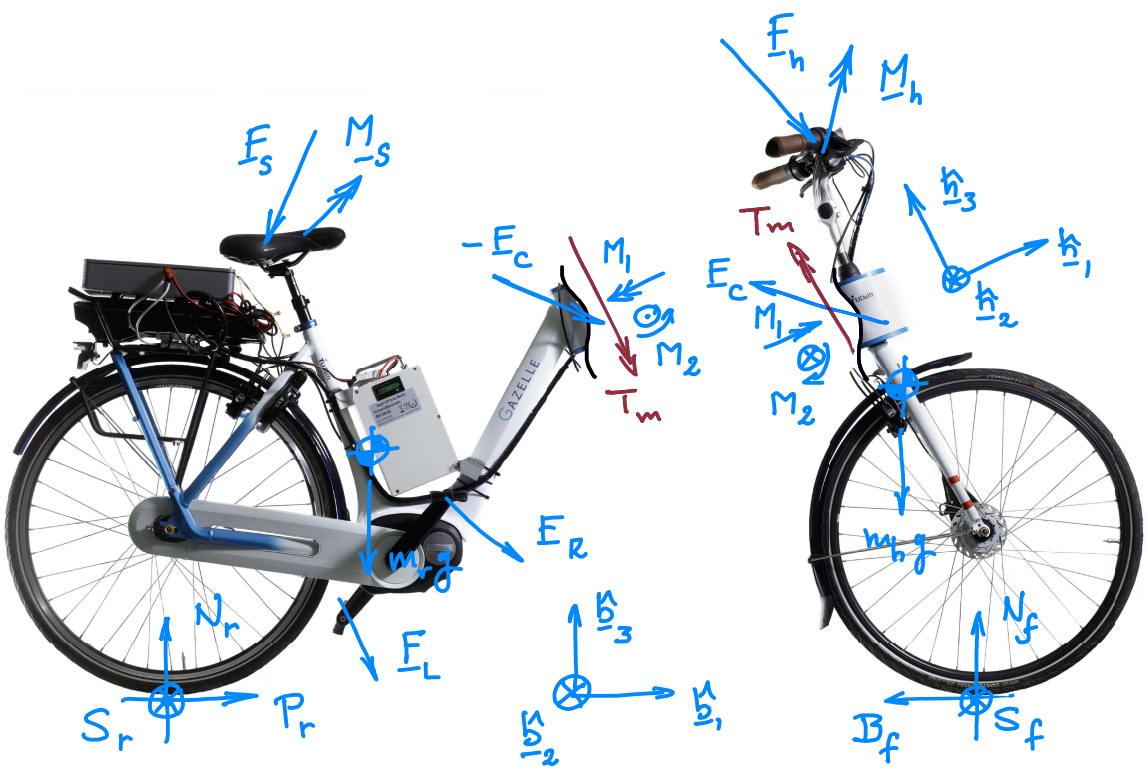
Advanced Dynamics

Heike Vallery Arend L. Schwab

Third fully revised edition, including
part I: Rigid-Body Dynamics
part II: Multibody Dynamics



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The cover image shows the steer-assist bicycle developed at the TU Delft in collaboration with the Dutch Royal Gazelle bicycle company. This prototype has a smart motor in the head tube, which can provide an actively controlled steer torque T_{m} (highlighted in red) to enhance the lateral stability of the bicycle at low-to-moderate forward speed. This active lateral-stability control is intended to reduce the number of falls in cycling. The prototype was built with funding support from the Sport Innovator Award 2016. Photo Sam Rentmeester.

Published by Stichting Newton-Euler, Delft, The Netherlands

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Printed in August 2024.
This is a reprint of the third edition, which was first printed August 2020.

Previous editions:
First edition, printed August 2017.
Second edition, first printed August 2018, re-printed August 2019.

The sketches in this book were drawn by Arend L. Schwab.

Typeface 11pt. *System* LaTeX

Print: Pumbo.nl

ISBN: 978-90-8309-060-3

Preface

Mechanics of rigid bodies moving in three-dimensional spaces represents a fundamental pillar for many engineering sciences.

Part I of this book is aimed particularly at undergraduate students who already have some background in statics and dynamics of particles and planar systems, but who wish to extend their knowledge towards three-dimensional problems involving rigid bodies, and towards variational methods.

Part II of this book is aimed particularly at graduate students interested in further deepening and extending their knowledge on these topics, applied to multibody systems.

For all introduced concepts, many practical examples and problems for self-study are included. Problems and examples are categorized according to their level. This level is depicted by a certain number of gimbals:

Problem 99.1 Quick knowledge clarification or check; answers can (almost) directly be found in text.

Problem 99.2 Full problem requiring several steps.

Problem 99.3 Advanced problem, for readers who want extra knowledge or challenge.

For selected problems, answers and solutions are provided in the end of this book. Those problems are marked in the end with ▷ for an answer and with ▶ for a full solution. Also, some problems have hints, which are printed as footnotes and upside-down.

Particular emphasis is placed on gradually enabling readers to independently solve real engineering problems, with confidence in the correctness of their solution. Therefore, this book provides ample and detailed guidance on how to check plausibility and rigorously question own results. Explicit recommendations are provided at the end of the first five chapters of part I. Also many problems are provided that, instead of relying on worked-out exemplary solutions, are designed to guide readers in the process of verifying own results.

The initial basis for part I of this book were the lecture and lecture slides of the course "Rigid-Body Dynamics" (formerly named "Advanced Dynamics") at the Faculty of Mechanical, Maritime and Materials Engineering at TU Delft, by the authors. An earlier version of part I appeared as the first edition of this book in 2017.

The basis for part II of this book were the lecture and lecture slides of the course "Multibody Dynamics" at the same faculty at TU Delft, by the author Arend Schwab. This part was added in the second edition.

Both lectures continue to exist, and the two parts of this book function as readers. Therefore, the authors have taken care to keep both parts of this book readable independently, although readers may find it useful to also consult the respective other part of the book when looking for background or further information. This independence necessarily means that some of the content provided in part I is repeated in part II in slightly modified form.

This third, fully revised edition differs substantially from the previous ones. The notation in general has been improved, in particular the way in which 3-D coordinate systems are described with the usage of triads. Moreover, the notation used in part I, rigid-body dynamics, and part II, multibody dynamics, are now more in agreement. Finally, the number of problems throughout the whole book has been enlarged and now covers a larger range of difficulty level. In part I, many guided problems have been added. These guided problems find their origin in homework assignments and assist the reader, in a step-by-step manner, to solve a complex problem. In part II, a considerable number of advanced problems with answers and solutions have been added. These problems find their origin in homework assignments and exam questions. Extensive solutions are included to guide the reader in understanding the material at hand in self-study.

Editorial support and review for earlier editions of part I was provided by Pier de Jong, Bram Smit, Johan Schonebaum, and Frederik Lachmann. The initial transcription from the lecture was performed by Johan Schonebaum. Many problems and examples were created with contributions from Wouter Wolfslag, Daniel Lemus, Bram Smit, and Patricia Baines. Critical review was also performed by Wouter Wolfslag and Mukunda Bharatheesha. Several attentive students identified errata in the first edition and thereby contributed to the second edition, especially Roelf-Jilling Wolthuis, Louisa Preis, and Martijn van Veen.

Richard van der Linde contributed to the multibody dynamics lecture. Editorial support to revise and re-format the lecture notes was

provided by Pier de Jong.

For this third edition, Daniel Lemus has contributed ideas for improved notations of triads. Pier de Jong continued to support the authors by answering formatting questions.

Jaap Meijaard has provided meticulous reading and excellent suggestions for improvement for all editions.

The authors would like to thank all contributors.

The authors have done their best to write carefully, and have placed no deliberate errors in the book, but are quite familiar with their own imperfections. They therefore ask the readers to be indulgent and assure them that e-mails from them calling attention to errors or containing suggestions for improvement of the book will be gratefully received and very much appreciated.

Heike Vallery, `h.vallery@tudelft.nl`
Arend L. Schwab, `a.l.schwab@tudelft.nl`

Delft, The Netherlands
August, 2020

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Part I

Rigid-Body Dynamics

Overview of Part I

The first chapter of this part contains a brief review of general mechanics background and also introduces some main notations used in the book.

Following up on these basics, Chapter 2 reviews basic dynamic concepts involving particles and systems of particles, using fixed coordinate systems.

When transitioning from 2-D to 3-D rigid-body dynamics, description of orientation requires particular attention. Therefore, two chapters are dedicated only to kinematics involving rotating bodies and coordinate systems, namely Chapter 3 and Chapter 4.

To derive the equations of motion of mechanical systems, four alternative methods will be introduced: Work and Energy in Chapter 5, Newton-Euler in Chapter 6, Virtual Work in Chapter 7, and Lagrange's method in Chapter 8. Practical guidance and advice is provided on when to choose which method for a particular problem at hand.

An introduction to numerical simulation of dynamic systems is provided in Chapter 9. This chapter can mostly be read independent of the previous, and some readers may prefer to read it first instead of last. That way, the different methods to derive the equations of motion can directly be translated to numerical simulations of the treated mechanical systems.

For further reading, we recommend classical mechanics and dynamics books, of which we like to mention here: Whittaker [75], Hamel [28], Sommerfeld [67], Pars [54], and Goldstein [25]. In particular we like to draw attention to the very readable book by Cornelius Lanczos on variational principles in mechanics [39].

1 Notations and Background

This chapter introduces some general notations used in this book, and it reviews a few selected concepts of mechanics that will be needed frequently in the following chapters. For readers unfamiliar with these topics, more detailed information is for example given in [4].

All following content relies on solid knowledge of the mathematical tools presented in Appendix A and Appendix B. Deeper coverage of the mathematical prerequisites can for example be found in [40, 57, 70].

1.1 Notations

1.1.1 Typesetting of Scalars, Vectors, and Matrices

Scalars, vectors, unit vectors and matrices in this book are typeset as depicted in Table 1.1.

Table 1.1: Notation of scalars, vectors, unit vectors and matrices

	This book	Handwriting
Scalars	F	$\mathcal{F}$
Vectors	$\boldsymbol{F}$	$\underline{\mathcal{F}}$
Unit vectors	$\hat{\boldsymbol{e}}$	$\underline{\hat{\mathcal{e}}}$
Matrices	$\mathbf{R}$	$\mathcal{R}$

Note that we often use the same letter when we refer to a vector's magnitude, so for example a could be used to refer to the magnitude $|\boldsymbol{a}|$ of a vector $\boldsymbol{a}$. However, such a relationship must still always be explicitly defined.

1.1.2 Sub- and Superscripts

A position vector that points to a point A from a point P will be denoted as $\boldsymbol{r}_{A/P}$ (which reads: "position of A *with respect to* P"). Analogous notation will be used for other relative quantities, such as linear or angular velocity.

◯ **Problem 1.1** The *simplest walker* [19, 63], a 2-D model of a bipedal mechanism, consists of only three particles: two feet, each of mass m, and a hip mass M. Massless legs connect the particles and are hinged at the hip (see Figure 1.1). The model can walk down a slope of angle γ under the influence of gravity g.

Use vector addition (see Appendix A.1) to calculate the location $\boldsymbol{r}_{D/A}$ of the swing foot with respect to the stance foot, as a function of the position vector $\boldsymbol{r}_{B/A}$ of the hip with respect to the stance foot A and the relative position vector $\boldsymbol{r}_{D/B}$ of the swing foot with respect to the hip.[a]

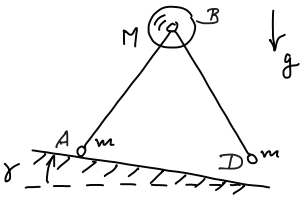


Figure 1.1: 2-D walker.

[a] *Hint*: Compare to Example A.1.

For components of m-dimensional vectors, we use the *index notation*:

$$\boldsymbol{r} = \begin{pmatrix} r_1 & r_2 & \dots & r_m \end{pmatrix}^{\mathrm{T}}. \tag{1.1}$$

Elements of $m \times n$-dimensional matrices receive two indices, for row and column:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}. \tag{1.2}$$

In case we operate in $\mathbb{R}^3$ and with Cartesian coordinate systems, we also often use the names of the axes as indices (for example x, y, z) instead of the numerical values 1, 2, 3, to allow quick association.

In this book, we will frequently make use of different coordinate systems and associated triads, which are a set of three unit vectors that define the axis directions. To indicate that the components of a vector are expressed in a particular triad, we use a left superscript. Thus, the vector ${}^{\mathcal{B}}\boldsymbol{r}$ is the same vector as ${}^{\mathcal{N}}\boldsymbol{r}$, just for the former the components are expressed in a triad $\mathcal{B}$, and for the latter the components are expressed in a triad $\mathcal{N}$. So, the difference is in the projection onto different axes (see also Appendix A.4). In case we use an inertial triad $\mathcal{N}$, we

often drop the superscript and only write $\boldsymbol{r}$. This is sufficient for the first two chapters in this book, where we do not yet deal with rotating coordinate systems or triads. Mapping between different triads will be explained in Chapter 3. Also more sub- and superscript notations are introduced in the following chapters.

1.1.3 Drawing Vectors

If we label an arrow in a drawing with a vector symbol (so a boldface variable), for example $\boldsymbol{F}$, then this label refers to the definition of the vector including its magnitude *and* direction. The direction drawn for the arrow is merely an illustration. This notation is particularly used if the direction of the vector is unknown or changes.

In contrast, if we label an arrow with a scalar variable name F (so typeset regular), this scalar value is the signed magnitude of the vector, and the direction of the vector is defined by the drawn direction of the arrow. We do allow this scalar value F to take on a negative value (even though the magnitude of a vector is strictly seen always positive), in order to enable inversion of the vector's direction later on: When defining a force vector or vector component, we do not need to know in which direction it points, we only need to know its *line of action*. This implies that an arrow labeled with $-F$ is equivalent to an arrow pointing in the opposite direction and labeled F. Particularly when a vector is split into components along specific axis directions, we will use scalar labels for these components.

Example 1.1 A body is cut free from a frictionless surface, the ground. Figure 1.2 shows three different ways to draw the forces between body and ground.

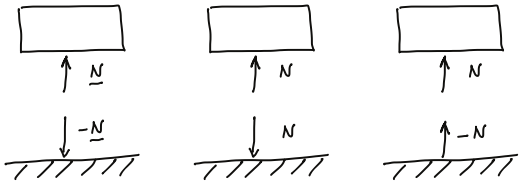


Figure 1.2: Equivalent drawings of force vectors, with vector (left) and scalar labels.

If the force vector acting on the body is labeled by $\boldsymbol{N}$, then the reaction force (following Newton's third law, see Section 1.2.5) to this vector, drawn as acting on the ground, *must* be labeled with $-\boldsymbol{N}$ (note the minus sign), to make clear that the direction is opposite (Figure 1.2, left). If the force

vector acting on the body is labeled by a scalar N and drawn upward, and its reaction force is drawn as pointing downward, then that reaction *must* be labeled by N as well, not by $-N$ (Figure 1.2, center). Alternatively, and still formally correct, one could draw the same reaction force vector as pointing upward and label it by $-N$ (Figure 1.2, right). However, this is discouraged as it can be confusing.

A vector pointing into or out of of the paper plane (Figure 1.3, top row) is drawn following the analogy of a dart: We indicate vector components pointing into the plane by $\otimes$ and those pointing out by $\odot$.

Vectors that encode rotational information, such as moments (see Section 1.2.1), will be drawn as in the bottom row of Figure 1.3, with a double arrowhead in the plane. The *right-hand rule* for rotations states that if the thumb points in direction of the arrow, the curled fingers of one hand indicate the positive direction of a *moment* or angle (Figure 1.3, center).

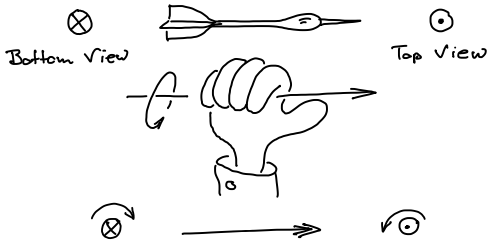


Figure 1.3: Notation for a vector that is perpendicular to the plane of drawing imagined as a dart (top); the right-hand rule to indicate positive direction of a rotation/moment about an axis (center and bottom).

1.1.4 Units

In mechanics, we often deal with quantities that have a measurement unit, like forces, velocities, or positions. Each such quantity is the *product* of a number and a measurement unit. This also holds for components of vectors or matrices. Generally, the letters of units are printed in roman (upright), those of scalar variables in italic.

Example 1.2 Force can be expressed as the product of a numerical value and the measurement unit Newton: $F = 3 \cdot \mathrm{N} = 3\,\mathrm{N}$

Expressions may contain mixtures of quantities given with their numerical value and quantities given as variables. Also in those cases, one must pay attention to consider both factors (numerical value and unit) that compose each quantity, and *not* to insert or omit units.

Example 1.3 For a particle of mass $m = 3\,\mathrm{kg}$ having acceleration vector $\boldsymbol{a}$, the resultant force vector $\boldsymbol{F}$ on the particle can be calculated as:
Correct is: $\boldsymbol{F} = m \cdot \boldsymbol{a} = 3\,\mathrm{kg} \cdot \boldsymbol{a}$.
*In*correct is: ~~$\boldsymbol{F} = 3 \cdot \boldsymbol{a}\,\mathrm{N}$~~.

Example 1.4 The derivative $\dot{\theta}$ of an angle θ is given as a constant value of $\dot{\theta} = 2\,\mathrm{rad/s}$. At time $t = 0$, the angle is zero as well. So, the angle θ is a function of time t:

$$\theta = \int \dot{\theta}\,\mathrm{d}t = 2\,\mathrm{rad/s} \cdot t. \tag{1.3E}$$

At $t = 3\,\mathrm{s}$, the angle is calculated as:

$$\theta = 2\,\mathrm{rad/s} \cdot 3\,\mathrm{s} = 2 \cdot 3 \cdot \mathrm{rad/s} \cdot \mathrm{s} = 6\,\mathrm{rad}. \tag{1.4E}$$

Problem 1.2 A particle performs a harmonic oscillation, such that its location coordinate x is a function of amplitude x_0, angular frequency Ω, and time t:

$$x = x_0 \sin(\Omega t). \tag{1.5P}$$

Fill in suitable units for x, x_0, Ω, and t and check the equation for consistency. Be aware that the argument of the sin-function does not have units.[a]

[a] *Hint:* Recall the definition of the unit radian, implying that $1\,\mathrm{rad} = 1$.

Following the ISO norm [34] and the "Red Book" [13], for quantities that have a measurement unit, the numerical value of a quantity Q is denoted by $\{Q\}$, and the measurement unit is denoted by $[Q]$, such that

$$Q = \{Q\} \cdot [Q]. \tag{1.6}$$

Therefore, the numerical value can also be denoted as $\{Q\} = Q/[Q]$.

Example 1.5 A mass $m = 3.5\,\mathrm{kg}$ has the numerical value $\{m\} = 3.5$ and the unit $[m] = \mathrm{kg}$. The same quantity value can be expressed as $3500\,\mathrm{g}$. Then, the numerical value is $\{m\} = 3500$, and $[m] = \mathrm{g}$.

Square brackets are a common source of mistakes. Such brackets only have meaning when placed around the physical quantity. Square brackets should *not* be placed around the unit.

Example 1.6 For time t,
good use is: $t/\mathrm{s} = 4$, or $[t] = \mathrm{s}$,
not good use is: ~~$t = 4\,[\mathrm{s}]$~~, or ~~$t\,[\mathrm{s}]$~~.

Incorrect square brackets are often found in axes labels of plots. The SI [5] advises division, as in a label like (for the example of time): t/s. Also possible are round brackets [33, 56]: time t (s).

A general advice is to work as long as possible with variables, and only to substitute numbers and units in the final solution. Units almost always help as part of plausibility checks for calculations.

◯ **Problem 1.3** Which of these expressions make correct use of units, for mass m, time t, force F, angular velocity ω, and angle θ?

A. $m = 20$
B. $m = 3\,\mathrm{N}$
C. $F = 3 \cdot m\,[\mathrm{N}]$
D. $\mathrm{F} = 20N$
E. $[m] = \mathrm{kg}$
F. $m/[m] = 20$
G. $t = 3\,\mathrm{s}$
H. $\omega = 2\,\mathrm{rad/s}$
I. $\theta = \omega \cdot t = 2 \cdot t\,[\mathrm{s}]$
J. $\theta = \omega \cdot t, [\omega] = \mathrm{rad/s}, [t] = \mathrm{s}, [\theta] = \mathrm{rad}$
K. $\{m\} = 20$
L. $\omega = t^2$
M. $\omega = 1\,\mathrm{rad/s} \cdot t^2$
N. $\omega = 1\,\mathrm{rad/s^3} \cdot t^2$
O. $f(t) = \cos(1\,\mathrm{rad/s} \cdot t)$
P. $g(t) = e^{-t}$
Q. $h(t) = \sin(\omega t) = \sin(2t)$

1.2 Basic Definitions and Tools Used in Mechanics

Readers are expected to have already knowledge on the branch of mechanics that is called *statics*. Statics deals with mechanical systems that are at rest or move with constant linear velocity. It defines conditions for forces and moments acting on or within these systems such that the systems remain in this state. Statics and other branches of mechanics share many important definitions and tools. In this section, some of these concepts will be reviewed.

1.2.1 Definition of a Moment

The *moment*, often also called *torque*[1], of a force $\boldsymbol{F}$ with respect to point P is defined as:

$$\boldsymbol{M}_P := \boldsymbol{r}_{A/P} \times \boldsymbol{F}, \tag{1.7}$$

[1] The term torque is often used for a couple and for a torsional moment; a moment about a longitudinal axis. An example is the torque a motor applies to a shaft.

where the vector $\boldsymbol{r}_{A/P}$ is a vector from point P to point A, the point of application of the force. Note that from the definition of the cross product, it follows that A may in fact be any *arbitrary* point on the line of action of $\boldsymbol{F}$ (see Problem 1.5). Also note that the order in which this cross product is written matters, as $\boldsymbol{F} \times \boldsymbol{r}_{A/P} = -\boldsymbol{M}_P$.

Problem 1.4 Calculate the moment about point P using (1.7), for force $\boldsymbol{F} = (3,\, 4,\, 0)^{\mathrm{T}}\,\mathrm{N}$, if the position vector of the point of application of this force, A, with respect to point P is $\boldsymbol{r}_{A/P} = (2,\, 3,\, 0)^{\mathrm{T}}\,\mathrm{m}$. Draw all three vectors in the XY-plane and apply the right-hand rule of Figure 1.2 to check the direction of the moment vector. Then, choose another point A' on the line of action of the force to calculate the moment and compare results.

If two force vectors act on a rigid body and if these two forces have parallel lines of action, equal magnitude, but opposite direction, then the sum of these two forces (also called a *force couple*) is the zero vector. However, the forces still cause a moment vector. This *couple moment* vector does not depend on the point of reference P, so it is a *free vector*.

