivalent force system.

CONCLUSION

Unlike two-force systems, more complex force systems do not allow the assessment of their stability by inspection. This paper presented a general method to determine the stability of complex force systems, based on the notion of dynamic equivalence, where a resultant force is considered dynamically equivalent to a given force system acting on a rigid body if the contributions to the body's stability of the resultant force and the original force system are equal. This demand is stronger than the demand for static equivalence. Static equivalence yields a resultant force vector and its line of action. The location on this line remains undetermined, as it does not affect the nominal state of a rigid body. However, for the assessment of the stability of this nominal state, the application point on the line of action is essential. Demanding dynamic equivalence pinpoints the attachment point of the resultant force on the body in a unique manner. This point was called the dynamically equivalent point of attachment.

It was shown that the dynamically equivalent resultant force of two given constant forces applies at the intersection of its line of action and the circle put up by the application points of the given forces and the intersection of their lines of action. This result yields a convenient graphical method for finding the dynamically equivalent application point of the resultant force. The determination of the combined center of mass can be considered as a special case of this theorem.

Two examples were given that illustrate the versatility and the significance of the proposed treatise. The first example considered the suspension of a body by springs only. It was shown that the body was statically balanced for rotation, while the center of rotation proved to be a virtual stationary point. The second example treated the roll stability of a ship. Using the proposed methodology, the metacentric height was determined in a natural and convenient way.

Future research will be directed towards the implication of dynamic equivalence of spatial force systems. Another field of application is present in robotic end effectors. In order to determine the stability of a grasp [7], the nature of the grasp forces is predominant, *i.e.* sliding of sticking; with fixed or floating line of action; zero or non-zero free length springs, etc. Consequently, future work comprises the dynamically equivalent composition of other than constant and central linear forces, and the application to the synthesis of stable grasps.

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NOMENCLATURE

a	distance between fixed spring attachment points
$\boldsymbol{a}_i$	position vector of a fixed spring attachment point relative to a fixed reference frame
$\boldsymbol{a}_{i/c}$	position vector of a fixed spring attachment point relative to a local reference frame
A	cross sectional area
$\boldsymbol{A}$	matrix used to effect the planar form of vector multiplication
b	ship width
c	distance of center of gravity above ship floor
d	distance between two points
$\boldsymbol{e}$	unit vector
$\boldsymbol{E}_i$	identity matrix of rank i
F	magnitude of force
$\boldsymbol{F}$	force vector
g	acceleration of gravity
h	submerged depth of ship
i	index, counter
I_c	mass moment of inertia with respect to point C
I_{oo}	area moment of inertia of horizontal waterline area
k	spring stiffness
$\boldsymbol{K}$	tangent stiffness matrix
ℓ	length of ship
m	mass
$\boldsymbol{M}$	mass matrix
n	number of external forces
p	fraction
r	link length
$\boldsymbol{r}_i$	position vector of a moving spring attachment point relative to a fixed reference frame
$\boldsymbol{r}_{i/c}$	position vector of a moving spring attachment point relative to a local reference frame
$\boldsymbol{R}$	rotation matrix
V	potential, potential energy
Vol	displaced water volume
w	depth of center of gravity below waterline
Δ	first order variation
ρ	density
φ	variable angle

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