Problem 1.5 Use the definition of the cross product to show that when calculating the moment vector $\boldsymbol{M}_P$ of a force vector $\boldsymbol{F}$ about point P according to (1.7), it does not matter which vector $\boldsymbol{r}_{A/P}$ you choose from point P to the line of action of the force.[a]

[a] *Hint*: Try these two approaches: a) Draw the parallelogram spanned by position and force vector and show that neither the magnitude nor the direction of the cross product changes when the force shifts along its line of action. b) Use (A.2) and the right-hand rule. Describe an arbitrary points on the line of action of a force as a linear combination of a position and a direction vector. Then, calculate the cross product of the point with the force vector.

1.2.2 Force System Resultants: Equipollence

When several forces and moments act on a rigid body, it is always possible to formulate an equivalent system that consists of 1) one single resultant force vector acting at an arbitrary point O and 2) one couple moment vector. Equivalence, or more precisely *equipollence*, means that the effect of the original and of the reduced force/moment system on the rigid body is identical. This leads to the following steps:

1. To determine the magnitude and direction of the resultant force $\boldsymbol{F}_{\mathrm{r}}$, all N original force vectors $\boldsymbol{F}_i$ need to be summed:

$$\boldsymbol{F}_{\mathrm{r}} = \sum_{i=1}^{N} \boldsymbol{F}_i. \tag{1.8}$$

2. The associated resultant couple moment vector $\boldsymbol{M}_\mathrm{r}$ equals the moment caused by the N original forces and K original couple moments about the same point O:

$$\boldsymbol{M}_\mathrm{r} = \sum_{j=1}^{K} \boldsymbol{M}_j + \sum_{i=1}^{N} \boldsymbol{r}_i \times \boldsymbol{F}_i, \tag{1.9}$$

where the $\boldsymbol{r}_i$ are the position vectors of the points of application of the N original forces with respect to point O.

Simplifying even further, it is also always possible to find an equivalent system that contains only one single force vector and a single couple moment vector *that is parallel to the force vector's line of action.*

Problem 1.6 For an original system of N forces $\boldsymbol{F}_i$, $i = 1 \ldots N$, set up an algorithm (set of equations) that allows determining the resultant force vector, one point on its line of action, and a resultant couple moment vector that is parallel to this line of action. Test using an example.[a]

[a] *Hint*: The resultant force vector determines the direction of any possible couple moment. Therefore, express the couple moment vector as a product of its magnitude and direction, which contains only one scalar unknown. To determine the location of the line of action of this force as well as the magnitude of the couple moment, equate the moment of all original forces and moments about a point O to the moment of the resultant force and the couple moment vector. Any point on the line of action of the force is a possible point of application. So, you could choose a position vector that is perpendicular to the line of action and only has unknown length.

1.2.3 Newton's First Law and Static Equilibrium

In 1687, Isaac Newton (1642 - 1727) stated his three laws of motion [49]. The first law reads:

1. *"Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon."*

So, a particle that moves at constant linear velocity $\boldsymbol{v}$ will keep the same velocity unless a force $\boldsymbol{F}$ acts upon it. Note that a particle at rest has a constant velocity of zero.

$$\sum \boldsymbol{F} = 0 \leftrightarrow \dot{\boldsymbol{v}} = 0. \tag{1.10}$$

Based on this law, static equilibrium of a rigid body is defined as a state where the sum of all forces acting on the system is zero, as is the sum of all moments acting on the body about any arbitrary point.

Note that the term equilibrium does not necessarily imply *stable* equilibrium. For example, a ball in a convex valley resides in a stable equilibrium, while the same ball balancing on top of a concave mountain peak is in an unstable equilibrium. Stability can be investigated by analyzing the system's response to infinitesimally small perturbation from its state of rest.

1.2.4 Center of Mass

The position vector $\boldsymbol{r}_{C/A}$ from a point A to the overall *center of mass* (CoM) of a system of N particles with masses m_i is defined by:

$$\boldsymbol{r}_{C/A} := \frac{\sum_{i=1}^{N} m_i \boldsymbol{r}_{i/A}}{\sum_{i=1}^{N} m_i}, \tag{1.11}$$

where $\boldsymbol{r}_{i/A}$ is the position vector of the i-th particle with respect to the same point A.

When calculating the center of mass for a rigid body, the summation turns into an integral:

$$\boldsymbol{r}_{C/A} := \frac{\int_m \boldsymbol{r}_{P/A}\,\mathrm{d}m}{\int_m \mathrm{d}m}, \tag{1.12}$$

where $\boldsymbol{r}_{P/A}$ is the position vector of the center of mass P of an infinitesimally small volume element of mass $\mathrm{d}m$ in the body. The integral resolves to a triple integral for the case of three-dimensional bodies.

In order to calculate the center of mass for a composite body where the mass and center of mass of each individual element are known, (1.11) can again be used. When dealing with hollow sections in systems with homogeneous mass distribution, the calculation can be simplified by considering first a solid structure and then considering each hollow section as a body with negative mass.

Note that the *centroid* (geometric center), the center of mass, and the *center of gravity* (point of application of the resultant gravitational force) may all be different points. This is the case if mass distribution is not homogeneous and/or if the gravitational field is not homogeneous.

Example 1.7 Three particles of masses m_1, m_2 and m_3 are at locations with position vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$, respectively, with respect to a point O. The values are given as:

$$m_1 = 2\,\text{kg},\ m_2 = 3\,\text{kg},\ m_3 = 5\,\text{kg}, \text{ and}$$
$$\boldsymbol{r}_1 = \begin{pmatrix}5 & 10 & 1\end{pmatrix}^{\mathrm{T}}\ \text{m},\ \boldsymbol{r}_2 = \begin{pmatrix}-10 & 0 & 6\end{pmatrix}^{\mathrm{T}}\ \text{m},\ \boldsymbol{r}_3 = \begin{pmatrix}2 & 0 & 4\end{pmatrix}^{\mathrm{T}}\ \text{m}.$$

Their combined center of mass is given by:

$$\begin{aligned}\boldsymbol{r}_C &= \frac{1}{m_1+m_2+m_3}\left(m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 + m_3\boldsymbol{r}_3\right)\\ &= \frac{1}{(2+3+5)\,\text{kg}}\left(2\,\text{kg}\begin{pmatrix}5\\10\\1\end{pmatrix}\text{m} + 3\,\text{kg}\begin{pmatrix}-10\\0\\6\end{pmatrix}\text{m} + 5\,\text{kg}\begin{pmatrix}2\\0\\4\end{pmatrix}\text{m}\right)\\ &= \begin{pmatrix}-1 & 2 & 4\end{pmatrix}^{\mathrm{T}}\cdot\,\text{m}.\end{aligned}$$

Problem 1.7 Use integration to calculate the center of mass of several common bodies, such as a cone, a solid hemisphere, and a thin-walled hollow cone. Assume homogeneous mass distribution.

Problem 1.8 A group of students constructs a top (Figure 1.4) from

- three orthogonally arranged wooden skewers, each of mass m_s, length l, and negligible thickness, and
- five pieces of cork, each of mass m_k and negligible dimensions. The center cork is positioned at a distance Δ from the center A of skewer 1.

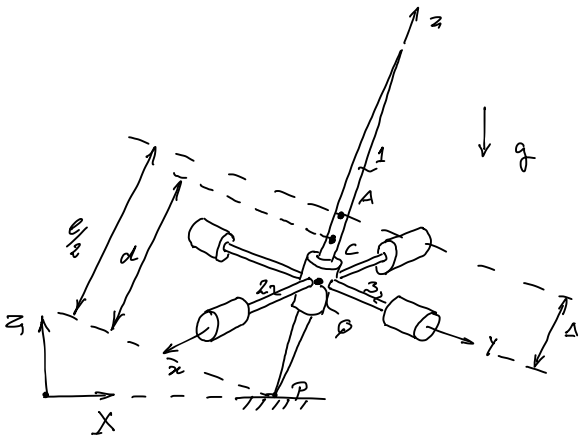


Figure 1.4: Top of skewers and corks.

(DIY-tip: use a screwdriver to drill a hole through the cork, after which it is easier to put the skewer through the cork)

The construction is idealized, such that the skewers intersect in point Q and the top's center of mass C is on the z-axis defined by skewer 1. The axes x and y point along skewers 2 and 3. Gravity with acceleration constant g acts in negative vertical direction. An inertial XYZ coordinate system has its origin on the ground and Z points up. A triad $\mathcal{N}$ is associated with XYZ, and a triad $\mathcal{B}$ is associated with xyz. After building the top, students realize that the top's overall mass is M^*, and the center of mass is not on the z-axis, but at the location C^* with relative position vector ${}^{\mathcal{B}}\boldsymbol{r}_{C^*/P} = (x_{C^*}, y_{C^*}, z_{C^*})^{\mathrm{T}}$ with respect to point P. The students want to add two more pieces of cork, each of mass $\frac{M^*}{10}$, to the skewers at yet unknown locations D and E, in order to put the center of mass back on z.

Calculate the components of suitable relative position vectors ${}^{\mathcal{B}}\boldsymbol{r}_{D/P}$ and ${}^{\mathcal{B}}\boldsymbol{r}_{E/P}$ of these two pieces with respect to point P. Give these components as functions of x_{C^*}, y_{C^*}, z_{C^*}, M^*, l, and Δ.

Remember that the cork pieces can only be put onto skewers, and that they have negligible dimensions. ▷▶

1.2.5 Newton's Third Law of Motion

Newton's third law of motion [49] states:

3. *"To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts."*

So, if an object a exerts a force $\boldsymbol{F}_a$ on another object b, then b exerts an equal and opposite force $\boldsymbol{F}_b$ on a:

$$\boldsymbol{F}_a = -\boldsymbol{F}_b. \tag{1.13}$$

The forces share the same line of action. Philosophically speaking: "A force is never alone"; the action-reaction forces form a unit, the force pair. When we discuss "generalized forces" in Chapter 7 and Chapter 8, we will see that a force pair often even forms a single generalized force.

1.2.6 Definition of Internal and External Forces

Two types of forces are distinguished which act on a particle that is part of a system of N particles: *external* forces $\boldsymbol{F}_i$ and *internal* forces $\boldsymbol{f}_{ij}$, such that the sum of all forces acting on a particle is:

$$\boldsymbol{F}_i + \sum_{j=1}^{N} \boldsymbol{f}_{ij}. \tag{1.14}$$

The external force vector $\boldsymbol{F}_i$ is defined as the sum of all forces that act on particle i by sources outside the system.

Internal forces $\boldsymbol{f}_{ij}$ are the forces that act between particles in a system. For example, if a spring interconnects two particles in one system, then the spring force is called an internal force. The subscript ij means "acting on particle i, exerted by particle j". Note that $\boldsymbol{f}_{ij} = \boldsymbol{0}$ for $i = j$. Figure 1.5 illustrates the particles and forces.

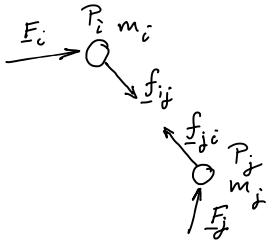


Figure 1.5: Two particles i and j with internal and external forces acting on them.

According to Newton's third law (1.13), there must be an equal and opposite reaction exerted on particle j, namely

$$\boldsymbol{f}_{ji} = -\boldsymbol{f}_{ij}. \tag{1.15}$$

Therefore, over the entire system of particles, the sum of all these internal forces is zero:

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \boldsymbol{f}_{ij} = 0. \tag{1.16}$$

Note that the interpretation of a force as being internal or external depends on the chosen definition of system boundaries. We tend to see gravity as an external force, because we consider Earth just in terms of providing us with an inertial coordinate system, but not as being part of our mechanical system. In case we do consider Earth as part of the system (like in a study on planetary motions), then gravity suddenly becomes an internal force. This demonstrates that the idea of internal and external is simply a matter of system definition.

1.2.7 Moments of Internal and External Forces

The resultant moment of internal forces acting on a system is zero, as will be shown in the following.

Figure 1.6 shows again two particles i and j of a system of particles with internal and external forces acting on them. We define a relative position vector $\boldsymbol{\rho}_i$ of the i-th particle with respect to an arbitrary reference point Q, such that

$$\boldsymbol{r}_i = \boldsymbol{r}_Q + \boldsymbol{\rho}_i. \tag{1.17}$$

This implies that $\boldsymbol{\rho}_i$ points from Q to the particle. This is for brevity of notation, as $\boldsymbol{\rho}_i$ is shorter than the more consistent naming $\boldsymbol{r}_{i/Q}$ of the relative position of i with respect to Q.

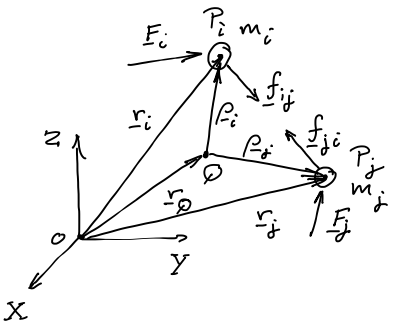


Figure 1.6: Definitions of the absolute and relative position of particles i and j, as well as of internal and external forces acting on the particles.

To obtain the resultant moment of all forces acting on particle i about the arbitrary point Q, we take the cross product of the particle's relative position vector $\boldsymbol{\rho}_i$ and the external and internal forces:

$$\boldsymbol{M}_{Q,i} = \boldsymbol{\rho}_i \times \left(\boldsymbol{F}_i + \sum_{j=1}^{N} \boldsymbol{f}_{ij} \right). \tag{1.18}$$

Summing over all particles gives the resultant moment of all forces acting on the system:

$$\begin{aligned} \boldsymbol{M}_Q &= \sum_{i=1}^{N} \boldsymbol{\rho}_i \times \left(\boldsymbol{F}_i + \sum_{j=1}^{N} \boldsymbol{f}_{ij} \right) \\ &= \sum_{i=1}^{N} (\boldsymbol{\rho}_i \times \boldsymbol{F}_i) + \sum_{i=1}^{N} \left(\boldsymbol{\rho}_i \times \sum_{j=1}^{N} \boldsymbol{f}_{ij} \right). \end{aligned} \tag{1.19}$$

Because of Newton's law of action and reaction (1.15), we know that for each internal force $\boldsymbol{f}_{ij}$, there is always a counter-force $\boldsymbol{f}_{ji}$ acting in

opposite direction, with the same magnitude and on the same line of action. That means that whichever point is chosen as a reference point, the moment pairs about this point will cancel:

$$\boldsymbol{\rho}_i \times \boldsymbol{f}_{ij} = -\boldsymbol{\rho}_j \times \boldsymbol{f}_{ji}. \tag{1.20}$$

Therefore, the sum of moments of all internal forces is zero, and (1.18) simplifies to

$$\boldsymbol{M}_Q = \sum_{i=1}^{N} (\boldsymbol{\rho}_i \times \boldsymbol{F}_i). \tag{1.21}$$

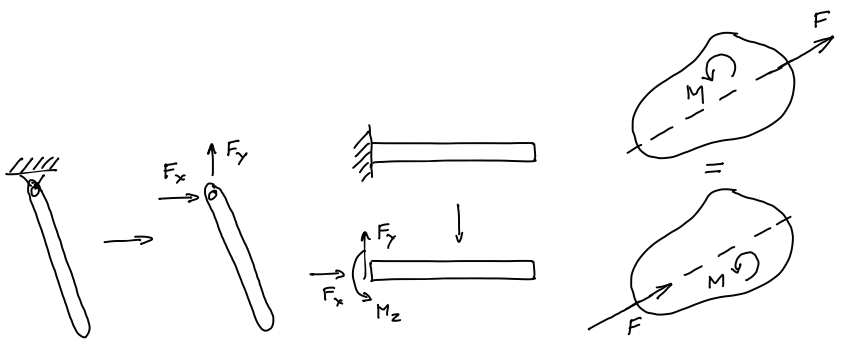
This proves that internal forces cannot cause net moments on a system, regardless of the choice of reference Q.

1.2.8 Free-Body Diagrams

A *free-body diagram* (FBD) helps to understand and visualize a problem and is used to define system boundaries and variables. It is a simplified drawing of (a part of) a mechanical system with all forces, moments and dimensions. The name is slightly misleading, as FBDs cannot only be drawn for single rigid bodies. They can also be established for parts of bodies or for systems of bodies or particles. These are the steps to draw a FBD:

1. Draw the system in a free state, i.e. "cut" the system at convenient locations and draw outlined shapes of the separate pieces. Each cut will introduce new external (formerly internal) action-reaction forces at the system boundaries. Some helpful guidelines:
   a) Always draw the system in a generic state. So for example if you draw the FBD of a pendulum, draw it with a generic angle, and not in upright or downright position, as those are special cases.
   b) Whenever possible, choose your system boundaries such that you expose only action-reaction forces that you are actually interested in calculating. Otherwise, extra equations and unknowns are introduced, complicating calculations.
2. Establish at least one coordinate system: Choose a clearly defined position for the origin. Check if rotating and/or translating the coordinate system can make things easier.

3. Indicate all known and unknown external forces and moments that act on the system, on the correct locations (e.g. external loads, support reactions, weight). Some rules:
   a) Do not show internal forces or moments.
   b) If a connection prevents movement in a particular direction, then forces/moments are drawn in that direction, see Figure 1.7a and Figure 1.7b.
   c) If a segment is split in two, the forces and moments acting on the two segments in the separated FBDs are equal in magnitude and opposite in direction.
   d) Forces acting on a rigid body may be shifted along their lines of action (sliding vectors), see Figure 1.7c.
   e) Couple moments acting on a rigid body may be placed anywhere (they are free vectors), see Figure 1.7c.
4. Label all forces and moments with unique names.
5. Label all necessary dimensions (distances and angles) to calculate moments of forces with respect to relevant reference points like the center of mass.



(a) For a pendulum, the support is replaced by forces transmitted by the hinge joint.

(b) For a clamped beam, the supporting wall is replaced by the forces and moment it exerts.

(c) Forces may be shifted along their lines of action; moments are free vectors.

Figure 1.7: Examples for drawing components of FBDs (notice that these are *not* complete FBDs).

A FBD must include all relevant information of the original drawing, particularly all information to establish the sums of forces and moments.

Example 1.8 Figure 1.8 shows a homogeneous block sliding down a slope (left), and its FBD (right).

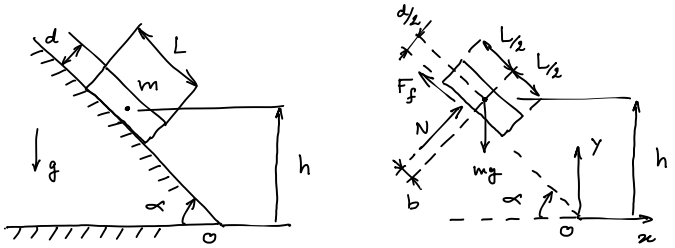


Figure 1.8: Sliding block and corresponding FBD.

A frequent mistake when drawing FBDs for dynamical systems is to already consider the specific movement the system performs. For example, if a particle rotates about one axis, one may be tempted to draw a force in radial direction based on previous knowledge on dynamics and the intuition that there must be a radial force sustaining this rotation. This is a critical mistake, as it defies the purpose of a FBD, reducing it to a confirmation of one's own intuition. Such a procedure exposes subsequent calculation to omissions. A FBD must contain all the *possible* reaction forces generated at the system boundaries, only based on the nature of the connection, and disregarding information that is given on the system's movement. Consequently, an FBD drawn to determine static equilibrium looks identical to one intended to derive a system's *equations of motion* (EoM), as will be done in later chapters.

Example 1.9 A cylindrical bar (Figure 1.9) has a body-fixed coordinate system uvw associated with triad $\mathcal{B}$. The bar can rotate with variable angular speed $\dot{\psi}$ about the Z-axis of an inertial coordinate system XYZ, associated with triad $\mathcal{N}$. The Z- and w-axis always coincide and the angle between the Y- and v-axis is denoted by ψ.

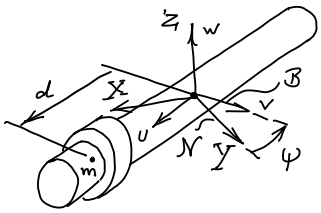


Figure 1.9: A rotating bar and sliding ring.

A ring with mass m can slide without friction on the bar and has a position vector ${}^{\mathcal{B}}\boldsymbol{\rho} = (d,\, 0,\, 0)^{\mathrm{T}}$, with respect to the origin of the coordinate systems. Herein, $d = d(t)$ is a variable in time t. The radius, width, and thickness of the ring are *not* negligible. There is no gravity.

Draw a FBD of the ring, in a suitable projection into a 2-D plane.

Exemplary solution

It is convenient to choose an orthographic projection into the uw-plane. The FBD (Figure 1.10) reflects that except for sliding in u-direction and rotating about that axis, the ring cannot move relative to the bar. Representing these constraints for the other two translations and for the other two rotations, forces and moments need to be drawn in those directions, respectively.

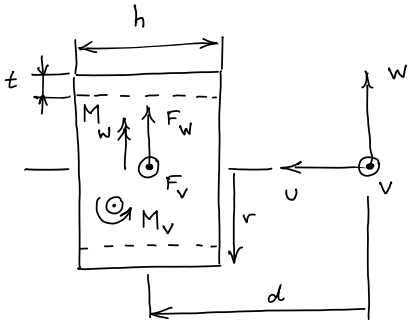


Figure 1.10: Free-body diagram of the ring.

Problem 1.9 A solid wheel (Figure 1.11) rolls without slipping in a steady circular motion with constant angle θ and constant ground radius R in positive direction about the Z-axis (Figure 1.11). Its center of mass C has speed v_C. The wheel is modeled as a thin disk with radius r and mass m.

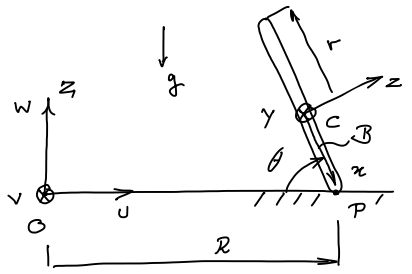


Figure 1.11: A disk rolling in a steady motion.

A coordinate system uvw rotates around an inertial Z-axis, such that the

disk's current ground contact point P is always on the u-axis.

Draw a FBD of the wheel, projected into the uw-plane. Use only scalar labels by resolving all force and moment vectors into their components, in such a way that there is a minimal number of labels. ▶

Drawing FBDs often leaves a choice how to apply the principle of equipollence, as reviewed in Section 1.2.2. To indicate that a body can neither translate in one particular direction x nor rotate about an axis y perpendicular to x, there are two main choices to represent both constraints:

- Drawing a force in direction of x at an unknown location, and specifying as unknowns the magnitude of the force and a parameter describing its location in z-direction. In that case, no additional moment vector needs to be drawn. However, one implicitly assumes that the force is never zero in combination with a nonzero moment, which needs verification. See for example Figure 1.8.
- Letting the force act at a pre-specified location (for example a rigid body's center of mass) and also adding a moment vector in y-direction. Then, the two unknowns are the magnitudes of the force and of the moment vector. See for example Figure 1.10.

The second option (chosen in Figure 1.10 for the M_w, F_v- and M_v, F_w-pairs) is more generic and safer in terms of avoiding possible mistakes in assumptions. However, in the case of unilateral contact, where it is not possible to exert a pure moment *without* a force, it is advisable to choose the first option, particularly when aiming for easy possibilities to later check results. This option was chosen for the N, b-pair in Figure 1.8.

Example 1.10 A cone rolls on the ground (Figure 1.12). It has mass m, uniform density ρ, base radius R, height H and center of mass C.

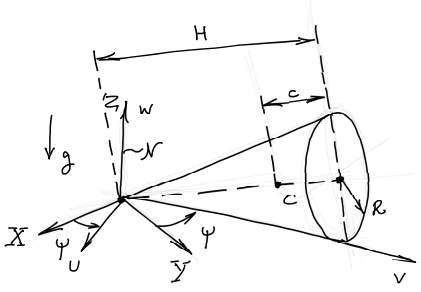


Figure 1.12: Cone

The figure indicates an inertial XYZ coordinate system, and a rotating

coordinate system uvw. The v-direction is always aligned with the cone's ground contact line, and w is always vertical. The cone rolls (possibly with slipping) on the XY-plane, and its tip always remains in the origin O. It is subjected to gravity with field strength g in negative Z-direction.

Draw a free-body diagram (FBD) of the cone projected onto the vw-plane. Show distributed loads only by their resultant, and leave out all components of these reactions which are zero.

Exemplary solution

A possible FBD for the cone is shown in Figure 1.13.

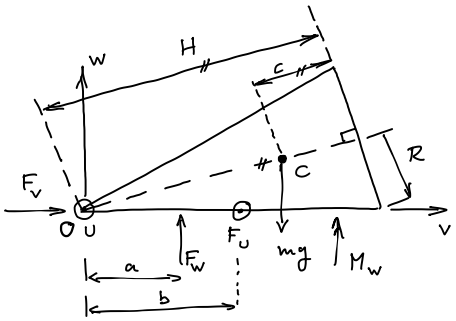


Figure 1.13: Exemplary FBD for the rolling cone.

One may ask why it contains a moment component about the w-axis, but not about the other axes. To understand this, one has to consider the type of constraint and use equipollence, as reviewed in Section 1.2.2: The resultant vertical ground reaction force F_w, representing the centroid of the distributed load acting on the cone, has been drawn at a (yet unknown) distance a from the origin O. This way, there is no additional constraint moment about the u-axis to be considered. Alternatively, using equipollence one could draw this resultant force acting in the origin, in which case an additional moment component about the u-axis would appear.

The occurrence of a couple moment component about the u-axis without any resultant vertical force is impossible given the unilateral contact. Drawing the force at the unknown distance a and solving for a has a clear advantage for results checking: A value for a being negative or larger than $\sqrt{H^2 + R^2}$ would be implausible, because the resultant ground reaction force can only be applied within the base of support of the cone.

A similar relationship holds for the force component in u-direction and the moment about the w-axis, although here it is possible that a pure (friction) couple moment can be transmitted without any resultant force F_u being present. The resulting redundancy would need to be resolved for a given situation by an additional condition during the calculations. For example,

by stating that the couple moment about the w-axis is only considered in case F_u resolves to be zero, or by forcing b to be zero.

About the v-axis, no moment can be transmitted as none of the potential distributed loads in the line contact has any lever arm.

Problem 1.10 The cone of Example 1.10 is now at rest. Use the FBD and static equilibrium in that case to calculate the distance a between the cone's tip and its *center of pressure*, so the resultant vertical ground reaction force.

1.3 Introduction to Dynamics

1.3.1 Kinematics and Kinetics

Dynamics is the branch of mechanics that deals with moving systems. It allows to analyze movements, as well as to determine the relationship between the movement and the forces and moments causing this movement. Accordingly, dynamics is split into two main branches, kinematics and kinetics:

- *Kinematics* is the domain in mechanics that describes motions, without considering the cause of these motions.
- *Kinetics* is the domain in mechanics that finds the relationships between motion and its causes (i.e. forces and moments). From these relationships, we will derive the equations of motion (EoM).

1.3.2 Kinematics in Fixed Coordinate Systems

In a fixed (or inertial) Cartesian coordinate system, the position vector $\boldsymbol{r}_P$, velocity $\boldsymbol{v}_P$ and acceleration $\boldsymbol{a}_P$ of a point P are defined as follows:

$$\textit{Position} \qquad \boldsymbol{r}_P := X_P\hat{\boldsymbol{n}}_1 + Y_P\hat{\boldsymbol{n}}_2 + Z_P\hat{\boldsymbol{n}}_3, \tag{1.22}$$

$$\textit{Velocity} \qquad \boldsymbol{v}_P := \dot{\boldsymbol{r}}_P = \dot{X}_P\hat{\boldsymbol{n}}_1 + \dot{Y}_P\hat{\boldsymbol{n}}_2 + \dot{Z}_P\hat{\boldsymbol{n}}_3, \tag{1.23}$$

$$\textit{Acceleration} \qquad \boldsymbol{a}_P := \dot{\boldsymbol{v}}_P = \ddot{\boldsymbol{r}}_P = \ddot{X}_P\hat{\boldsymbol{n}}_1 + \ddot{Y}_P\hat{\boldsymbol{n}}_2 + \ddot{Z}_P\hat{\boldsymbol{n}}_3. \tag{1.24}$$

The vectors $\hat{\boldsymbol{n}}_1 = (1,0,0)^{\mathrm{T}}$, $\hat{\boldsymbol{n}}_2 = (0,1,0)^{\mathrm{T}}$, $\hat{\boldsymbol{n}}_3 = (0,0,1)^{\mathrm{T}}$ are unit vectors in the direction of respectively the inertial X-, Y- and Z-axes. These vectors are constant, their derivatives with respect to time are zero vectors. The scalars X_P, Y_P and Z_P are *coordinates* of P in XYZ.

Velocity and *speed* are not synonymous. Velocity is a vector with magnitude and direction. Speed is the (signed) magnitude of velocity.

◯ **Problem 1.11** Calculate the velocity vector of the center of mass of a system of particles as a function of the particles' individual velocity vectors.[a]

[a] *Hint*: Do this by taking the derivative of (1.11) with respect to time.

◎ **Problem 1.12** A particle's movement is described by its Cartesian position vector as a function of time t:

$$\boldsymbol{r}(t) = \begin{pmatrix} \sin(3t/\mathrm{s}) & 3\cos(2t/\mathrm{s}) & \sin^2(2t/\mathrm{s}) \end{pmatrix}^{\mathrm{T}} \mathrm{m}. \tag{1.25P}$$

Calculate the particle's acceleration $\ddot{\boldsymbol{r}}(t)$ as a function of time. Use the product and the chain rule, as described in Appendix B.1. Note that you can simplify your result using a trigonometric addition formula. ▷

1.3.3 Signs

Clear definition of signs is paramount in obtaining reliable results. Frequently, students believe that a sign mistake is a minor mistake, or that it can be fixed later on by intuition, when the final result appears to be in the wrong direction. Both assumptions are utterly untrue and dangerous for many real-world problems. Therefore, we will spend some time to revise rules for how "positive" and "negative" are determined.

For drawing vectors and labeling them, we already covered some principles in 1.1.3. These are particularly relevant for FBDs.

◯ **Example 1.11** The cover of this book shows (almost) complete FBDs of the rear frame and of the handlebar assembly of a bicycle with a special steer-assist system. The action-reaction forces and moments that are drawn *between* the two parts illustrate the difference between vector labels and scalar labels on arrows in terms of signs, as explained in Section 1.1.3. For example, while the reaction to force $\boldsymbol{F}_C$ must be $-\boldsymbol{F}_C$, with a minus sign, the reaction to the motor moment T_m is drawn in the opposite direction and also labeled with a scalar T_m.

For components of position, velocity and acceleration vectors, special care must be taken to keep definitions consistent, such that indeed $\boldsymbol{v}_P = \dot{\boldsymbol{r}}_P$ and $\boldsymbol{a}_P = \dot{\boldsymbol{v}}_P$, following (1.23) and (1.24).

Sign mistakes can be avoided by following these rules:

1. *Variable* distances or angles are defined by a line or arc with a single arrowhead. The arrowhead ends where the point or line segment is in a generic state. The arrow starts at a clearly defined reference, for example a resting position for a distance, or a vertical or horizontal line for an angle.

2. *Constant* distances or angles are generally measured by lines or arcs, respectively, with arrowheads on *both* sides. This is in fact not a strict rule but a guideline. There are some exceptions, for example when indicating a radius, following rules of technical drawing. In such cases, only one arrowhead is drawn.
3. Axis names should not be used to also denote dimensions. A new name should be chosen for each specific dimension. For example, the X-coordinate of point P should not be called X as well (like the axis direction), a better name is X_P.
4. Coordinates, so elements of a position vector, are defined positive in positive axis direction, so if an arrow is to be labeled with a positive coordinate, the arrowhead must be pointing in positive axis direction. In case a point P is drawn at a location having a negative X-coordinate, we still follow rule 1, which means that the label of the arrow includes a negative sign, as in $-X_P$.
5. Variables that are *not* coordinates or derivatives of already defined variables may be defined positive in any direction[2]. This includes angles.

Redundant measures are allowed, for example to measure the position of a point with respect to two different references. However, one should be aware of such redundancy to not miscount the number of unknowns and equations for a given problem.

Example 1.12 In the Kenyan savannah, a lion (simplified as point L) and a rhinoceros (point R) are approaching a waterhole at a speed v_L and v_R, respectively, as depicted in Figure 1.14. An antelope (point A) is simultaneously running away from the waterhole at speed v_A.

The three animals have the respective distances d_L, d_R, and d_A from the waterhole. They are not changing directions but running in straight lines. All the above-mentioned distances and speeds are positive in the given situation.

For now, we focus only on the lion and want to express its position and velocity vectors as functions of the above variables.

Exemplary solution

We start by defining an inertial coordinate system. A convenient origin is the center O of the waterhole. As directions, we choose the X-axis pointing East, and the Y-axis pointing North. So, the right-hand rule dictates that the Z-axis points vertically up.

[2] We will see later that variables that describe kinematic configuration without being Cartesian coordinates can still be interpreted as *generalized coordinates*.

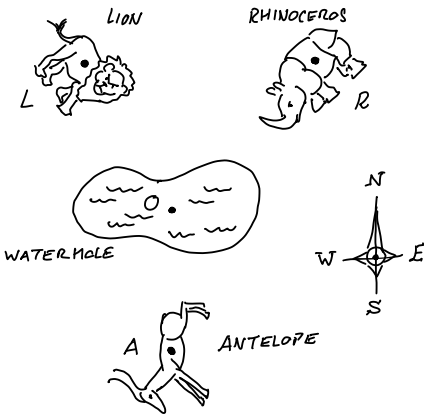


Figure 1.14: A lion, a rhinoceros, and an antelope at a waterhole.

In the first drawing of Figure 1.15, we label the distance d_L, using an arrow with its tail connected to the fixed reference O, and its arrowhead ending at the variable location of point L, the lion, following rule 1 above.

The arrowhead for the angle β follows the same rule. We choose to measure the angle from the horizontal in such a way that clockwise movement of the lion leads to an increase in β. This choice of direction is allowed following rule 5. Even though the animal does not change direction, there is no constraint that keeps it from doing so. So, we choose not to follow rule 2, but to label the angle also with only a singe arrowhead. This also enables later extension of the problem to more generic movement with non-constant angle.

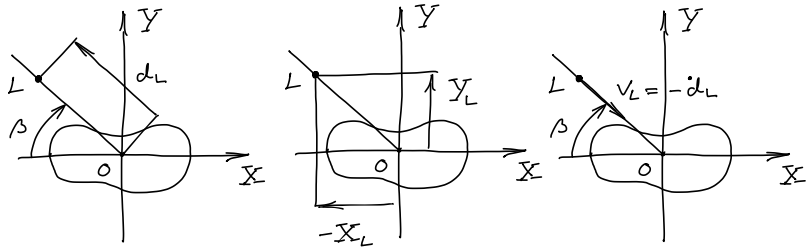


Figure 1.15: Describing kinematics of the lion. Left: Location in terms of distance and angle. Center: Cartesian coordinates. Right: Velocity

In the second drawing, we label the coordinates of the lion, X_L and Y_L, following rule 3. Since the lion is on the left of the origin, while X points right, the arrow has to be labeled with a negative sign, following rule 4.

The third drawing shows the lion's velocity. By definition, the derivative of a variable is positive if that variable increases (See Appendix B.1). There-

fore, a vector labeled with $\dot{d}_L$ needs to point outward. However, the lion's speed v_L was defined positive for inward movement, so an arrow labeled with this variable needs to be drawn pointing inward.

The position and velocity vectors can be populated with the correct components. For position, we obtain the alternative expressions:

$$\boldsymbol{r}_L = \begin{pmatrix} X_L \\ Y_L \\ Z_L \end{pmatrix} = \begin{pmatrix} -d_L \cos\beta \\ d_L \sin\beta \\ 0 \end{pmatrix} \tag{1.26E}$$

For velocity, there are various alternative expressions, which all need to be consistent regarding their signs:

$$\boldsymbol{v}_L = \begin{pmatrix} v_L \cos\beta \\ -v_L \sin\beta \\ 0 \end{pmatrix} = \dot{\boldsymbol{r}}_L = \begin{pmatrix} \dot{X}_L \\ \dot{Y}_L \\ \dot{Z}_L \end{pmatrix} = \begin{pmatrix} -\dot{d}_L \cos\beta + d_L \cancelto{0}{\dot{\beta}} \sin\beta \\ \dot{d}_L \sin\beta + d_L \cancelto{0}{\dot{\beta}} \cos\beta \\ 0 \end{pmatrix} \tag{1.27E}$$

The drawing in Figure 1.15 also shows that $v_L = -\dot{d}_L$, so this is consistent.

Problem 1.13 Describing the movement of the rhinoceros and the antelope in Example 1.12 requires careful handling of signs.

First, complete the three drawings in Figure 1.15 with the equivalent variables describing the kinematics of these two animals. Use the angle α for the rhino. There is no need to consider an angle for the antelope.

Then, replace the $\overset{?}{\pm}$ in the below equations by a "+" or a "−"-sign:

The rhino's position vector is:

$$\boldsymbol{r}_R = \begin{pmatrix} \overset{?}{\pm}\, X_R \\ \overset{?}{\pm}\, Y_R \\ \overset{?}{\pm}\, Z_R \end{pmatrix} = \begin{pmatrix} \overset{?}{\pm}\, d_R \cos\alpha \\ \overset{?}{\pm}\, d_R \sin\alpha \\ 0 \end{pmatrix} \tag{1.28P}$$

The rhino's velocity vector is:

$$\boldsymbol{v}_R = \begin{pmatrix} \overset{?}{\pm}\, v_R \cos\alpha \\ \overset{?}{\pm}\, v_R \sin\alpha \\ 0 \end{pmatrix} = \overset{?}{\pm}\, \dot{\boldsymbol{r}}_R = \begin{pmatrix} \overset{?}{\pm}\, \dot{X}_R \\ \overset{?}{\pm}\, \dot{Y}_R \\ \overset{?}{\pm}\, \dot{Z}_R \end{pmatrix} = \begin{pmatrix} \overset{?}{\pm}\, \dot{d}_R \cos\alpha \overset{?}{\pm} d_R \cancelto{0}{\dot{\alpha}} \sin\alpha \\ \overset{?}{\pm}\, \dot{d}_R \sin\alpha \overset{?}{\pm} d_R \cancelto{0}{\dot{\alpha}} \cos\alpha \\ 0 \end{pmatrix} \tag{1.29P}$$

The antelope's position vector is:

$$\boldsymbol{r}_A = \begin{pmatrix} \overset{?}{\pm} X_A \\ \overset{?}{\pm} Y_A \\ \overset{?}{\pm} Z_A \end{pmatrix} = \begin{pmatrix} 0 \\ \overset{?}{\pm} d_A \\ 0 \end{pmatrix} \tag{1.30P}$$

The antelope's velocity vector is:

$$\boldsymbol{v}_A = \begin{pmatrix} 0 \\ \overset{?}{\pm} v_A \\ 0 \end{pmatrix} = \overset{?}{\pm}\, \dot{\boldsymbol{r}}_A = \begin{pmatrix} \overset{?}{\pm} \dot{X}_A \\ \overset{?}{\pm} \dot{Y}_A \\ \overset{?}{\pm} \dot{Z}_A \end{pmatrix} = \begin{pmatrix} 0 \\ \overset{?}{\pm} \dot{d}_A \\ 0 \end{pmatrix} \tag{1.31P}$$

▶

◯ **Problem 1.14** For all coordinates $X_L, X_R, X_A, Y_L, Y_R, Y_A$ and their derivatives in Example 1.12 and Problem 1.13, indicate whether their values are positive, zero, or negative in the described situation. ▷

◯ **Problem 1.15** The animals of Example 1.12 and Problem 1.13 are not running at constant speed. The lion and the rhino are decelerating, so reducing their speed, at a rate of a_L and a_R, respectively. The antelope is accelerating at a rate of a_A. The three rates are all defined positive, so $\{a_L, a_R, a_A\} > 0$.

First, draw the acceleration with the above labels into Figure 1.15.

Second, write down the acceleration vector components as functions of a_L, a_R, a_A. Also specify for all variables $\dot{v}_L$, $\dot{v}_R$, $\dot{v}_A$, $\ddot{X}_L$, $\ddot{X}_R$, $\ddot{X}_A$, $\ddot{Y}_L$, $\ddot{Y}_R$, $\ddot{Y}_A$ whether they currently have positive, zero, or negative values.

1.3.4 Newton's Second Law of Motion

Newton's second law of motion [49] is specifically relevant to the field of dynamics, forming its fundament:

2. *"The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed."*

This means that when a resultant force acts upon an object, it will evoke a change of the object's velocity vector that is proportional in magnitude to the force and that also occurs in the direction in which the force acts. The constant of proportionality is defined as the mass m of the object. So, Newton's second law resolves to the known equation:

$$\sum \boldsymbol{F} = m\dot{\boldsymbol{v}} = m\boldsymbol{a}. \tag{1.32}$$

Departing from the object being a single particle, we will show in the next chapter that the formula also holds for a system of particles or rigid bodies, where $\boldsymbol{a}$ is the acceleration of the CoM of the system.

1.4 Solving Problems in Mechanics

1.4.1 Question Answering Strategy

For most problems in mechanics, the recommended strategy is to:

- start the answer to each question by defining the strategy, using only symbolic equations. Only fill in details (such as vector components or even numerical values) when really needed to compute/simplify the answer.
- start at the end: With equations that contain the variable *asked for*. Looking at those equations, and identifying which terms are known and unknown, typically works better than trying to find equations that contain parameters *given* in the problem and then hoping to reach the correct answer.
- draw a FBD, even if it is not directly asked for. It will help to precisely understand the question.
- define *all* variables not mentioned in the text explicitly by a figure or algebraically, in order to be fully aware of their meaning.
- include units in all calculations with numerical values.

1.4.2 Checking Plausibility

In the end or during the calculations one needs to check if the obtained (intermediate) results are correct. Some specific ideas for checking plausibility and identifying inconsistencies are to:

- use units as a quick check: if units on both sides of an equal sign do not match, the equation is incorrect.
- use clearly separate notations for vectors as a quick sanity check: if a calculation involves division by a vector, or taking the cross product of two scalars, it is incorrect.
- do the same question with a *different coordinate system*.
- check *special cases* where the answer is easier to find, for example when some values or derivatives are 0. An intuitive special case for dynamic systems is often static equilibrium.

- solve the problem *numerically*, e.g. via MATLAB^{TM}, and compare results.
- check vector *directions* independent of magnitudes
- check *signs*.
- differentiate a previously integrated term again.
- check dimensions of a variable, e.g. are vectors and scalars added?
- check different approaches (e.g. to determine the equations of motion, this book introduces four different approaches).
- compare to peers and discuss.

Example 1.13 This Example is about the ring on a rotating bar of Example 1.9 on page 18. A student is uncertain about the correctness of the equation of motion she derived in the body-fixed u-direction:

$$F_u = -m(\ddot{d}\sin\psi + 2\dot{d}\dot{\psi}\sin\psi + d\ddot{\psi}\sin\psi + d\dot{\psi}^2\cos\psi), \tag{1.33E}$$

where F_u is the sum of all forces acting on the ring in u-direction.

Check the plausibility of this equation with three different, *independent* plausibility checks. Base one of the checks on FBD of the ring. For each check, first state a precise condition you expect to hold for the equation, and a reasoning why you expect this to hold. Only then, perform the check.

Exemplary solution

- A first check can be done on units.
  Prediction and Reasoning: Units must be identical for any two terms that are added or equated, so all terms in the equation must have the same unit (Note that sine or cosine of an angle have no units.):

  $$[F_u] = [m]\cdot[\ddot{d}] = [m]\cdot[\dot{d}]\cdot[\dot{\psi}] = [m]\cdot[d]\cdot[\ddot{\psi}] = [m]\cdot[d]\cdot[\dot{\psi}^2]. \tag{1.34E}$$

  Execution and Conclusion: It is easiest to check with SI units,

  $$[F_u] = \mathrm{N}, [m] = \mathrm{kg}, [d] = \mathrm{m}, [\dot{d}] = \mathrm{m/s}, [\ddot{d}] = \mathrm{m/s^2},$$
  $$[\dot{\psi}] = \mathrm{rad/s} = \mathrm{s^{-1}}, [\ddot{\psi}] = \mathrm{s^{-2}}, [\dot{\psi}^2] = \mathrm{s^{-2}}.$$

  Substituting these units, the left side of (1.34E) is a force with unit N, and all terms on the right side of the equation have the same unit, which also resolves to N (as $\mathrm{N} = \mathrm{kg\,m/s^2}$).
  So, this is plausible.
- A second check can be done for the special case of static equilibrium:
  Prediction and Reasoning: Special case: In static equilibrium, so if all derivatives are zero ($\ddot{d} = 0, \dot{d} = 0, \dot{\psi} = 0, \ddot{\psi} = 0$), the sum of forces (so also the component F_u) acting on the ring must be zero.

Execution and Conclusion: Substituting zeros for all derivatives gives

$$F_u = -0 \cdot \sin\psi - 2 \cdot 0 \cdot 0 \cdot \sin\psi - d \cdot 0 \cdot \sin\psi - d \cdot 0^2 \cdot \sin\psi = 0. \quad (1.35E)$$

This fulfills the prediction and is therefore plausible.

- A third check can be done based on the FBD:
  Prediction and Reasoning: The FBD shows that the sum of forces in u-direction must always be zero, because there is no friction: $F_u = 0$. This must hold for any physically plausible kinematic parameters.
  Execution and Conclusion: A possible combination could be $\psi = 0$, $d > 0$, and $\dot{\psi} > 0$. We do not even need to think about what feasible accelerations and velocities in this case could be, because they do not appear anymore once we substitute $\psi = 0$ in (1.33E):

$$F_u = -md\dot{\psi}^2 < 0. \quad (1.36E)$$

This contradicts the prediction, so (1.33E) must be incorrect.

1.4.3 Frequent Mistakes

Some mistakes are particularly frequent, so it is likely that a calculation failed a plausibility check because:

- Coordinate system were not clearly defined, and vectors expressed in different coordinate systems mixed incorrectly. To remedy, it is important to define all used coordinate systems (origin and axis directions) and to *draw them* in a general state.
- Formulae were applied without checking their underlying assumptions. The first step is awareness that many formulae are true for special cases only, and the second is to thoroughly check whether the given problem really falls into that special case.
- Variables were implicitly (and inconsistently) defined. Variable definitions should be made explict.
- Vectors and scalars were not cleanly separated. To remedy, clear notations and typesetting often help.
- Units were omitted throughout calculations, and only later regenerated based on prior assumptions on units of the resulting variable. This hinders the use of units to spot mistakes.
- scalar equations from 2-D dynamics were used, thereby omitting terms that only exist in 3-D.
- out-of-plane reaction forces and moments were omitted in FBDs.

1.5 Summary

The importance of making and adhering to proper definitions, particularly of signs, can hardly be overestimated. Lack of explicit definitions is one of the most common sources of uncertainty and mistakes.

Free-body diagrams are the first step to solving most problems in mechanics. An FBD must 1) be in a generic state, and include 2) a coordinate system, 3) all external forces and moments, 4) unique labels for these, and 5) dimensions needed to calculate moments.

When drawing FBDs, actual movement of the system must *not* be considered. The focus should instead be on representing any cut kinematic constraints of the system via forces and moments.

If the line of action passes of a force $\boldsymbol{F}$ passes through point A, the moment with respect to point P is:

$$\boldsymbol{M}_P := \boldsymbol{r}_{A/P} \times \boldsymbol{F}, \tag{1.37}$$

With this and the principle of equipollence, equivalent force/moment systems acting on a rigid body can be calculated.

The center of mass with respect to point A of a system of mass m, which could be a system of N particles or a rigid body, is defined as

$$\boldsymbol{r}_{C/A} := \frac{\sum_{i=1}^{N} m_i \boldsymbol{r}_{i/A}}{\sum_{i=1}^{N} m_i} \text{ , or } \boldsymbol{r}_{C/A} := \frac{\int_m \boldsymbol{r}_{P/A}\, \mathrm{d}m}{\int_m \mathrm{d}m} \text{ , respectively.} \tag{1.38}$$

Newton's First Law states that a particle that moves at constant linear velocity $\boldsymbol{v}$ will keep the same velocity unless a force $\boldsymbol{F}$ acts on it.

Newton's Third Law is the law of action and reaction: If an object a exerts a force on another object b, then b exerts an equal and opposite force on a.

Kinematics is the domain of dynamics that describes motion. Velocity and acceleration of a point P are defined as derivatives of the position vector with respect to time:

$$\boldsymbol{v}_P := \dot{\boldsymbol{r}}_P\,, \quad \boldsymbol{a}_P := \dot{\boldsymbol{v}}_P = \ddot{\boldsymbol{r}}_P. \tag{1.39}$$

In the special case where the position vector components are coordinates in an inertial Cartesian coordinate system, the derivatives can be calculated by taking derivatives of the components.

1.6 Problems

1.6.1 Guided Problems

Problem 1.16 A Tippe Top [68] (Figure 1.16) is a special kind of top [12]. When it is spun on its spherical end under the influence of gravity with field strength g, it will invert itself and will continue to spin on the small peg.

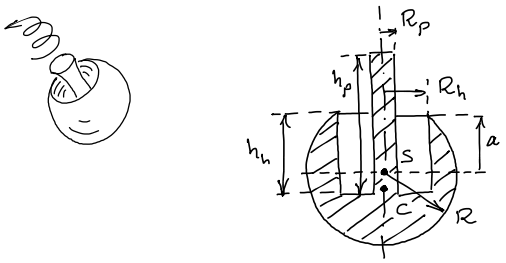


Figure 1.16: Tippe Top.

The top is modeled as shown in the section view. The spherical part has radius R and has a cap cut off at a from the sphere center. The cylindrical hole has radius R_h and height h_h, and the peg has radius R_p and height h_p. The complete top has mass M.

a. Calculate the position vector $\boldsymbol{r}_{K/C}$ from the center of mass C to the contact point K between ground and top as a function of the position vectors $\boldsymbol{r}_{C/S}$ and $\boldsymbol{r}_{S/K}$. Make a drawing in a suitable 2-D-projection that shows the three vectors and the three points in a generic configuration of the top.
b. Draw a free-body diagram (FBD) of the top. Do this in a generic state, and choose a projection that allows you to draw all vector components in the plane or orthogonal to it. Use only scalar labels. Choose a rotating coordinate system xyz to define your projection. Let the coordinate axis directions define a triad $\mathcal{F}$. Clearly define the origin of your coordinate system, and state whether it translates or not, for general motion of the top. Label all relevant points and dimensions.
c. Use equipollence and the cross product to derive an equivalent system where all forces and moments act in the center of mass of the top. Calculate the components of the resultant force and moment vectors as functions of the given constants M, g, R, the distance d between the center S of the sphere and the center of mass C of the top, and the variables you defined in your own FBD. Use your chosen coordinate system, so express all vector components in $\mathcal{F}$.

d. Defining that a solid sphere with a cap of height $(R-a)$ cut off has a remaining volume of V_s, and that its center of mass is at a distance b below the sphere's center, calculate the distance d between the center S of the sphere and the center of mass C of the tippe top as a function of V_s, b, h_h, h_p, R_h, R_p, a and R.

e. For the volume V_s of a solid sphere that has a cap of height $(R-a)$ cut off, one can use the formula

$$V_\mathrm{s} = \tfrac{1}{3}\pi\,(R+a)^2\,(2R-a)\,. \tag{1.40P}$$

Check (1.40P) for plausiblity using at least three different checks, without re-calculating the equation yourself yet.

f. In an unknown triad $\mathcal{G}$, someone calculated three vectors: The gravitational acceleration ${}^{\mathcal{G}}\boldsymbol{g} = (0,\,-g,\,0)^\mathrm{T}$, the contact total friction force ${}^{\mathcal{G}}\boldsymbol{F}_\mathrm{f} = (F_{\mathrm{f}x},\,0,\,F_{\mathrm{f}z})^\mathrm{T}$ acting on the top, and the contact normal force ${}^{\mathcal{G}}\boldsymbol{F}_\mathrm{n} = (0,\,F_{\mathrm{n}y},\,0)^\mathrm{T}$. Show mathematically that the gravitational acceleration vector is parallel to the normal force vector. Then show that the friction force is perpendicular to the normal force.

g. In dynamics, we often encounter Greek symbols. Someone labels the angle between the peg axis and the vertical θ (so-called *nutation*), the *spin* angle around the peg axis with ψ, and the rotation angle about the vertical axis (so-called *precession*) with φ. Furthermore, an angular speed is labeled with ω. These letters will be used especially in Chapter 3 and Chapter 4.
How are these Greek lowercase letters pronounced and what are the uppercase symbols? What is the alternative symbol for φ?

h. Figure 1.17 shows the three angles in three different *orthographic projections* of the tippe top into two-dimensional planes. Four (incompletely drawn) moving coordinate systems are defined: XYZ, $x'y'z'$, x'', y'', z'', and x''', y''', z'''. For each of the projections, apply the right-hand rule to draw all missing axes of the coordinate system $x''y''z''$ that are contained within the drawn plane or orthogonal to it.

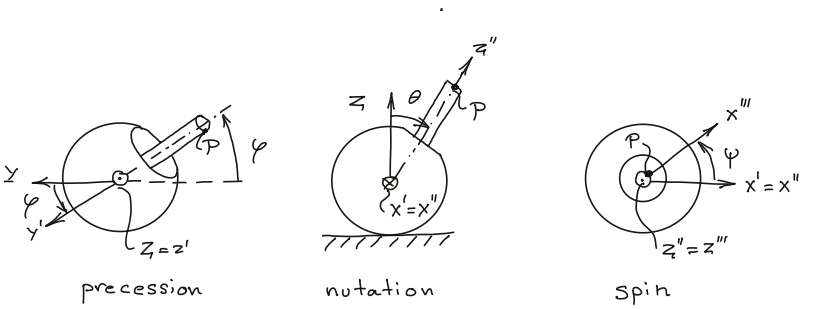


Figure 1.17: Tippe Top projections.

i. A student applies the Newton-Euler method (a technique to be treated in Chapter 6) and derives the equations of motion of the top. One of these equations results to be

$$Md(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta - \ddot{\varphi}\sin\theta - 2\dot{\varphi}\dot{\theta}\cos\theta) = N - Mg, \qquad (1.41\text{P})$$

with variables as defined before, gravitational acceleration g, and N the normal force exerted on the top by the ground in upward direction. The student wants to verify the result. Conduct at least three different, independent plausibility checks on this equation.

j. Calculate the volume V_s via integration, and compare your result to (1.40P). Make one or multiple drawings where you include and label *all* variables used in your calculations (including those that are infinitesimally small).

k. Use integration to calculate the distance b as a function of R and a. Make again drawings to define all variables.

Problem 1.17 A force $\boldsymbol{F}$ applied at point P causes a moment $\boldsymbol{M}_O = \boldsymbol{r}_{P/O} \times \boldsymbol{F}$ about point O with position vector $\boldsymbol{r}_{P/O}$ (see Figure 1.18). The vectors have components

$$\boldsymbol{M}_O = \begin{pmatrix} M_X \\ M_Y \\ M_Z \end{pmatrix}, \qquad \boldsymbol{r}_{P/O} = \begin{pmatrix} X_P \\ Y_P \\ Z_P \end{pmatrix}, \qquad \boldsymbol{F} = \begin{pmatrix} F_X \\ F_Y \\ F_Z \end{pmatrix}. \qquad (1.42\text{P})$$

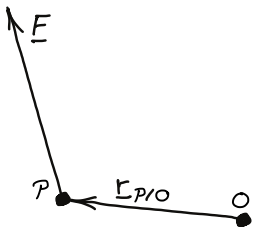


Figure 1.18: Illustration of a force and position vector of its point of application.

Consider two cases:

- Case I: $\boldsymbol{r}_{P/O}$ known, $\boldsymbol{F}$ unknown,
- Case II: $\boldsymbol{r}_{P/O}$ unknown, $\boldsymbol{F}$ known.

a. Show that for both cases we cannot determine the unknown vector unambiguously. Do this by rewriting the cross product to a representation with the tilde matrix and try to solve for the respective unknown vector components. Alternatively, directly use the rank of the tilde matrix to come to a conclusion.

b. For a unique solution, we choose the constraint that the force is perpendicular to the position vector, $\boldsymbol{r}_{P/O} \perp \boldsymbol{F}$. Describe this constraint by a vector equation involving $\boldsymbol{r}_{P/O}$ and $\boldsymbol{F}$.
c. Combine the moment equation with the constraint from part b and rewrite the equations to the form $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$. Do this for both cases, such that $\boldsymbol{x}$ is $\boldsymbol{r}_{P/O}$ or $\boldsymbol{F}$, respectively [a].
d. Now the numerical values of the moment are known:

$$\boldsymbol{M}_O = \begin{pmatrix} 6 & 1 & -2 \end{pmatrix}^{\mathrm{T}} \mathrm{N\,m}. \tag{1.43P}$$

Also one of the other vectors' values are known:

- Case I: $\boldsymbol{r}_{P/O} = (1,\, 2,\, 4)^{\mathrm{T}}\,\mathrm{m}$,
- Case II: $\boldsymbol{F} = (-2,\, 6,\, -3)^{\mathrm{T}}\,\mathrm{N}$.

Substitute the values in the expression from part c and solve for the unknown vector to find the numerical values[b].

[a] *Hint*: Matrix $\mathbf{A}$ should contain (among others) a tilde matrix.
[b] *Hint*: Use MATLAB™'s \-operator.

Problem 1.18 A camera of mass m is suspended on three cables, as shown in Figure 1.19.

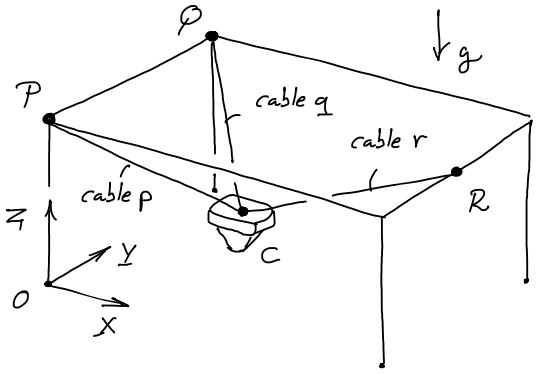


Figure 1.19: Camera suspension system

The position of the camera with respect to the origin O of the XYZ coordinate system is given as $\boldsymbol{r}_{C/O} = \boldsymbol{r}_C = (\frac{l}{3},\, \frac{w}{3},\, \frac{h}{2})^{\mathrm{T}}$, in the indicated XYZ coordinate system, with constant parameters l, w, and h describing the dimensions of the suspending frame. The positions of the endpoints of cables p, q and r with respect tot O are given as $\boldsymbol{r}_P = (0,\, 0,\, h)^{\mathrm{T}}$, $\boldsymbol{r}_Q = (0,\, w,\, h)^{\mathrm{T}}$ and $\boldsymbol{r}_R = (l,\, \frac{w}{2},\, h)^{\mathrm{T}}$, respectively. Gravity, with constant g, acts in the negative Z direction. Simplify the camera as a particle, and at rest.

We want to determine the force in cable r. To increase reliability of the

result, we use two different methods. The first is straightforward but requires more computational effort, the second is more elegant and costs less effort, but requires more insight.

a. Draw the free-body diagram of the camera.
Method 1
b. Use vector addition to calculate the vectors $\boldsymbol{r}_{P/C}$, $\boldsymbol{r}_{R/C}$, and $\boldsymbol{r}_{Q/C}$, with components expressed as functions of l, w, and h.
c. Calculate the length of cable p as a function of l, w, and h.
d. Calculate the components of the direction vector $\hat{\boldsymbol{u}}_{\mathrm{r}}$ in direction of cable r, as functions of the given constants, and such that the vector has a positive Z component. Do the same for the two other cables.
e. Express the force vector $\boldsymbol{F}_{\mathrm{r}}$ that cable r exerts on point C as a function of $\hat{\boldsymbol{u}}_{\mathrm{r}}$ and the (yet unknown) magnitude $F_{\mathrm{r}} = |\boldsymbol{F}_{\mathrm{r}}|$ of the force in cable r. Do the same for the two other cables.
f. Express the resultant force vector $\Sigma\boldsymbol{F}_{\mathrm{c}}$ acting on the camera as a function of the cable force magnitudes $F_{\mathrm{p}} = |\boldsymbol{F}_{\mathrm{p}}|$, $F_{\mathrm{q}} = |\boldsymbol{F}_{\mathrm{q}}|$, $F_{\mathrm{r}} = |\boldsymbol{F}_{\mathrm{r}}|$, and the constants w, h, l, m and g.
g. Rewrite the vector equation $\Sigma\boldsymbol{F}_{\mathrm{c}} = \boldsymbol{0}$ for static equilibrium such that it takes the form $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, with $\boldsymbol{x} = (F_{\mathrm{p}}, F_{\mathrm{q}}, F_{\mathrm{r}})^{\mathrm{T}}$.
h. Solve $\Sigma\boldsymbol{F}_{\mathrm{c}}$ to determine F_{p}, F_{q}, and F_{r} as functions of l, w, h, m, g. The MATLAB$^{\mathrm{TM}}$ symbolic toolbox can simplify calculations.
Method 2
i. Calculate a unit vector $\hat{\boldsymbol{e}}$ that is orthogonal to cables p and q, as a function of the given constants l, w, h.
j. Determine the projection ΣF_{ce} of $\Sigma\boldsymbol{F}_C$ in direction of $\hat{\boldsymbol{e}}$ as a function of F_{p}, F_{q}, F_{r} and of the given constants. Explain why two of the unknowns do not appear in the function.
k. Use $\Sigma F_{ce} = 0$ to determine F_{r} as a function of l, w, h, m, g.
l. Given the two methods above, how would you generally choose coordinate directions to minimize calculation effort if some force vectors are irrelevant?

1.6.2 Practice Problems

Problem 1.19 This Problem is about the top of corks and skewers of Problem 1.8 on page 12. Calculate the vector $\boldsymbol{r}_{A/P}$ as a function of the vector $\boldsymbol{r}_{C/P}$ and the two scalars l and d. ▷

Problem 1.20 Three particles form a T-shaped rigid body floating in space. The particles have mass m_1, m_2, and m_3, respectively, and are connected by two massless rods (Figure 1.20). There is no gravity. A body-fixed xyz coordinate system is drawn where the x-direction is aligned with the long rod of length L, and the y-direction is aligned with the short rod of

length $L/2$. The origin of xyz is in the center of mass of the body. The axis directions define a body-fixed triad $\mathcal{B}$.

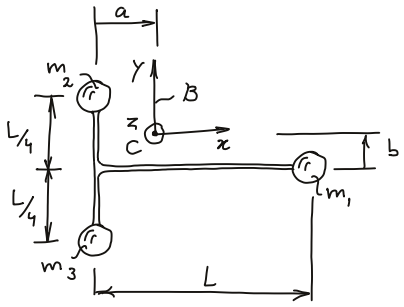


Figure 1.20: T-shaped rigid body consisting of three particles.

Calculate the distances a and b that define the location of the body's center of mass C, as functions of the given geometry and the particles' masses. ▷

Problem 1.21 Which statement(s) make(s) correct use of units for mass m, speed v, time t, and length h? Check all that apply.

A. $m =$kg
B. $m =$[kg]
C. $v = 20h\,\mathrm{m/s}$
D. $\cos t = 1$
E. None of the above

▷

Problem 1.22 The thin-walled cylindrical rotor in Figure 1.21, of mass m and radius R, is in *static unbalance*. This means that the rotor's center of mass D is not on the x-axis. Static unbalance can cause undesired vibrations when the rotor is in motion.

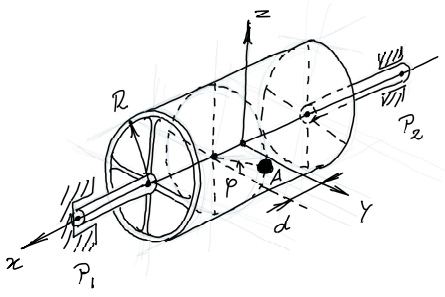


Figure 1.21: Unbalanced rotor.

A weight of mass m_a is to be added to the rotor at a strategic place on the shell of the rotor A, described by the angle φ and the distance d. The weight should compensate for static unbalance, such that the center of mass of the combined body (rotor plus mass m_a), with position vector $\boldsymbol{r}_C$, is on the x-axis. This is done so that the bearings P_1 and P_2 do not have to support any net force due to rotation of the rotor.

Point D has the position vector $\boldsymbol{r}_\mathrm{D} = R \cdot (\frac{1}{40}, \frac{1}{50}, \frac{1}{50})^\mathrm{T}$ in the rotor-fixed xyz. The weight is considered a particle that is rigidly attached to the rotor.

Determine m_a and *one possible* location A to compensate for static unbalance of the rotor. Provide m_a, φ, and d as functions of R and m. ▶

Problem 1.23 For all bodies shown below, free-body diagrams are to be drawn. Choose one or multiple projections that allow you to draw components of vectors either contained in or orthogonal to the paper plane, such that there is no ambiguity in direction. Also, use only scalar labels, so split vectors into multiple components when needed.

a. Draw the FBD of a spacecraft (Figure 1.22) of mass m floating in outer space.

Figure 1.22: Spacecraft.

b. Three gimbal rings are mounted with friction-free rotational joints (Figure 1.23). An external moment about the vertical axis is applied on the outer gimbal ring. Draw the FBD of the *middle* gimbal ring.

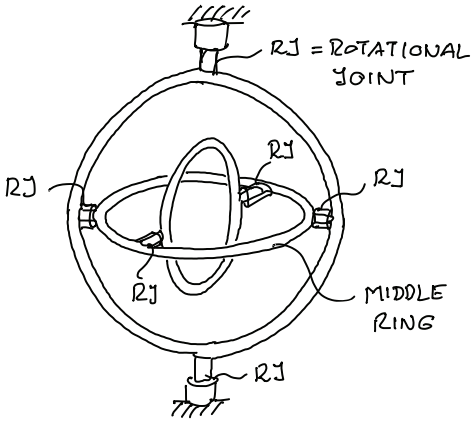


Figure 1.23: Gimbal rings.

c. A grasper is opened and empty at the moment as drawn in Figure 1.24. Each of the grasper's beaks is controlled by an electromotor on its rotational hinge. Draw the FBD of the upper beak.

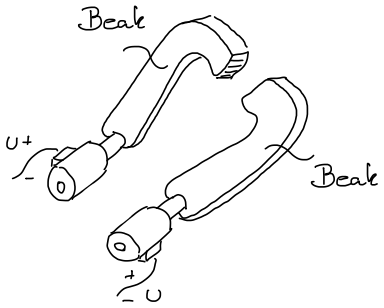


Figure 1.24: Grasper.

d. Draw the FBD of a piston as shown in Figure 1.25.

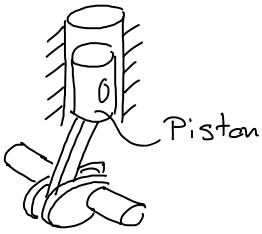


Figure 1.25: Piston.

e. A trombone is a brass musical instrument. The player can regulate the pitch by sliding a telescoping mechanism to a position (see Figure 1.26). Draw the FBD for the sliding element when the player is playing the instrument.

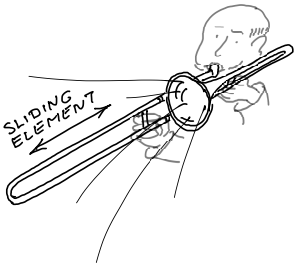


Figure 1.26: Trombone.

Problem 1.24 A person is riding a skateboard (Figure 1.27). The rider can steer the skateboard by leaning sideways over a tilt angle γ.

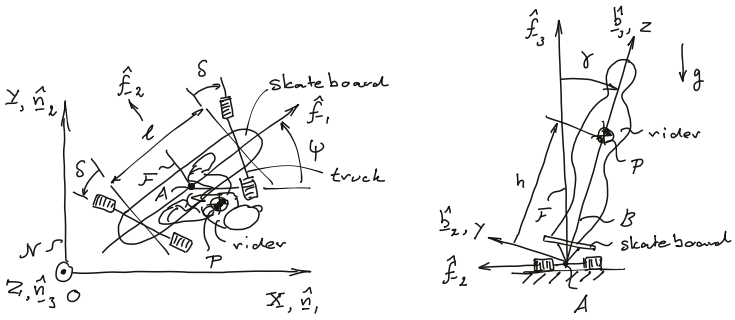


Figure 1.27: A person riding a skateboard.

The rider is modeled as rigidly attached to the board and positioned at the center of the board A.

The arrangement of the *trucks* (hinged axles), which connect the wheels to the board, are such that there is a fixed kinematic relation between the tilt angle γ of the board and the steer angle of the front and rear truck δ (Figure 1.27). For small angles, this kinematic relation can be approximated by $\delta = k\gamma$, were the constant k only depends on the fixed geometric arrangement of pivot axes of the trucks with respect to the skateboard.

The wheels roll and do not slip in either lateral or longitudinal direction[a]. The distance between the centers of the wheel axles is l. The distance between A and P is h. We neglect friction in the bearings.

A person-fixed triad $\mathcal{B}$ is defined such that $\hat{\boldsymbol{b}}_3$ points upwards from the feet to the head, $\hat{\boldsymbol{b}}_1$ points in direction of the nose (in anatomical terms, the *anterior* direction), and $\hat{\boldsymbol{b}}_2$ points sideways towards person's left.

The inertial XYZ-coordinate system has directions of $\mathcal{N}$ and origin O.

An intermediate triad $\mathcal{F}$ describes the motion of the skateboard on the plane. The unit vector $\hat{\boldsymbol{f}}_1$ points in the direction of the forward speed of the board, and $\hat{\boldsymbol{f}}_3$ is aligned with the inertial triad vertical $\hat{\boldsymbol{n}}_3$.

Gravity with acceleration constant g acts in negative Z-direction. The ground is flat.

Draw a FBD of the skateboard including the rigid rider in a top view, like the left picture in Figure 1.27.

Use only scalar labels for your vectors. Also include all dimensions that would be needed to calculate relevant moments for Euler's second law. Do not actually calculate the moments. ▶

[a] This is not fully realistic; to enable turning with wheels of finite width, there has to be local slip

◯ **Problem 1.25** This Problem is about the skateboarder of Problem 1.24 on page 39. Which statement(s) is/are always true about the (relative) po-

sition vectors $\boldsymbol{r}_{P/A}, \boldsymbol{r}_{P/O}, \boldsymbol{r}_{A/O}$? Check all that apply.

A. $\boldsymbol{r}_{P/A} = \boldsymbol{r}_{P/O} - \boldsymbol{r}_{A/O}$
B. $\boldsymbol{r}_{P/A} \times \boldsymbol{r}_{A/O} = \boldsymbol{0}$
C. $\boldsymbol{r}_{P/A} \cdot \boldsymbol{r}_{A/O} = 0$
D. None of the above

▷

Problem 1.26 A simple 3-axis accelerometer (Figure 1.28) consists of a mass that is elastically suspended within a cage. This cage is attached to a rotating and translating object. Acceleration of the mass is related to deformations of the tension springs, which can be used to measure the object's movements. However, the springs also deform due to gravity acting on the mass.

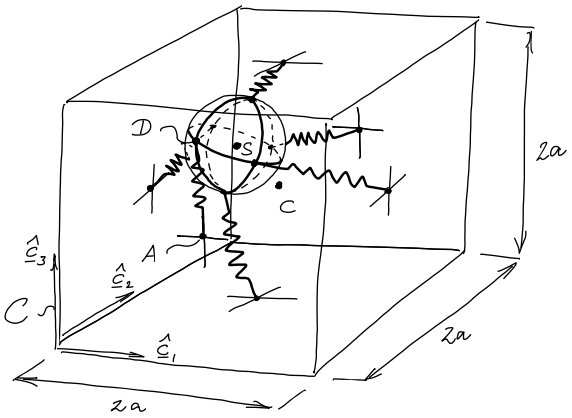


Figure 1.28: An accelerometer, consisting of a spherical mass suspended in a (transparent) cube by six springs.

We model the cage as a massless hollow cube with center C and edge length $2a$. The cage-fixed triad $\mathcal{C}$ has the direction vectors $\hat{\boldsymbol{c}}_1, \hat{\boldsymbol{c}}_2, \hat{\boldsymbol{c}}_3$.

We model the mass as a rigid homogeneous sphere of mass m, radius R, and center S. The sphere-fixed triad $\mathcal{K}$ has the direction vectors $\hat{\boldsymbol{k}}_1, \hat{\boldsymbol{k}}_2, \hat{\boldsymbol{k}}_3$. When all springs have equal length, triad $\mathcal{C}$ and triad $\mathcal{K}$ are the same.

Point O is the origin of the inertial axes X, Y, Z, which define triad $\mathcal{N}$.

Gravity with acceleration constant g acts in negative Z-direction.

Draw FBDs of the cage and of the sphere, assuming that the cage is rigidly attached to another body at its face that contains point A. Here, we consider the case where there is *no* gravity. For the FBD, you may make the simplification that the springs are very stiff, such that their directions do not change with respect to the cage. Remark: Do *not* make this simplification in later uses of this problem. Use only scalar labels for your vectors. Show only

orthographic projections, where all forces and moments are either contained in the paper plane or orthogonal to it. ▶

Problem 1.27 This Problem is about the accelerometer of Problem 1.26 on page 41. The spring between points A and D has linear characteristics, with spring constant k. When no force acts on it, the spring has length l_n, and in static equilibrium (neglecting gravity), the spring has a different length l_p, because all springs are in pre-tension.

Calculate the force vector that this spring exerts on the sphere, as a function of the vector $\boldsymbol{r}_{D/A}$ and the constants k, l_p, l_n. ▷

Problem 1.28 This Problem is about the accelerometer of Problem 1.26 on page 41. The accelerometer is mounted on a robot arm, which is subjected to the gravitational field of Earth, with acceleration constant g acting in negative Z-direction. The sensors measure that the six springs apply a resultant spring force of $\boldsymbol{F}$ on the cage.

The sphere's center has an acceleration $\boldsymbol{a}_S$ of (check all that apply):

A. $\boldsymbol{a}_S = -\boldsymbol{F}/m - g\hat{\boldsymbol{n}}_3$
B. $\boldsymbol{a}_S = -\boldsymbol{F}/m + g\hat{\boldsymbol{n}}_3$
C. $\boldsymbol{a}_S = -\boldsymbol{F}/m$
D. $\boldsymbol{a}_S = \boldsymbol{F}/m - g\hat{\boldsymbol{n}}_3$
E. $\boldsymbol{a}_S = \boldsymbol{F}/m + g\hat{\boldsymbol{n}}_3$
F. $\boldsymbol{a}_S = \boldsymbol{F}/m$
G. None of the above

▷

Problem 1.29 This Problem is about the accelerometer of Problem 1.26 on page 41. The accelerometer is simplified and used in a one-dimensional movement. It is oriented as shown in the kinematic diagram in Figure 1.29, and it does not change this special-case orientation. The coordinates Z_C, x_S, and d are defined in the figure.

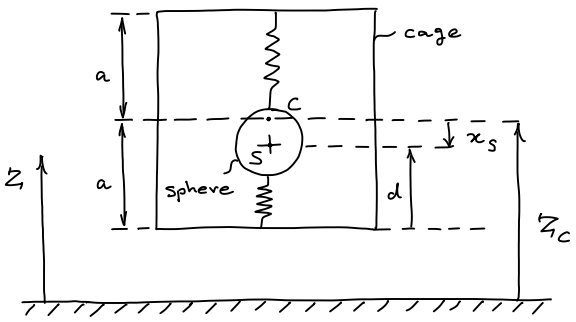


Figure 1.29: Kinematic diagram of simplified accelerometer in 1-D.

Someone already drew a FBD and determined that the sum of external forces acting on the mass in upward direction Z is ΣF_u. The following equation(s) is/are correct (Check all that apply):

A. $\Sigma F_u = m\ddot{x}_S$
B. $\Sigma F_u = m\ddot{Z}_C$
C. $\Sigma F_u = 0$
D. $\Sigma F_u = m\ddot{d}$
E. $\Sigma F_u = -m\ddot{d}$
F. None of the above

▷

Problem 1.30 Re-Do Example 1.12 with a North-East-Down (NED) coordinate system, where the X-axis points North and the Y-axis points East. Focus again on the signs and conduct plausibility checks.

Problem 1.31 A person is performing tricks on a scooter with springboard. Figure 1.30 shows the system in 3-D(left), and a detailed sideview of the scooter (right).

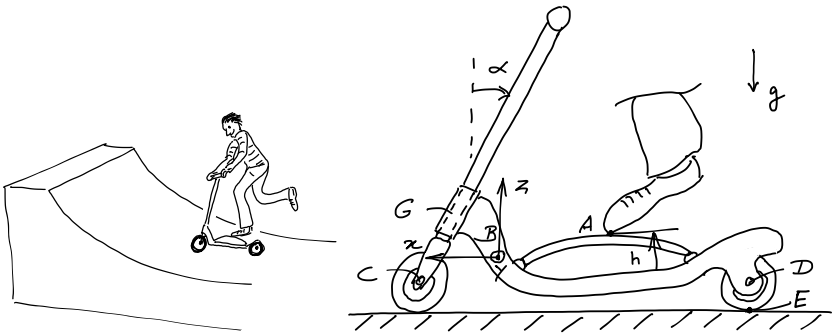


Figure 1.30: Springboard scooter.

The scooter consists of one massless leafspring and four rigid bodies that all have mass and inertia: Two wheels of equal radius and mass m_{w}, a footboard of mass m_{f}, and a steering handle. A local xyz coordinate system is connected to the footboard and has its origin in the footboard's center of mass B.

The four bodies are all connected to each other by friction-free hinges: The wheels are connected to the footboard such that they can only rotate about their respective axles C and D, which are parallel to the y-axis and pass through the centers of mass of the wheels. The steering handle is connected to the footboard by a hinge G that allows only rotation about an axis that is inclined by an angle α with respect to the z-axis.

The scooter's wheels roll on the ground without slip and have only point

contact, the rear wheel in point E. The person is touching the leafspring only with one toe, idealized as one point A. The person also holds the handle of the scooter.

Gravity acts in the downward direction with field strength g.

Draw the FBDs of two systems, namely of the rear wheel and of a system consisting of the footboard and the leafspring connected to each other. Do this by completing Figure 1.31, which already contains the outlined shapes of these two systems in a projection into the xz-plane.

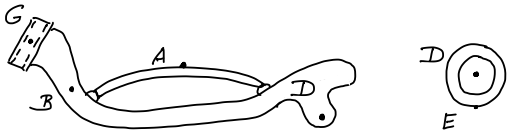


Figure 1.31: Scooter's footboard and rear wheel as free bodies.

Represent distributed loads by their resultant and split vectors into their components such that you use only scalar labels for all drawn arrows. Remember to use clear, unique labels. ▶

2 Particle Dynamics in Fixed Coordinate Systems

This chapter will deal with dynamics of particles and systems of particles using fixed coordinate systems[1], which forms the basis for the later analysis of more complex systems with rotating reference systems.

The Newton-Euler approach is explained for particles and systems of particles in Section 2.1 (The principle will be extended to rigid bodies in Chapter 6). The principle of work and energy is explained for particles and systems of particles in Section 2.2 (It will be extended to rigid bodies in Chapter 5).

2.1 The "Newton-Euler" Approach

This section reviews a fundamental approach to derive the equations of motion of particles and systems of particles.

2.1.1 Linear Momentum

The right side of Newton's second law of motion (1.32) can also be rewritten using *linear momentum*. Linear momentum is defined as the product of mass m and velocity $\boldsymbol{v}$. Note that *momentum* is not the same as *moment*. For a particle i, located at $\boldsymbol{r}_i$, having mass m and velocity $\boldsymbol{v}_i$, the linear momentum $\boldsymbol{p}_i$ is defined as

$$\boldsymbol{p}_i := m_i \boldsymbol{v}_i = m_i \dot{\boldsymbol{r}}_i. \tag{2.1}$$

For a system of N particles, the individual linear momenta $\boldsymbol{p}_i$ according to (2.1) can be summed to an overall vector $\boldsymbol{p}$ of the system:

$$\boldsymbol{p} := \sum_{i=1}^{N} \boldsymbol{p}_i. \tag{2.2}$$

[1] In this book, only right-handed coordinate systems are used, and we recommend the reader to do the same to ensure that all formulae are valid.

2.1.2 Principle of Linear Impulse and Momentum

Newton's second law (1.32) states that the force $\boldsymbol{F}_i$ on a particle equals the rate of change of the particle's linear momentum:

$$\boldsymbol{F}_i = \dot{\boldsymbol{p}}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \boldsymbol{v}_i\right). \tag{2.3}$$

If the particle is part of a system of particles, we can apply summation to both sides of the equation. Because the sum of internal forces is zero (see (1.16)), Newton's second law of motion shows that the system's overall rate of change of the linear momentum $\boldsymbol{p}$ with respect to time equals the sum of the *external* forces acting on it:

$$\sum_{i=1}^{N} \boldsymbol{F}_i = \sum_{i=1}^{N} \dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}. \tag{2.4}$$

As a special case, this implies that if there are no external forces acting on a mechanical system in a particular direction, the system's linear momentum in this direction remains constant. Using the definition (1.11) of the center of mass of a system of particles and the definition $m := \sum_{i=1}^{N} m_i$, we see that for constant masses m_i,

$$\sum_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_i = m\ddot{\boldsymbol{r}}_C = m\boldsymbol{a}_C. \tag{2.5}$$

Combined with (2.4) and (2.3) for mass m, this yields

$$\sum_{i=1}^{N} \boldsymbol{F}_i = \sum_{i=1}^{N} m_i \boldsymbol{a}_i = m\boldsymbol{a}_C. \tag{2.6}$$

So, Newton's second law tells us that the center of mass of a system of particles will accelerate as if the entire mass of the system was concentrated in it and all external forces were acting on it.

When integrated over time between times t_0 and t_f, Newton's Second Law is called the *principle of linear impulse and momentum*:

$$\int_{t_0}^{t_\mathrm{f}} \sum_{i=1}^{N} \boldsymbol{F}_i \mathrm{d}t = \boldsymbol{p}(t_\mathrm{f}) - \boldsymbol{p}(t_0) = \Delta\boldsymbol{p}. \tag{2.7}$$

Hereby, *impulse* is defined as the integral of force over time. So, the principle states that the increment in linear momentum is equal to the impulse of the resultant force that was applied during the same time interval. This form of the relationship is particularly relevant for collisions.

We saw in this section that internal forces cannot effect a change in linear momentum of the system as a whole. Note that this does not mean that internal forces cannot have any effect. On the contrary, if you look *inside* the system, at relative movement of particles, internal forces *can* change motion.

Example 2.1 Two persons on skates are standing on a frictionless surface, each holding the end of a connecting rope in their hands. They can pull themselves towards each other by means of this rope, so the internal forces in the rope do have an effect. However, the skaters do not change the position of their combined center of mass.

Problem 2.1 A cannon of mass M stands at rest on a frictionless surface. Then, it shoots a ball of mass m. Assuming the relative velocity of the ball with respect to the cannon has a magnitude of v_r, what is the resulting rebound speed of the cannon?[a]

[a] *Hint*: After defining a coordinate system, determine the velocity of the center of mass of the combined system consisting of ball and cannon, using Newton's Second Law. Then, re-use your result from Problem 1.11 to relate this velocity to the individual velocities of ball and cannon.

2.1.3 Angular Momentum of Particles

The previous subsection showed the relation between the sum of forces on the center of mass of a system of particles and linear momentum of this system. The following subsections will show the relation between the sum of moments acting on a system and a quantity that, intuitively speaking, describes the amount of rotational movement of a mechanical system: *Angular momentum*.

Angular momentum of a particle i about a reference point Q can be seen as the moment of this particle's linear momentum about point Q.

First, we consider only the important special case in which the reference point Q is fixed, denoted by O. The angular momentum $\boldsymbol{H}_{O,i}$ of a particle about O, is defined as the cross product of the particle's position vector $\boldsymbol{r}_i$ with respect to point O, and the particle's linear

momentum vector $\boldsymbol{p}_i$:

$$\boldsymbol{H}_{O,i} := \boldsymbol{r}_i \times \boldsymbol{p}_i = \boldsymbol{r}_i \times (m_i \dot{\boldsymbol{r}}_i), \tag{2.8}$$

For a general point Q, the angular momentum $\boldsymbol{H}_{Q,i}$ of a particle about Q, is defined here as:

$$\boldsymbol{H}_{Q,i} := \boldsymbol{r}_{i/Q} \times (m_i \dot{\boldsymbol{r}}_{i/Q}). \tag{2.9}$$

Note that we are using here a relative and not an absolute velocity in the definition. If Q is a fixed point, the definitions are equivalent.

For a system of particles, the particles' individual angular momenta are summed to:

$$\boldsymbol{H}_Q := \sum_{i=1}^{N} \boldsymbol{H}_{Q,i}. \tag{2.10}$$

Example 2.2 In some machines the power is controlled by a so-called "*centrifugal governor*". The original type, introduced by *James Watt*, is shown in Figure 2.1.

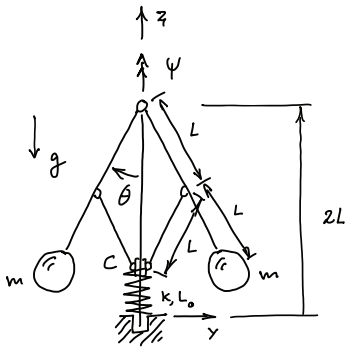


Figure 2.1: Centrifugal governor.

The central spindle, to which the arms carrying the two balls are hinged, rotates at a rate proportional to the speed of the engine. When this rotation rate is constant, the balls, under the action of gravity, a spring force, and centrifugal "forces", will converge to a definite "equilibrium" position that depends on the rotational speed. If the speed increases, the balls diverge outwards, raising the collar C to which some system of levers can be connected, which can turn a valve so as to reduce the power supply. Conversely, when the speed diminishes, the collar descends, and the supply is reinforced.

The inclination of the two identical upper arms to the spindle is denoted by the angle θ, the rotation of the spindle by the angle ψ. The rotating

coordinate system xyz with associated triad $\mathcal{F}$ is defined in such a way that the balls are always contained in the yz-plane and z is pointing upward.

The balls are modeled as particles of mass m and all other parts of the mechanism as massless bodies. The length of the arms carrying the balls is $2L$ and the length of the rods connecting the arms with the collar is L. Between the collar C and the fixed base, a linear spring with spring stiffness k and unstressed length L_0 is attached. Gravity acts in the downward direction with field strength g.

The left ball has the position vector ${}^{\mathcal{F}}\boldsymbol{r}_1 = (0, -2L\sin\theta, 2L(1-\cos\theta))^{\mathrm{T}}$ and the velocity ${}^{\mathcal{F}}\dot{\boldsymbol{r}}_1 = (2L\dot{\psi}\sin\theta, -2L\dot{\theta}\cos\theta, 2L\dot{\theta}\sin\theta)^{\mathrm{T}}$ (See Example 4.5).

The angular momentum component H_z of the system about the z-axis can be written as $H_z = P\dot{\theta} + Q\dot{\psi}$. Determine P and Q as functions of the parameters m, L, g, k, L_0 and the angle θ.

Exemplary solution

The angular momentum component about the z-axis is exactly the same for both particles, because of symmetry. So, angular momentum H_z can be calculated from the third component of the left particle's angular momentum vector $\boldsymbol{H}_{O1}$ about the origin O of the xyz coordinate system:

$$H_z = 2\,{}^{\mathcal{F}}\boldsymbol{H}_{O1} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.11E}$$

The angular momentum of the particle with respect to O is the cross product of its position vector $\boldsymbol{r}_1 = \boldsymbol{r}_{1/O}$ and its linear momentum, so:

$$\begin{aligned} {}^{\mathcal{F}}\boldsymbol{H}_{O1} &= {}^{\mathcal{F}}\boldsymbol{r}_1 \times (m\,{}^{\mathcal{F}}\dot{\boldsymbol{r}}_1) && \text{(2.12E)} \\ &= \begin{pmatrix} 0 \\ -2L\sin\theta \\ 2L\,(1-\cos\theta) \end{pmatrix} \times m \begin{pmatrix} 2L\dot{\psi}\sin\theta \\ -2L\dot{\theta}\cos\theta \\ 2L\dot{\theta}\sin\theta \end{pmatrix} && \text{(2.13E)} \\ \Rightarrow H_z &= 2 \cdot 4mL^2\dot{\psi}\sin^2\theta && \text{(2.14E)} \\ &= \underbrace{0}_{P}\cdot\dot{\theta} + \underbrace{8mL^2\sin^2\theta}_{Q}\dot{\psi}. && \text{(2.15E)} \end{aligned}$$

Comparing coefficients gives:

$$P = 0, \tag{2.16E}$$

$$Q = 8mL^2\sin^2\theta. \tag{2.17E}$$

Problem 2.2 A wheel consists of 20 particles of equal mass that are uniformly distributed around the circumference of the otherwise massless

wheel. The wheel rotates about its rotational symmetry axis, which is fixed in space. In which direction does the vector of angular momentum of the wheel, with respect to its fixed center, point? Use the cross product in your reasoning. Does your answer change if one of the particles is heavier than the others or if the arrangement is not uniform anymore?

A second important special reference point Q is the system's center of mass C. A convenient property of this point is that angular momentum can be calculated either using absolute velocities $\dot{\boldsymbol{r}}_i$ or relative velocities $\dot{\boldsymbol{r}}_{i/C}$, the result is the same (see Problem 2.3):

$$\boldsymbol{H}_C = \sum_{i=1}^{N} (\boldsymbol{r}_{i/C} \times (m_i \dot{\boldsymbol{r}}_{i/C})) = \sum_{i=1}^{N} (\boldsymbol{r}_{i/C} \times (m_i \dot{\boldsymbol{r}}_i)). \tag{2.18}$$

Problem 2.3 Prove the equivalence of the definitions in (2.18). Make use of vector addition and of the definition of the center of mass of a system of particles. ▶

Knowing a system's angular momentum with respect to its center of mass, it is often still desirable to calculate angular momentum with respect to a different reference point. Helpful reference points are often fixed points. Another frequent case is that the system is part of a larger system, and one wants to calculate the overall system's angular momentum with respect to its overall center of mass.

In such cases, we can make use of the relationship (see Problem 2.5 for a proof) for a (sub)system of mass m and center of mass C:

$$\boldsymbol{H}_Q = \boldsymbol{H}_C + \boldsymbol{r}_{C/Q} \times \left(m \dot{\boldsymbol{r}}_{C/Q}\right). \tag{2.19}$$

Problem 2.4 Check basic plausibility of (2.19) by looking at these two trivial special cases: a) The system's center of mass C coincides with point Q, b) The subsystem consists of one single particle[a].

[a] *Hint*: What is $\boldsymbol{H}_C$ for a single particle?

Problem 2.5 Prove (2.19). Make use of vector addition and of the definition of the center of mass of a system of particles, and of (2.9), (2.10) and (2.18). ▶

2.1.4 Euler's Second Law of Motion

We recall that Newton's Second Law for one particle of mass m_i and position vector $\boldsymbol{r}_i$ states that

$$m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i + \sum_{j=1}^{N} \boldsymbol{f}_{ij}, \tag{2.20}$$

with external resultant force $\boldsymbol{F}_i$ and internal forces $\boldsymbol{f}_{ij}$. The main step to derive Euler's Second Law is to apply the cross product on both sides:

$$\boldsymbol{r}_{i/Q} \times (m_i \ddot{\boldsymbol{r}}_i) = \boldsymbol{r}_{i/Q} \times \left(\boldsymbol{F}_i + \sum_{j=1}^{N} \boldsymbol{f}_{ij} \right), \tag{2.21}$$

If we sum over all particles, we can simplify the right side, because we saw in (1.21) in Section 1.2.7 that internal forces cause no net moments:

$$\sum_{i=1}^{N} \left(\boldsymbol{r}_{i/Q} \times (m_i \ddot{\boldsymbol{r}}_i) \right) = \sum_{i=1}^{N} \left(\boldsymbol{r}_{i/Q} \times \boldsymbol{F}_i \right) = \boldsymbol{M}_Q, \tag{2.22}$$

We now look for a relationship between the rate of change of angular momentum and the left side of this equation.

To that end, we apply the product rule to obtain the time derivative of $\boldsymbol{H}_Q$ in (2.10), using the definition (2.9). We assume constant masses m_i, so we obtain

$$\dot{\boldsymbol{H}}_Q = \sum_{i=1}^{N} \dot{\boldsymbol{H}}_{Q,i} = \sum_{i=1}^{N} \left(\dot{\boldsymbol{r}}_{i/Q} \times (m_i \dot{\boldsymbol{r}}_{i/Q}) + \boldsymbol{r}_{i/Q} \times (m_i \ddot{\boldsymbol{r}}_{i/Q}) \right). \tag{2.23}$$

The cross product of a vector with itself is zero, so (2.23) simplifies to

$$\dot{\boldsymbol{H}}_Q = \sum_{i=1}^{N} \left(\boldsymbol{r}_{i/Q} \times \left(m_i \ddot{\boldsymbol{r}}_{i/Q} \right) \right). \tag{2.24}$$

This is not yet the term on the left side of (2.22). However, we can apply vector addition,

$$\ddot{\boldsymbol{r}}_i = \ddot{\boldsymbol{r}}_{i/Q} + \ddot{\boldsymbol{r}}_Q, \tag{2.25}$$

and the definition of the center of mass

$$\sum_{i=1}^{N} m_i \boldsymbol{r}_{i/Q} = m \boldsymbol{r}_{C/Q} \tag{2.26}$$

to obtain Euler's Second Law for a generic point Q:

$$\dot{\boldsymbol{H}}_Q + \boldsymbol{r}_{C/Q} \times (m\ddot{\boldsymbol{r}}_Q) = \boldsymbol{M}_Q. \tag{2.27}$$

So, for a generic point of reference, the rate of change of a system's angular momentum is not always equal to the external moments applied. However, there are two important special cases where it is: First, point Q can be chosen as a non-accelerating point O, which means $\ddot{\boldsymbol{r}}_Q = \boldsymbol{0}$. In that case:

$$\dot{\boldsymbol{H}}_O = \boldsymbol{M}_O. \tag{2.28}$$

The second special case is point Q being the system's center of mass C, because then $\boldsymbol{r}_{C/Q} = \boldsymbol{0}$:

$$\dot{\boldsymbol{H}}_C = \boldsymbol{M}_C. \tag{2.29}$$

Note that a practically less relevant third special case is given if the vectors $\boldsymbol{r}_{C/Q}$ and $\ddot{\boldsymbol{r}}_Q$ are aligned.

This set of three equations, in either of the forms (2.27), (2.28), or (2.29), also called *Euler's second law of motion*, describes rotational movement of a system of particles. Combined with Newton's law for a system of particles (2.6), the resulting six equations form a set of equations of motion for the combined movement of the particles of the system in 3-D: One equation per dimension for linear motion, and one equation per dimension for rotational motion.

Remember that the simple equality between rate of change of angular momentum and external moments, (2.28) or (2.29), which is also called Euler's second law of motion, only holds in two special cases: (1) the reference point Q is a fixed (non-accelerating) point O, or (2) the reference point Q is the center of mass C.

Also remember that internal forces cannot cause net moments, and therefore they cannot effect a change in angular momentum of the system as a whole. Therefore, when calculating $\boldsymbol{M}_O$ or $\boldsymbol{M}_C$ in (2.28) or (2.29), it suffices to sum only all *external* moments acting on the system.

In analogy to the principle of linear impulse and momentum (see Section 2.1.1), Euler's Second Law when integrated over time is called the *principle of angular impulse and momentum*. Hereby, *angular impulse* is defined as the integral of the moment over time, and it determines angular momentum change during the same time interval:

$$\int_{t_0}^{t_\mathrm{f}} \boldsymbol{M}_C \mathrm{d}t = \boldsymbol{H}_C(t_\mathrm{f}) - \boldsymbol{H}_C(t_0) = \Delta \boldsymbol{H}_C. \tag{2.30}$$

◯ **Problem 2.6** For a system of particles where the only external forces are gravitational forces, what can you say about the system's rate of change of angular momentum with respect to its center of mass?

◯ **Problem 2.7** A cat is falling off a building, back first. The cat manages to rotate its body and safely lands on its feet. Ignore air drag. Is it correct to state that the cat used its muscle forces to change its angular momentum with respect to its center of mass during the fall? Why/why not? ▷

◯ **Problem 2.8** Two particles 1 and 2 are connected by a rigid rod and form one system. In scenario a), an external force of magnitude F is applied to particle 1. In an alternative scenario b), a force of the same direction and magnitude is applied to particle 2 instead.

- Will the acceleration of the system's center of mass differ between the two scenarios?
- Will the system's rate of change of angular momentum with respect to its center of mass differ?
- Will the system's rate of change of angular momentum with respect to a fixed point differ?

Show why for each answer. Can you imagine special cases where the answer to the above three questions is always "No"? Can you imagine special cases where the answer to the above three questions is always "Yes"?

2.2 Work and Energy

Another way to describe the kinetics of (a system of) particles is using *work and energy*. This will be introduced in this section and discussed more in Chapter 5.

2.2.1 Power and Work of a Force

The instantaneous *power* P of a force $\boldsymbol{F}$ is defined as the scalar product of $\boldsymbol{F}$ and the velocity $\boldsymbol{v}$ of its point of application:

$$P := \boldsymbol{F} \cdot \dot{\boldsymbol{r}} = \boldsymbol{F} \cdot \boldsymbol{v}. \tag{2.31}$$

Power has no direction; it is a scalar, not a vector.

To calculate the *work* done on a particle by the application of a force between times t_0 and t_1, power is integrated over time:

$$W := \int_{t_0}^{t_1} P \,\mathrm{d}t = \int_{t_0}^{t_1} \boldsymbol{F} \cdot \boldsymbol{v} \,\mathrm{d}t. \tag{2.32}$$

Like power, work is a scalar quantity. As $\mathrm{d}\boldsymbol{r} = \boldsymbol{v}\,\mathrm{d}t$, work W of a force $\boldsymbol{F}$ is the integral of $\boldsymbol{F}$ with respect to the path:

$$W = \int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}, \tag{2.33}$$

where the point of application travels from point A to point B, with infinitesimal position vector increments $\mathrm{d}\boldsymbol{r}$.

That means a force $\boldsymbol{F}$ can only do work W on a mechanical system if there is a displacement of the force vector's point of application and when the dot product of $\boldsymbol{F}$ and $\mathrm{d}\boldsymbol{r}$ is not always zero between times t_0 and t_1.

Example 2.3 If an apple is held above the ground and then dropped, the work done on the apple as it falls is equal to the weight (a force) of the apple integrated over the path to the ground.

Note that *only* in the special case where the force is constant *and* pointing in direction of the movement (such as in Example 2.3), (2.33) simplifies and one can calculate the work as being equal to the magnitude of the force times distance traveled.

Example 2.4 A person carries a suitcase along a horizontal path. The upward force the person applies on the suitcase is always perpendicular to application point displacement increment $\mathrm{d}\boldsymbol{r}$, such that the scalar product (2.33) is zero. That means the upward force does *no* work along this path.

Problem 2.9 A ball rolls down an incline that has an angle of 45° with the vertical. Calculate the distance the ball traveled as a function of height difference covered. Calculate the work done on the ball by its own weight, as a function of this height difference. Show that this work is *not* equal to the magnitude of the weight force times distance traveled. Take care about the signs throughout your calculation.

Problem 2.10 Among other forces, a force of magnitude F acts on a particle. The force always acts in the direction of the particle's velocity vector, and F is a function of time t, with $F = F_0(1 + \sin(2\pi f t))$, with constant frequency f and amplitude F_0. The particle has constant speed v_0. Calculate the work done by the force F as a function of t and the given constants. Show that this work is generally *not* equal to the force amplitude times distance traveled.

2.2.2 Principle of Work and Energy for a Single Particle

Newton's second law for a particle with constant mass m, with a resultant force vector $\boldsymbol{F}$ acting on it, states

$$\boldsymbol{F} = \dot{\boldsymbol{p}} = m\ddot{\boldsymbol{r}}. \tag{2.34}$$

If we integrate this over the path between points A and B, we obtain work W on the left side of (2.34). If we integrate the right side of (2.34) as well, we obtain (see also Problem B.4)

$$\begin{aligned}\int_A^B \boldsymbol{F}^{\mathrm{T}}\,\mathrm{d}\boldsymbol{r} &= m\int_A^B \ddot{\boldsymbol{r}}^{\mathrm{T}}\,\mathrm{d}\boldsymbol{r}\\ &= m\int_{t_0}^{t_1}\ddot{\boldsymbol{r}}^{\mathrm{T}}\dot{\boldsymbol{r}}\,\mathrm{d}t = m\left[\frac{1}{2}\dot{\boldsymbol{r}}^{\mathrm{T}}\dot{\boldsymbol{r}}\right]_{t_0}^{t_1} = \frac{1}{2}m(v_1^2 - v_0^2),\end{aligned} \tag{2.35}$$

where v_0 and v_1 are the respective speeds of the particle at time instants t_0 and t_1, corresponding with the particle being at locations A and B.

We define the *kinetic energy* T of the particle with mass m and velocity vector $\boldsymbol{v} = \dot{\boldsymbol{r}}$ with magnitude $v = |\boldsymbol{v}|$ to be the work needed to bring the particle from rest to this velocity. From (2.35), we see (with $v_1 = v$ and $v_0 = 0$) that this quantity is equal to

$$T = \frac{1}{2}m|\boldsymbol{v}|^2 = \frac{1}{2}mv^2. \tag{2.36}$$

Note that in contrast to linear or angular momentum, kinetic energy is a *scalar*, which means that it has no direction.

Example 2.5 This Example is about the governor of Example 2.2 on page 48. The kinetic energy T of the system can be written in the form $T = \frac{1}{2}A\dot{\theta}^2 + \frac{1}{2}B\dot{\psi}^2$. Determine A and B as functions of the parameters m, L, g, k, L_0 and the generalized coordinate θ and explain why the kinetic energy is no function of the generalized coordinate ψ.

Exemplary solution

Because of symmetry, kinetic energy T_1 of particle 1 equals kinetic energy T_2 of particle 2, so

$$T = T_1 + T_2 = 2T_1. \tag{2.37E}$$

The balls are modeled as particles, so kinetic energy is

$$T = \sum_{i=1}^{2}\tfrac{1}{2}m\left|\boldsymbol{v}_i\right|^2 = 2\cdot\tfrac{1}{2}m\left|{}^{\mathcal{F}}\dot{\boldsymbol{r}}_1\right|^2 \tag{2.38E}$$

$$= 2\cdot\tfrac{1}{2}m\left(4L^2\dot{\psi}^2\sin^2\theta + 4L^2\dot{\theta}^2\cos^2\theta + 4L^2\dot{\theta}^2\sin^2\theta\right) \tag{2.39E}$$

$$= \tfrac{1}{2}\cdot\underbrace{8mL^2}_{A}\dot{\theta}^2 + \tfrac{1}{2}\cdot\underbrace{8mL^2\sin^2\theta}_{B}\dot{\psi}^2. \tag{2.40E}$$

Comparing coefficients gives $A = 8mL^2$ and $B = 8mL^2\sin^2\theta$.

The kinetic energy is no function of ψ, because the magnitude of the velocity vector of either particle does not depend on ψ.

When the initial velocity is not zero, we find from the integral of (2.35) that the work W done on any particle by the resultant force $\boldsymbol{F}$ acting on it during an arbitrary time interval equals the *increment* ΔT of the particle's kinetic energy, which is the difference between the kinetic energy T_B at position B and the kinetic energy T_A at position A:

$$W = \int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = T_B - T_A = \Delta T, \tag{2.41}$$

or in short

$$W = \Delta T. \tag{2.42}$$

This is the *principle of work and kinetic energy* for a single particle. Note that we derived it only using Newton's second law.

2.2.3 Work of Internal and External Forces

In order to investigate whether internal forces can do work on mechanical systems, we consider a particle i with mass m_i and speed v_i, which is part of a system of N particles, as depicted in Figure 1.6 on page 15. The resultant force acting on the i-th particle over a position increment $\mathrm{d}\boldsymbol{r}_i$ includes both the resultant external force $\boldsymbol{F}_i$ that acts on this particle *and* the sum of forces $\boldsymbol{f}_{ij}$ exerted on the i-th particle by any other j-th particle in the system. Therefore, the work W_i done on the particle is

$$W_i = \int_{A_i}^{B_i} \left(\boldsymbol{F}_i + \sum_{j=1}^{N} \boldsymbol{f}_{ij} \right) \cdot \mathrm{d}\boldsymbol{r}_i. \tag{2.43}$$

It is *not* possible to further reduce this equation, because the work done by internal forces does *not* cancel. This is in contrast to what we found in Section 1.2.7 for resultant moments of internal forces. So, internal forces can do work.

◯ **Problem 2.11** The two skaters on a frictionless flat surface from Example 2.1 use the rope they are holding to start pulling towards each other.

Draw individual FBDs of each skater. Identify all external forces for each skater and explain whether these forces do positive, negative, or zero work when the skaters move towards each other, and when they move apart.

Next, draw a FBD of the combined system, consisting of the two skaters, and identify again all external forces.

Finally, describe the displacement of the points of application and explain

why the work of the *external* forces is zero. Also explain why the work of the *internal* forces in the combined system is *not* zero.

◯ **Problem 2.12** A wheel is rolling without slip. Explain with a short reasoning whether the friction forces between wheel and ground do work. ▷

2.2.4 Principle of Work and Energy for Systems of Particles

In a system of N particles (see Figure 1.6 on page 15), where particle i has the individual kinetic energy

$$T_i = \frac{1}{2} m_i v_i^2, \tag{2.44}$$

the principle of work and energy (2.42) applies to the particle as

$$W_i = \Delta T_i. \tag{2.45}$$

We may apply summation over the left and right side of (2.45). By defining the kinetic energy of the system of particles as the sum of all individual kinetic energies,

$$T := \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2, \tag{2.46}$$

we obtain the principle of work and energy for a system of particles:

$$W = \sum_{i=1}^{N} \int_{A_i}^{B_i} \left(\boldsymbol{F}_i + \sum_{j=1}^{N} \boldsymbol{f}_{ij} \right) \cdot \mathrm{d}\boldsymbol{r}_i = \sum_{i=1}^{N} \Delta T_i = \Delta T. \tag{2.47}$$

This means the change in kinetic energy of a system of particles is given by the work of all external *and* internal forces acting on this system.

◯ **Example 2.6** Consider again the two skaters on a frictionless flat surface from Example 2.1 and Problem 2.11. If the skaters start from rest, their speeds (and thus kinetic energy) will increase. Interpreting the skaters as particles number 1 and 2 within one system, the work done by all external forces acting on this combined system is zero. Still, kinetic energy changes, due to the work of *internal* forces $\boldsymbol{f}_{12}$ and $\boldsymbol{f}_{21}$ transmitted by the rope.

2.2.5 Conservative Systems

As an introduction, we approach the topic intuitively. A force is called *conservative* if it is a function of position only and it *conserves* mechanical energy.

◯ **Example 2.7** If a particle of mass m is thrown vertically into the air and rises up to a height of 5 m against the force of gravity, it will lose speed and thereby kinetic energy. However, energy is still *conserved* since one can fully recuperate this mechanical energy by letting the particle fall from the 5 m height back to the ground. In contrast, if a friction force acts on a particle that is sliding over the ground, thereby causing the particle to brake, then mechanical energy is transformed into thermal energy. This energy cannot be recuperated when the particle moves back into the other direction; instead, even more mechanical energy would be transformed into heat.

The work done by conservative forces, such as gravity in Example 2.7, can be expressed as a function of the start and end positions of a system. This saves the integration work since the path the particle takes from start to the end does not affect the work done on the system. In our example, the work done on the system (lifting the particle against gravity) will be the same if we go up and down with the particle two or more times in a row.

Consequently, a conservative force would not do any work along an arbitrary *closed path*, because then the start and end position are identical. Thus, according to the principle of work and energy, the kinetic energy of a system with only conservative forces acting on it will return to its initial value if the initial and final configuration of the system are identical.

◯ **Problem 2.13** A particle is moved from position A to position B along three possible trajectories. A conservative force acts on it. Can you say anything about the work that this force does on the particle along each of these three trajectories?

To check whether or not a force is conservative, the key is the concept of *potential energy*.

Without loss of generality, we first assume there is only a single conservative force acting on a mechanical system. To express the *potential* of this conservative force to do work on the system (in the future), a scalar potential function V is introduced, equivalent to energy. Given that the work of a conservative force purely depends on net displacement of its point of application $\boldsymbol{r}$ between two points, also the associated potential energy V must be a function of $\boldsymbol{r}$, so

$$V = V(\boldsymbol{r}) = V(x,\, y,\, z). \tag{2.48}$$

For the force to be conservative, it is not sufficient for it to be only position-dependent. The total energy of the system, consisting of kinetic

energy T and potential energy V, must equal a constant c at all times:

$$T + V = c. \tag{2.49}$$

In other words, the difference in kinetic energy ΔT plus the difference in potential energy ΔV is zero for any two instants in time:

$$\Delta T + \Delta V = 0. \tag{2.50}$$

Note that this implies that along any closed path (where ΔV will be zero), ΔT must be zero, such that the system is conservative.

If we substitute the principle of work and energy in (2.42) in (2.50), we obtain

$$W + \Delta V = 0. \tag{2.51}$$

◯ **Problem 2.14** What is the change in potential energy ΔV if a particle starts in one position and ends in the same position? What does your answer mean for the work W that a conservative force does along any path that starts and ends in the same position?[a]

[a] *Hint*: Recall that V is a function of position $\boldsymbol{r}$. Also, use the connection between W and ΔV.

We now consider infinitesimally small increments. This also eliminates the initial value of the potential V:

$$\mathrm{d}W + \mathrm{d}V = 0. \tag{2.52}$$

Substituting the definition of work, (2.33), this is equal to

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} + \mathrm{d}V = 0. \tag{2.53}$$

The infinitesimal change of potential energy $\mathrm{d}V$ can be calculated using the partial derivatives of V in respectively x, y and z direction:

$$\mathrm{d}V = \frac{\partial V}{\partial x}\,\mathrm{d}x + \frac{\partial V}{\partial y}\,\mathrm{d}y + \frac{\partial V}{\partial z}\,\mathrm{d}z = \begin{pmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial V}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = \nabla V \cdot \mathrm{d}\boldsymbol{r}. \tag{2.54}$$

Here, the nabla operator ∇ produces the gradient of $V(x, y, z)$. See Section B.5 for more explanation on the gradient.

Example 2.8 For a conservative mechanical system, the elevation in Figure B.1 on page 497 could be interpreted as proportional to a scalar potential V, and the arrows could be interpreted as 2-D force vectors $\boldsymbol{F}$ associated with this potential.

If we combine (2.53) with (2.54) we find

$$\boldsymbol{F} = -\nabla V. \tag{2.55}$$

This means that the system is conservative if and only if the force equals the negative gradient of a scalar function $V(\boldsymbol{r})$, the potential energy. So, only if one can find such a scalar function V for a particular force, then this force is a conservative force.

Examples for conservative forces are gravitational force or the force of a spring. An example of a non-conservative force is friction during sliding. Moreover, if all forces acting on a system are conservative, the system itself is conservative.

Problem 2.15 Draw some more vectors of the gradient field into the hiking map in Figure B.1 on Figure B.1. At which angle do the vectors intersect with the isos lines (representing potential)? How is the length of the arrows related to distance between iso lines? How precisely are the vectors related to gravity?

Problem 2.16 Try to draw an example of a force field that is position-dependent, but not conservative.[a] ▶

[a] *Hint*: In a conservative force field, the force vector is the negative gradient of the potential field. This means that when one moves in the direction of the force, potential is lost.

Problem 2.17 Show that friction during sliding motion is not a conservative force, by looking at work done along different paths taken (Note that a single counterexample suffices). ▶

Problem 2.18 Investigate whether the force vector $\boldsymbol{F} = c(x^2,\, x^2,\, 0)^{\mathrm{T}}$, with constant c, is a conservative force.[a]

[a] *Hint*: 1. Draw some force vectors of this field in the xy plane. Then look whether you can find any example where the force does work along a closed path. 2. See whether you can find a V via partial integration. What did you conclude from steps 1 and 2, do these findings match?

Problem 2.19 Investigate whether gravitational forces, sliding friction, and spring forces are conservative, by trying to find, for each force, a function

V that satisfies (2.55). ▷►

Example 2.9 This Example is about the governor of Example 2.2 on page 48. The potential energy V of the system can be written in the form $V = Cz^2 + Dz$, with the height $z = 2L(1 - \cos\theta)$ of the collar C and the balls. Determine C and D as functions of the parameters m, L, g, k, L_0.

Exemplary solution

Each of the two particles has gravitational potential energy V_g, which is exactly the same for either, because of symmetry. Moreover, the spring adds potential energy V_s to the system:

$$V = V_\mathrm{s} + 2V_\mathrm{g}. \tag{2.56E}$$

Substituting the appropriate expressions and given variables into (2.56E) leads to

$$V = \tfrac{1}{2}k\,(z - L_0)^2 + 2mgz, \tag{2.57E}$$

$$= \underbrace{\tfrac{1}{2}k}_{C}\ z^2 + \underbrace{(2mg - kL_0)}_{D}z + \underbrace{\tfrac{1}{2}kL_0^2}_{E(\text{constant})}\ . \tag{2.58E}$$

Comparing coefficients gives $C = \frac{1}{2}k$ and $D = 2mg - kL_0$.

2.3 Checking Plausibility and Frequent Mistakes

Some specific ideas for checking plausibility and checking inconsistencies, in addition to Section 1.4, are to:

- do the same problem using work and energy as well as using Newton-Euler.
- solve the equations of motion using MATLAB™ or a numerical program of your choice, and animate the simulation.
- numerically calculate the net energy at each time instant for a simulation. This is particularly useful for conservative systems.

Frequent mistakes specific to this chapter that may be the source of erroneous results are:

- specific, or even numerical values were filled in for positions or velocities at certain time instants *before* taking derivatives, such that the values appear constant and their derivatives zero.
- although point O was not fixed, (2.28) was applied.

- when applying Euler's second law in the forms of (2.28) or (2.29), moments and angular momenta were not calculated about the same point.
- kinetic energy was misinterpreted as a vector quantity.
- the work of a force was calculated incorrectly, in a scalar fashion by just multiplying force with a distance traveled, not taking into account signs nor force or velocity vectors directions, and not taking into account that the force vector might not be constant. To remedy, it is important to always take the scalar product of force and velocity or distance increment vectors, and use integration.

2.4 Summary

This chapter covered two main approaches to derive equations of motion for particles and systems of particles: Newton-Euler (or the principles of linear and angular momentum) and work and energy.

Linear and angular momentum of a system of N particles were respectively defined as

$$\boldsymbol{p} := \sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i, \qquad \boldsymbol{H}_Q := \sum_{i=1}^{N} \boldsymbol{\rho}_i \times \boldsymbol{p}_i. \tag{2.59}$$

Angular momentum only has a meaning when a reference point Q is defined. Newton's and Euler's second laws state that

$$\sum_{i=1}^{N} \boldsymbol{F}_i = \dot{\boldsymbol{p}}, \quad \text{and} \quad \sum_{i=1}^{N} \boldsymbol{M}_{O,i} = \dot{\boldsymbol{H}}_O \text{ or } \sum_{i=1}^{N} \boldsymbol{M}_{C,i} = \dot{\boldsymbol{H}}_C, \tag{2.60}$$

respectively, where only the *external* forces and moments acting on the system need to be considered: the effects of the internal ones cancel out. Euler's second law in this form can only be applied for a fixed point O or for the system's center of mass C.

Kinetic energy of a system of N particles was defined as

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i |\dot{\boldsymbol{r}}_i|^2, \tag{2.61}$$

and work of a force $\boldsymbol{F}_k$ as

$$W_k = \int_A^B \boldsymbol{F}_k \cdot \mathrm{d}\boldsymbol{r} = \int_{t_0}^{t_1} \boldsymbol{F}_k \cdot \dot{\boldsymbol{r}} \, \mathrm{d}t = \int_{t_0}^{t_1} P \, \mathrm{d}t. \tag{2.62}$$

The principle of work and kinetic energy states that the work of all n internal *and* external forces determines a system's change in kinetic energy:

$$\sum_{k=1}^{n} W_k = \Delta T. \tag{2.63}$$

In case that a potential field V can be found for each internal or external force $\boldsymbol{F}_k$ that does work on a system such that

$$\boldsymbol{F}_k = -\nabla V_k, \text{ with } V_k = V_k(\boldsymbol{r}), \tag{2.64}$$

then the system is conservative, which implies that the sum of the system's kinetic and potential energy is constant:

$$T + \sum_{k=1}^{n} V_k = \text{const.} \tag{2.65}$$

2.5 Problems

2.5.1 Guided Problems

Problem 2.20 In a right-handed, orthogonal, inertial coordinate system, the directions of the axes are given by unit vectors $\hat{\boldsymbol{n}}_1$, $\hat{\boldsymbol{n}}_2$, $\hat{\boldsymbol{n}}_3$. With respect to the origin O of the coordinate system, the position vector of a particle of mass m is given as a function of time t: $\boldsymbol{r}(t) = a\sin(\omega t)\hat{\boldsymbol{n}}_1 + b\cos(\omega t)\hat{\boldsymbol{n}}_2 + ct\hat{\boldsymbol{n}}_3$, where a, b, c and ω are positive constants.

a. State suitable SI-units for $\boldsymbol{r}$, t and parameters a, b, c, and ω. Give your answer in the form: $[a] = ...$, $[b] = ...$, etc.
b. Draw the coordinate system and the particle's motion for time ranging from $t = 0\,\text{s}$ to $t = 4\frac{\pi}{\omega}$. Clearly label the point where the particle is at time $t = 0\,\text{s}$, and also the point where it is at $t = \frac{\pi}{2\omega}$. Use axis ticks involving the constants a, b, c, ω to clarify the locations. Produce either a clear (e.g. isometric) 3-D projection, or use multiple 2-D projections.
c. Calculate the angular momentum vector $\boldsymbol{H}_O(t)$ of the particle with respect to the origin O as a function of t and the parameters a, b, c, ω, and m.
d. Calculate the rate of change of angular momentum $\dot{\boldsymbol{H}}_O(t)$ of the particle with respect to the origin as a function of t and the parameters a, b, c, ω, and m.
e. Calculate the acceleration vector $\boldsymbol{a}(t)$ of the particle as a function of t and the parameters a, b, c, and ω.
f. Use inverse dynamics to calculate the resultant force vector acting on the particle. Then calculate the moment this force generates around

the origin, as a function of t and the parameters a, b, c, ω, and m. Discuss your result in relationship to part d.

g. Calculate the power transmitted by the resultant force vector acting on the particle, as a function of time t (and of the constants). Also calculate the work done by this force over a finite time interval Δt.
h. Calculate the kinetic energy T of the particle, as well as the rate of change of kinetic energy $\dot{T}$, as functions of t (and of the constant parameters).
i. Check plausibility and consistency of your answers to the previous question parts. Do this for example by comparing them to each other, and by investigating special cases, such as $a = b$. Conduct at least five independent checks.
j. Under what condition on a, b and c are the velocity and the acceleration of the particle perpendicular to each other for all t? To verify your answer, make a drawing of the movement in that special case, including velocity and acceleration vectors at one location.
k. Can the resultant force vector acting on the particle be explained by a conservative force field? Clearly answer "yes" or "no", and also provide a mathematical proof and a graphical illustration for your response.
l. To check your drawing, plot the path of the particle. In MATLAB$^{\text{TM}}$, you can do this by first creating a linearly spaced time vector and choosing numerical values for all parameters. Then, plot the resulting components of $\boldsymbol{r}$ into 3-D axes using the command `plot3`.

Problem 2.21 A three-dimensional force vector field is given by

$$\boldsymbol{F}(x,y,z) = -\frac{c}{\sqrt{x^2+y^2+z^2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{2.66P}$$

with a constant c with $\{c\} = 1$, and position coordinates x, y, z in an inertial Cartesian coordinate system.

a. Determine $[c]$, so the unit of c (see Section 1.1.4).
b. Draw some force vectors of this field in a projection on the xy-plane. Then, do the same in projections onto the two other coordinate planes.
c. In your drawing, look whether you can find an example where the force does nonzero net work along a closed path.
d. Try to find a potential function V that can be associated with $\boldsymbol{F}$, meaning that $\boldsymbol{F}$ is the negative gradient of V. Compare your findings from this part and parts b and c. Do they match, considering what you know about conservative forces?
e. Give a real-life example of such a force field if possible.
f. Employ MATLAB$^{\text{TM}}$ or another software to help with the first part

of the question. To this end, first create matrices $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ that contain a grid for x, y and z (e.g. using the function `ndgrid`), all ranging from $-r$ to r, where r is a radius you can choose freely. Then create a matrix $\mathbf{U}$ that contains the x-components of the vector $\boldsymbol{F}$, a matrix $\mathbf{V}$ that contains the y-components and a matrix $\mathbf{W}$ that contains the z-components. Then, (e.g. using the function `quiver3`), plot the vector field in 3-D. Compare to your manual drawing.

g. Repeat parts a-d for the two-dimensional force vector field $\boldsymbol{F}(x, y) = c(x^2, x^2)^{\mathrm{T}}$, with $\{c\} = 1$.

Problem 2.22 *Kepler's second law* describes the fact that if a planet orbits around a sun under the influence of the sun's gravity, the area covered by lines connecting planet and sun during identical time intervals is always the same. That means that if the path is elliptical, the planet's speed must change.

We will try to derive Kepler's second law on an example system: A planet P orbits a star with an elliptic orbit (see Figure 2.2). The star is located in one of the focal points O of the orbit. The planet and star are modeled as particles with planet mass m and star mass M. We assume $M \gg m$, so the center of mass of the star is a fixed point.

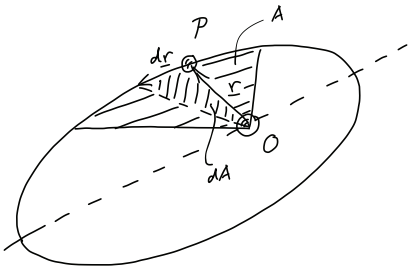


Figure 2.2: Planet in elliptic orbit around a star.

a. Draw a free-body diagram (FBD) of the planet projected onto the plane of the elliptic orbit.
b. Calculate the angular momentum $\boldsymbol{H}_O$ of the planet with respect to O as a function of $\boldsymbol{r}$, $\dot{\boldsymbol{r}}$ and m.
c. Calculate the sum of moments $\Sigma \boldsymbol{M}_O$ acting on the planet with respect to O using the FBD from part a.
d. Calculate the rate of change of angular momentum $\dot{\boldsymbol{H}}_O$ using Euler's second law.
e. Calculate the infinitesimally area $\mathrm{d}A$ as a function of $\boldsymbol{r}$ and $\mathrm{d}r$.
f. Rewrite $\mathrm{d}A$ from part e as a function of $\boldsymbol{r}$, $\dot{\boldsymbol{r}}$ and infinitesimal time $\mathrm{d}t$.
g. Rewrite $\mathrm{d}A$ from part f as a function of $\boldsymbol{H}_O$ and m.

h. Calculate the finite area A for a time interval $\Delta t = t_2 - t_1$ by integrating $\mathrm{d}A$ from t_1 to t_2.
i. Check your result from part h against Kepler's second law.

Problem 2.23 A particle of mass m slides down a frictionless slide, in a constant gravity field $-g\hat{\boldsymbol{n}}_3$. The surface of the slide is shaped such that its normal vector is always perpendicular to the direction $\hat{\boldsymbol{n}}_1$, see Figure 2.3. The slide has height H between the starting and finish positions. The slide is level (it has zero slope) both at the start and the end positions of the block. At the start, the particle has a velocity vector $\boldsymbol{v}_0 = (v_{X0},\, v_{Y0},\, 0)^{\mathrm{T}}$.

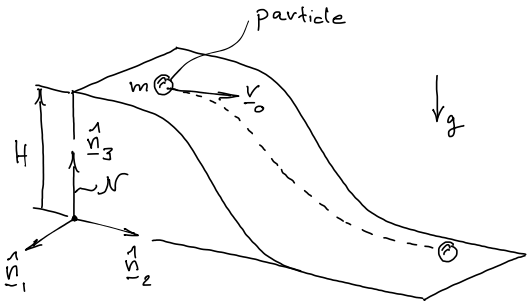


Figure 2.3: A particle sliding down a slide.

a. Draw a FBD of the particle for a general location on the slide.
b. By using conservation of energy, calculate the magnitude of the particle's velocity at the end position, as indicated in the figure.
c. Calculate the angle between the final velocity vector $\boldsymbol{v}_1$ and $\hat{\boldsymbol{n}}_2$. Note that you can re-use your answers to the previous questions.
d. Calculate the work done on the particle by the gravitational force during the movement.

Problem 2.24 For the suspended camera of mass m, as depicted in Problem 1.18 on page 35, we investigate the case where the camera moves.

a. The camera accelerates from a standstill, with initial acceleration vector $\boldsymbol{a}_{\mathrm{c}0} = (a_{X0},\, a_{Y0},\, 0)^{\mathrm{T}}$. Calculate the tension in cable r as a function of l, w, h, m, g, a_X, a_Y. Start your answer with a free-body diagram.
b. In order to move the camera, we need to change the respective lengths l_{p}, l_{q}, l_{r} of the three cables. For a generic velocity of the camera of $\boldsymbol{v}_{\mathrm{c}} = (v_X,\, v_Y,\, v_Z)^{\mathrm{T}}$, calculate the speeds $\dot{l}_{\mathrm{p}}$, $\dot{l}_{\mathrm{q}}$, $\dot{l}_{\mathrm{r}}$ with which the three cables need to elongate/shorten as functions of the camera's position and velocity components. Re-write your answer such that the "vector" containing all cable speeds can be calculated by taking the product of

a matrix with the velocity vector $\boldsymbol{v}_\mathrm{c}$.[a]

c. Motorized winches manipulate the lengths of the three cables. The acceleration vector needs to be $\boldsymbol{a}_\mathrm{c} = (a_X,\, a_Y,\, a_Z)^\mathrm{T}$. Calculate the three accelerations $\ddot{l}_\mathrm{p}$, $\ddot{l}_\mathrm{q}$, $\ddot{l}_\mathrm{r}$ with which the cables change length, as functions of the camera location, velocity, and acceleration components.

d. In order to check your above results, implement three functions in MATLAB$^\mathrm{TM}$ or another software: The first calculates the vector of cable lengths as functions of camera location, the second and third respectively provide cable velocities and accelerations as functions of camera velocity and acceleration.
Then, write a script that generates an arbitrary camera trajectory (for example linear or circular motion) and that uses your functions to calculate cable lengths, velocities and accelerations. To check, also calculate the numerical derivatives of cable lengths (for example using the function `gradient`) and check the analytical versus numerical results by plotting them on top of each other.

[a] *Hint*: First, express each cable length as a function of camera position. Then, calculate the derivative of each length with respect to time.

2.5.2 Practice Problems

Problem 2.25 Figure 2.4 shows a ballerina performing a jumping pirouette. A very simple model consists of a massless body and two point-masses, each of mass m_h, representing the hands. The symmetry axis is also the rotation axis, and it remains vertical throughout. The ballerina holds her hands in a symmetric position, each at the same shoulder height and horizontal distance away from the body.

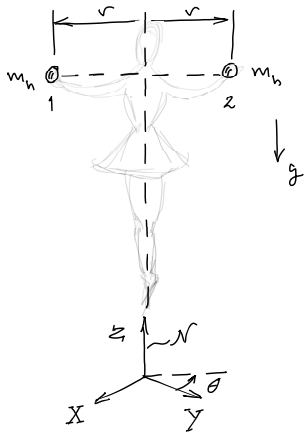


Figure 2.4: Jumping pirouette ballerina.

The ballerina pulls her arms in, by exerting a force on each point mass. The only external force is gravity that acts along the vertical axis of rotation. Explain whether this system conserves all linear momenta, angular momenta about the center of mass, and energy, in one sentence per quantity. ▶

Problem 2.26 This Problem is about the accelerometer of Problem 1.26 on page 41. We want to calculate the sphere's potential energy due to gravity. We consider a simplified case with only one-dimensional movement, as shown in Figure 2.5. The accelerometer maintains the same orientation throughout.

Now, the positive inertial Z-direction is down (not up), and the location of the mass is described by different coordinates.

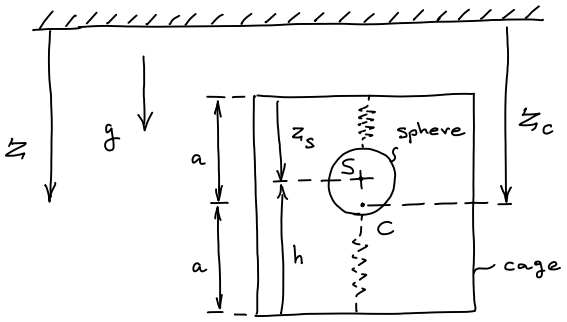


Figure 2.5: Simplified 1-D case of the accelerometer.

With an arbitrary constant energy V_0, the potential energy V of the sphere due to gravity can be correctly calculated as (Check all that apply):

A. $V = mgz_S$
B. $V = mg(Z_C - z_S)$
C. $V = mgh$
D. $V = mgZ_C + V_0$
E. $V = -mgz_S + V_0$
F. None of the above

▷

Problem 2.27 The catapult in Figure 2.6 consists of three springs, a cup, and a ring. Each spring is connected to the cup on one end and to the ring on the other end. The ring is connected to the ground in the XY plane of the inertial XYZ coordinate system.

An object of mass m is to be launched. It is placed in the cup in point D and released with zero initial speed. The initial position vector of D in the inertial coordinate system XYZ is provided as $\boldsymbol{p}_D = (0,\, 0,\, -4l_0)^{\mathrm{T}}$. Neglect the mass of the springs and the cup, and assume the object is a particle.

The connection points of the three springs at the ring are given by position

vectors $\boldsymbol{p}_A = r(-\frac{1}{2}\sqrt{3},\, \frac{1}{2},\, 0)^{\mathrm{T}}$, $\boldsymbol{p}_B = r(\frac{1}{2}\sqrt{3},\, \frac{1}{2},\, 0)^{\mathrm{T}}$ and $\boldsymbol{p}_C = r(0,\, -1,\, 0)^{\mathrm{T}}$, with constant parameter $r = 3l_0$. All springs are linear, have stiffness k, unstretched/relaxed length l_0, and are massless.

Gravity with field strength g acts in $(0,\, 0,\, -1)^{\mathrm{T}}$ direction.

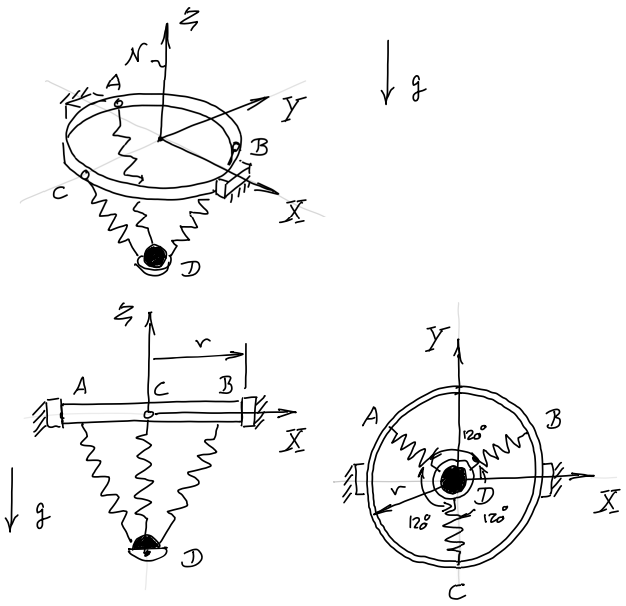


Figure 2.6: Catapult made with a ring, a cup to hold an object, and springs.

a. The object loses contact with the cup when the springs are horizontal. Use energy principles to calculate the height h above the XY-plane to which the object flies, when released from the above configuration at rest. Express h as a function of the parameters m, k, l_0 and g.
b. Calculate the work done by the springs during the launch.

3 Representing Orientations in 3-D

In this chapter, we will introduce the representation of vectors in differently oriented coordinate systems and triads in 3-D. First, we will introduce the rotation matrix for a single rotation. Then, successive rotations around multiple axes are described. This leads to Euler Angles, a way of representing the general rotation of a body by means of three successive rotations. Finally, we will show how the principal axis and angle of rotation can provide another representation.

3.1 Rotated Coordinate Systems and Triads

3.1.1 Definition of Coordinate Systems and Triads

We will make a distinction between the terms *coordinate system* and *triad of unit direction vectors* or short *triad* in such a way that a coordinate system requires the definition of axis directions as well as an origin, and the coordinate system changes once this origin displaces. A triad of unit direction vectors, however, only defines axis directions, so a translation of the origin does not create the need for a new triad definition. A triad can therefore be associated with one or more coordinate systems. Note that in this book, we will only consider right-handed coordinate systems and triads.

We will always denote the triad name by a calligraphic capital letter, and the unit vectors by the corresponding lowercase letter, so for example triad $\mathcal{D}$ has the unit direction vectors $\hat{\boldsymbol{d}}_1$, $\hat{\boldsymbol{d}}_2$ and $\hat{\boldsymbol{d}}_3$.

Special, most frequently used triads are inertial and body-fixed triads.

In Figure 3.1, two different coordinate systems are shown. The uppercase XYZ-coordinate system also defines the $\mathcal{N}$-triad. We often use the letter $\mathcal{N}$ to denote an i$\mathcal{N}$ertial triad. The axis directions of this inertial coordinate system or triad are the three unit vectors $\hat{\boldsymbol{n}}_1$, $\hat{\boldsymbol{n}}_2$ and $\hat{\boldsymbol{n}}_3$.

The lowercase xyz-coordinate system defines directions of the $\mathcal{B}$-triad. This *rotated* triad is generally a $\mathcal{B}$ody-fixed triad, which is why we denote it with the letter $\mathcal{B}$. We will do the following derivations using

this letter, but they are equally valid for any other rotated, not body-fixed triad. The axis directions of this rotated coordinate system or triad are the three unit vectors $\hat{\boldsymbol{b}}_1$, $\hat{\boldsymbol{b}}_2$ and $\hat{\boldsymbol{b}}_3$.

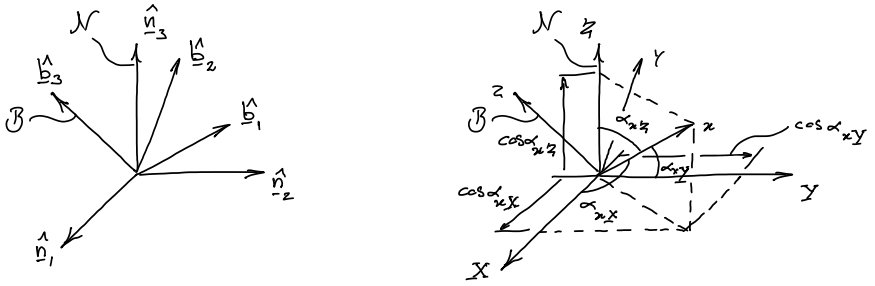


Figure 3.1: Rotation of triads and coordinate systems.

While in this book, we will frequently use more than two triads, there is rarely a need to define a coordinate system for each single triad[1].

3.1.2 Vector Projections

To find the components of a vector $\boldsymbol{r}$ expressed in an inertial triad $\mathcal{N}$, we project the vector onto the triad direction unit vectors. We denote this expression of the vector as ${}^{\mathcal{N}}\boldsymbol{r}$. We can also write down the components of the vector respective to the directions of a rotated triad such as $\mathcal{B}$; in that case we write ${}^{\mathcal{B}}\boldsymbol{r}$.

Example 3.1 If we are interested in finding the projection r_{b1} of $\boldsymbol{r}$ onto the vector $\hat{\boldsymbol{b}}_1$, we need to calculate the scalar product of $\boldsymbol{r}$ and $\hat{\boldsymbol{b}}_1$ (because $\hat{\boldsymbol{b}}_1$ is a unit vector):

$$r_{b1} = \boldsymbol{r} \cdot \hat{\boldsymbol{b}}_1. \tag{3.1E}$$

Note that irrespective of the triad in which it is expressed, the vector $\boldsymbol{r}$ will always be the same vector $\boldsymbol{r}$. Only the components of $\boldsymbol{r}$ (the specific numerical values one uses) depend on the triad in which the vector $\boldsymbol{r}$ is expressed. In other words: Both ${}^{\mathcal{B}}\boldsymbol{r}$ and ${}^{\mathcal{N}}\boldsymbol{r}$ refer to the same vector $\boldsymbol{r}$, but *components are expressed relative to other axis directions.*

Problem 3.1 If the length of a vector ${}^{\mathcal{B}}\boldsymbol{r}$ is known, what is the length of ${}^{\mathcal{N}}\boldsymbol{r}$? ▷

[1] That is because we mostly have coordinates of points described either in an inertial or in a body-fixed coordinate system. In case more is needed than one inertial and one body-fixed coordinate system, we will resort to new names for axes, such as rst or uvw.

Example 3.2 In an inertial coordinate system XYZ defining triad $\mathcal{N}$, consider a vector $\boldsymbol{r}$. Projected to the inertial axes, the vector is given as: ${}^{\mathcal{N}}\boldsymbol{r} = (r_{n1}, r_{n2}, r_{n3})^{\mathrm{T}}$. The vector lies in the XY-plane, so $r_{n3} = 0$.

A second triad $\mathcal{B}$ has its first direction aligned with the vector $\boldsymbol{r}$, and its third direction aligned with $\hat{\boldsymbol{n}}_3$.

First, draw the vector and both triads, in the XY-plane. Then use trigonometry to calculate ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_1$, ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_2$, ${}^{\mathcal{B}}\hat{\boldsymbol{b}}_1$, ${}^{\mathcal{B}}\hat{\boldsymbol{b}}_2$, ${}^{\mathcal{N}}\hat{\boldsymbol{n}}_1$ and ${}^{\mathcal{N}}\hat{\boldsymbol{n}}_2$.

Exemplary solution

The vector and the triads are depicted in Figure 3.2.

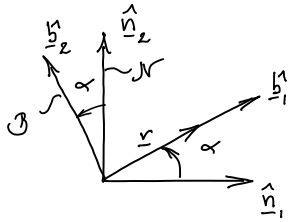


Figure 3.2: A vector and two triads.

From trigonometry we obtain

$$\alpha = \arctan\left(\frac{r_{n2}}{r_{n1}}\right). \tag{3.2E}$$

We can express the requested vectors with this angle as:

$${}^{\mathcal{N}}\hat{\boldsymbol{b}}_1 = \begin{pmatrix}\cos\alpha\\ \sin\alpha\\ 0\end{pmatrix}, {}^{\mathcal{N}}\hat{\boldsymbol{b}}_2 = \begin{pmatrix}-\sin\alpha\\ \cos\alpha\\ 0\end{pmatrix}, {}^{\mathcal{B}}\hat{\boldsymbol{b}}_1 = {}^{\mathcal{N}}\hat{\boldsymbol{n}}_1 = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}, {}^{\mathcal{B}}\hat{\boldsymbol{b}}_2 = {}^{\mathcal{N}}\hat{\boldsymbol{n}}_2 = \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}. \tag{3.3E}$$

An arbitrary vector $\boldsymbol{r}$ can be split into its components in the XYZ directions:

$${}^{\mathcal{N}}\boldsymbol{r} = r_{n1}\ {}^{\mathcal{N}}\hat{\boldsymbol{n}}_1 + r_{n2}\ {}^{\mathcal{N}}\hat{\boldsymbol{n}}_2 + r_{n3}\ {}^{\mathcal{N}}\hat{\boldsymbol{n}}_3 \tag{3.4}$$

Where ${}^{\mathcal{N}}\hat{\boldsymbol{n}}_1, {}^{\mathcal{N}}\hat{\boldsymbol{n}}_2$ and ${}^{\mathcal{N}}\hat{\boldsymbol{n}}_3$ represent the unit vectors in the three directions of the inertial coordinate system (expressed with components in that triad) and r_{n1}, r_{n2}, r_{n3} the components (projections) of $\boldsymbol{r}$ in the respective directions. The equivalent shorter representation of (3.4) is:

$${}^{\mathcal{N}}\boldsymbol{r} = \begin{pmatrix} r_{n1} & r_{n2} & r_{n3}\end{pmatrix}^{\mathrm{T}} \tag{3.5}$$

The vector $\boldsymbol{r}$ can alternatively be expressed using its projections on the $\mathcal{B}$ directions. This representation of the vector is denoted as ${}^{\mathcal{B}}\boldsymbol{r}$.

Then, the components are the projections of the vector $\boldsymbol{r}$ in the axis directions of the $\mathcal{B}$-triad, or

$$ {}^{\mathcal{B}}\boldsymbol{r} = \begin{pmatrix} r_{b1} \\ r_{b2} \\ r_{b3} \end{pmatrix} = r_{b1}\, {}^{\mathcal{B}}\hat{\boldsymbol{b}}_1 + r_{b2}\, {}^{\mathcal{B}}\hat{\boldsymbol{b}}_2 + r_{b3}\, {}^{\mathcal{B}}\hat{\boldsymbol{b}}_3. \tag{3.6} $$

Note again that it is in fact irrelevant in which triad we express the vectors, as long as we use the same triad for the projections on the left and on the right side of an equation.

Both representations, using one set or the other set of unit vectors, must give the same result, so we can write

$$ {}^{\mathcal{N}}\boldsymbol{r} = r_{n1}\, {}^{\mathcal{N}}\hat{\boldsymbol{n}}_1 + r_{n2}\, {}^{\mathcal{N}}\hat{\boldsymbol{n}}_2 + r_{n3}\, {}^{\mathcal{N}}\hat{\boldsymbol{n}}_3 = r_{b1}\, {}^{\mathcal{N}}\hat{\boldsymbol{b}}_1 + r_{b2}\, {}^{\mathcal{N}}\hat{\boldsymbol{b}}_2 + r_{b3}\, {}^{\mathcal{N}}\hat{\boldsymbol{b}}_3, \tag{3.7} $$

or with everything expressed in triad $\mathcal{B}$:

$$ {}^{\mathcal{B}}\boldsymbol{r} = r_{n1}\, {}^{\mathcal{B}}\hat{\boldsymbol{n}}_1 + r_{n2}\, {}^{\mathcal{B}}\hat{\boldsymbol{n}}_2 + r_{n3}\, {}^{\mathcal{B}}\hat{\boldsymbol{n}}_3 = r_{b1}\, {}^{\mathcal{B}}\hat{\boldsymbol{b}}_1 + r_{b2}\, {}^{\mathcal{B}}\hat{\boldsymbol{b}}_2 + r_{b3}\, {}^{\mathcal{B}}\boldsymbol{b}_3. \tag{3.8} $$

◯ **Problem 3.2** Given a vector ${}^{\mathcal{N}}\boldsymbol{r} = (0,\, 2,\, 4)^{\mathrm{T}}$, with components expressed in the $\mathcal{N}$-triad, use projection with the scalar product to calculate the component r_{b1} of this vector that points in the $\hat{\boldsymbol{b}}_1$-direction of a different triad $\mathcal{B}$, with ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_1 = (0,\, \frac{1}{2},\, \frac{1}{2}\sqrt{3})^{\mathrm{T}}$. Make a drawing and verify your result via trigonometry.

Showing that a vector component is a projection onto a given unit vector requires adding a subscript. In case the vector already has a subscript, this can become cumbersome. In such cases, it is recommended to define a new vector name without subscript. Details and conventions used in this book are summarized in Appendix C.

3.1.3 Rotation Matrices

In the following, we aim to find a systematic way to determine the relationship between the different representations ${}^{\mathcal{N}}\boldsymbol{r}$ and ${}^{\mathcal{B}}\boldsymbol{r}$ of the same vector $\boldsymbol{r}$.

By re-writing (3.8) in matrix format, we see that we can directly map the components of the vector from one triad to the other via matrix multiplication:

$$ {}^{\mathcal{B}}\boldsymbol{r} = \begin{pmatrix} {}^{\mathcal{B}}\hat{\boldsymbol{n}}_1 & {}^{\mathcal{B}}\hat{\boldsymbol{n}}_2 & {}^{\mathcal{B}}\hat{\boldsymbol{n}}_3 \end{pmatrix} \begin{pmatrix} r_{n1} \\ r_{n2} \\ r_{n3} \end{pmatrix} = \begin{pmatrix} {}^{\mathcal{B}}\hat{\boldsymbol{b}}_1 & {}^{\mathcal{B}}\hat{\boldsymbol{b}}_2 & {}^{\mathcal{B}}\hat{\boldsymbol{b}}_3 \end{pmatrix} \begin{pmatrix} r_{b1} \\ r_{b2} \\ r_{b3} \end{pmatrix}, \tag{3.9} $$

or shorter (note that the second matrix is the identity matrix):

$$ {}^{\mathcal{B}}\boldsymbol{r} = \begin{pmatrix} {}^{\mathcal{B}}\hat{\boldsymbol{n}}_1 & {}^{\mathcal{B}}\hat{\boldsymbol{n}}_2 & {}^{\mathcal{B}}\hat{\boldsymbol{n}}_3 \end{pmatrix} {}^{\mathcal{N}}\boldsymbol{r}. \tag{3.10} $$

By defining the matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$, we can write this as

$$ {}^{\mathcal{B}}\boldsymbol{r} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} {}^{\mathcal{N}}\boldsymbol{r}. \tag{3.11} $$

The matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$ is called a *rotation matrix*. It describes the orientation of the two triads with respect to each other.

The vector $\boldsymbol{r}$ stays the same and is not rotated; its components are only expressed in a different (rotated) triad[2].

The components of the rotation matrix are the components of a triad's unit vectors expressed in the respective other triad. To calculate the components of ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_1$, we project onto the directions $\hat{\boldsymbol{n}}_1$, $\hat{\boldsymbol{n}}_2$, and $\hat{\boldsymbol{n}}_3$. Projections are performed using the scalar product with the respective unit vectors, so

$$ {}^{\mathcal{N}}\hat{\boldsymbol{b}}_1 = \begin{pmatrix} \hat{\boldsymbol{b}}_1 \cdot \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{b}}_1 \cdot \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{b}}_1 \cdot \hat{\boldsymbol{n}}_3 \end{pmatrix} \tag{3.12} $$

As both the unit vectors of the rotated triad and the unit vectors in the inertial triad have magnitude 1, their scalar products equal the cosine of the angle they enclose (Figure 3.1).

Example 3.3 The projection of the first unit vector of triad $\mathcal{B}$ onto the second direction of triad $\mathcal{N}$ is

$$ \hat{\boldsymbol{b}}_1 \cdot \hat{\boldsymbol{n}}_2 = \cos\alpha_{xY}. \tag{3.13E} $$

Evaluating the scalar product, as in Example 3.3, for all components of the unit vector in (3.12) leads to the complete expression:

$$ {}^{\mathcal{N}}\hat{\boldsymbol{b}}_1 = \begin{pmatrix} \cos\alpha_{xX} \\ \cos\alpha_{xY} \\ \cos\alpha_{xZ} \end{pmatrix}. \tag{3.14} $$

We saw in (3.14) how the first unit vector of the $\mathcal{B}$-triad can be written in terms of the $\mathcal{N}$-directions. For the second and third directions, this

[2] Be aware that in the literature, often also a different definition of rotation matrices is used, namely where the vector *is* rotated, and the triad stays the same. The definitions are *not* equivalent and will in fact lead to matrices that are the transpose of one another.

can be done similarly. The same can be done inversely to write the unit vectors of the $\mathcal{N}$-triad in terms of the $\mathcal{B}$-directions. Then, following (3.10), the elements of ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$ are:

$$ {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = \begin{pmatrix} \cos\alpha_{Xx} & \cos\alpha_{Yx} & \cos\alpha_{Zx} \\ \cos\alpha_{Xy} & \cos\alpha_{Yy} & \cos\alpha_{Zy} \\ \cos\alpha_{Xz} & \cos\alpha_{Yz} & \cos\alpha_{Zz} \end{pmatrix} . \tag{3.15} $$

The cosines are called direction cosines for their respective axes; for example, the term $\cos\alpha_{Yx}$, the cosine of the angle between the positive Y- and x-axis directions, is called a direction cosine for these two axes. Therefore, the matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$, which contains all these cosines, is also called the *direction cosine matrix*.

To shortcut notation in this book, rotation matrices are sometimes written just in terms of the axis about which the rotation takes place, so if we rotate about an axis pointing in the first direction of a triad, the rotation matrix is denoted $\mathbf{C}_X$.

Example 3.4 A triad $\mathcal{B}$ is rotated with respect to a triad $\mathcal{N}$ by an angle φ about the X-axis, as illustrated in Figure 3.3.

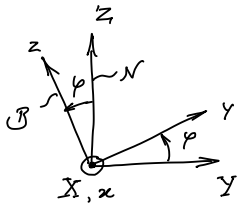


Figure 3.3: Rotation of a coordinate triad about the X-axis.

By evaluating cosines of all angles between the axes in (3.15), we obtain:

$$ {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = \mathbf{C}_X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix} . \tag{3.16E} $$

Note that this notation is only this book's definition of $\mathbf{C}_X$, not necessarily a general definition.

By inspecting (3.15) further and comparing with (3.14), we notice that the rows of ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$ contain the (transposed) unit vectors $\hat{\boldsymbol{b}}_1$, $\hat{\boldsymbol{b}}_2$ and $\hat{\boldsymbol{b}}_3$ expressed in the inertial $\mathcal{N}$-triad. That means the rotation matrix

can be written as

$$ {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = \begin{pmatrix} {}^{\mathcal{N}}\hat{\boldsymbol{b}}_1^{\mathrm{T}} \\ {}^{\mathcal{N}}\hat{\boldsymbol{b}}_2^{\mathrm{T}} \\ {}^{\mathcal{N}}\hat{\boldsymbol{b}}_3^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} {}^{\mathcal{B}}\hat{\boldsymbol{n}}_1 & {}^{\mathcal{B}}\hat{\boldsymbol{n}}_2 & {}^{\mathcal{B}}\hat{\boldsymbol{n}}_3 \end{pmatrix}. \tag{3.17} $$

So, by now, we have three alternative ways of constructing a rotation matrix: One of the two in (3.17), or evaluating all direction cosines in (3.15).

Example 3.5 Consider again Example 3.4, the case when a certain triad $\mathcal{B}$ is rotated with respect to a triad $\mathcal{N}$ by an angle φ about the X-axis. We can alternatively construct the rotation matrix $\mathbf{C}_X$ in (3.16E) from the unit vectors ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_1 = (1,\, 0,\, 0)^{\mathrm{T}}$, ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_2 = (0,\, \cos\varphi,\, \sin\varphi)^{\mathrm{T}}$, and ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_3 = (0,\, -\sin\varphi,\, \cos\varphi)^{\mathrm{T}}$.

Problem 3.3 Given a rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$ as

$$ {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.18P} $$

state the numerical components of the unit vectors ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_1$, ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_2$, and ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_3$ that point in x, y, and z-direction of the $\mathcal{B}$-triad, respectively. ▷

Problem 3.4 What is the magnitude of each column of a rotation matrix? ▷

Problem 3.5 Which result do you obtain if you calculate the scalar product of any two columns of a rotation matrix? ▷

Problem 3.6 If you know the first two columns of a rotation matrix, how can you calculate the third? ▶

Problem 3.7 Show that the rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = \mathbf{C}_Z$ for a rotation of triad $\mathcal{B}$ with respect to a triad $\mathcal{N}$ by an angle φ about the Z-axis is

$$ \mathbf{C}_Z = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.19P} $$

Start by drawing directions of both triads and the angle φ, projected into the XY-plane of the $\mathcal{N}$-triad (let the Z-axis point towards you). ▶

Problem 3.8 Determine ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = \mathbf{C}_Y$ when triad $\mathcal{B}$ is rotated with respect to a triad $\mathcal{N}$ by an angle φ about the Y-axis. Start by drawing the axis directions of both triads and the angle φ, projected into the XZ-plane of the $\mathcal{N}$-triad (let the Y-axis point towards you). ▶

3.1.4 Inverse Rotation Matrices

To obtain the rotation matrix for the inverse mapping, so ${}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}}$, we need to exchange all axis labels of the angles in the rotation matrix in (3.15). When this is performed, one obtains the transpose of ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$.

An important conclusion from these considerations is that the inverse of any rotation matrix equals its transpose:

$$ {}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}^{-1} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}^{\mathrm{T}}. \tag{3.20} $$

This is the defining property of an *orthogonal matrix*. Orthogonal matrices consist only of unit vectors of an orthogonal triad (the columns or rows are orthogonal to each other, and they have unit magnitude).

Problem 3.9 What is the result of the multiplication of an arbitrary rotation matrix with its transpose? ▷

3.2 Successive Rotations

3.2.1 Multiplication of Rotation Matrices

For a successive rotation from a triad $\mathcal{N}$ to a triad $\mathcal{B}$ via an intermediate triad $\mathcal{F}$, we can apply a mapping as in (3.11) twice in succession:

$$ {}^{\mathcal{F}}\boldsymbol{r} = {}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}}\,{}^{\mathcal{N}}\boldsymbol{r}, \qquad {}^{\mathcal{B}}\boldsymbol{r} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}\,{}^{\mathcal{F}}\boldsymbol{r}. \tag{3.21} $$

From this, we find that

$$ {}^{\mathcal{B}}\boldsymbol{r} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}\,{}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}}\,{}^{\mathcal{N}}\boldsymbol{r}, \tag{3.22} $$

which means that we can find a new mapping matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}\,{}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}}$, omitting the intermediate step via $\mathcal{F}$.

So, we see that successive rotations are nothing but a multiplication of the rotation matrices. We know that for matrix multiplications, the order matters; they are not commutative. In other words, in contrast to order-independent scalar products, the product $\mathbf{AB}$ of two matrices $\mathbf{A}$ and $\mathbf{B}$ is not necessarily the same as $\mathbf{BA}$.

Therefore, the order in which we multiply the rotation matrices is crucial; generally ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}\,{}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}} \neq {}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}}\,{}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}$. Only in some special cases, the sequence does not matter, for example when rotating about the same axis.

◯ **Problem 3.10** Take this book and draw a coordinate system on its cover, with x pointing right, y pointing up, and z pointing towards you. Then, take the book and rotate it by 90 degrees in positive direction about the x-axis. Afterwards, rotate it by another 90 degrees in positive direction about the current z-axis.

Bring the book back to its original position and repeat the procedure, but now exchange the order: start with z and then proceed with x. Do you get the same result?

◯ **Problem 3.11** Given are two rotation matrices ${}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}}$ and ${}^{\mathcal{F}}\mathbf{C}_{\mathcal{B}}$ as well as a position vector ${}^{\mathcal{B}}\boldsymbol{r}_{P/O}$ of a point P with respect to an origin O of a fixed coordinate system. Express the position vector ${}^{\mathcal{N}}\boldsymbol{r}_{P/O}$ as a function of ${}^{\mathcal{B}}\boldsymbol{r}_{P/O}$, ${}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}}$, and ${}^{\mathcal{F}}\mathbf{C}_{\mathcal{B}}$. ▷

3.2.2 Euler Angles

To describe the general orientation of an object or triad in 3-D, we often use a succession of three rotations about given coordinate axes, where these axes rotate with the object or triad. Euler showed that three is the minimal number to account for all possible rotations (see Section 18.1 for a proof), and the set of such three angles is called *Euler angles*. A convention is to use one of two specific sequences called *yaw-pitch-roll* (Z-Y-X) and *precession-nutation-spin* (Z-X-Z).

The set of yaw-pitch-roll angles, also called type I Euler angles or Z-Y-X, first rotates about the original Z-axis by the *yaw* angle ψ ("psi"), second about the (now rotated) y'-axis by the *pitch* angle θ ("theta"), and third about the body-fixed $x = x''$-axis by the *roll* angle φ ("phi").

An intuitive way of representing successive triad rotations by Euler angles are *cans-in-series* [62]. For type I, these are shown in Figure 3.4 and in more detail in Figure 3.5 for the airplane example. Note that in Figure 3.4 the right-hand rule of Figure 1.3 determines positive rotation direction.

Type I is frequently used in aerospace applications and in combination with a *NORTH-EAST-DOWN* (NED) coordinate system. In the initial configuration, so when all Euler angles are zero, a plane's nose points in direction of North (X), while the right wing points East (Y). The Z-axis points downwards (DOWN). This Z-axis is also the yaw axis of

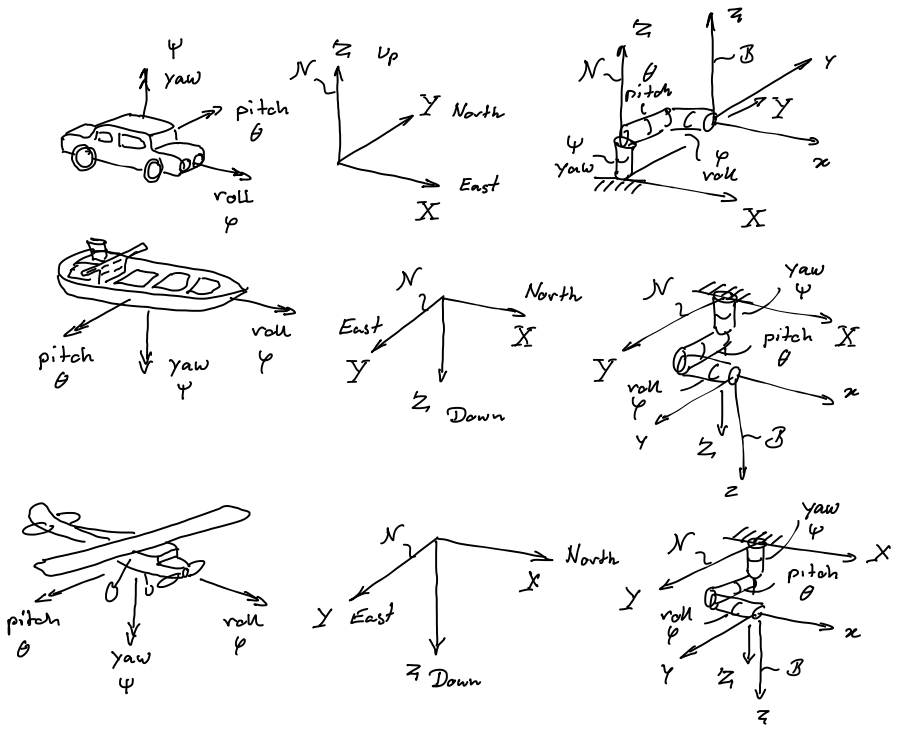


Figure 3.4: Typical coordinate systems and yaw-pitch-roll Euler angles for a car, a ship, and an airplane. Left: Body-fixed axis directions. Center: ENU and NED inertial coordinate systems. Right: Cans-in-series representation of yaw-pitch-roll.

the plane, the y'-axis is the pitch axis, and the body-fixed $x'' = x$-axis is the plane's roll axis. Type I Euler angles are also frequently used in combination with an *EAST-NORTH-UP* (ENU) coordinate system, for example for automobiles. Also in this case, x points in forward direction seen from the vehicle, only this direction points EAST when all Euler angles are zero. Type I Euler angles are usually restricted to the interval $[0,\, 2\pi)$ for yaw and roll, and to $[-\pi/2,\, \pi/2]$ for pitch.

We can construct the overall rotation matrix as a function of the Euler angles from the individual successive rotations. To this end, we define two more intermediate triads between the inertial triad $\mathcal{N}$ and the body-fixed triad $\mathcal{B}$: triad $\mathcal{F}$, which is rotated only by the yaw angle with respect to $\mathcal{N}$, and triad $\mathcal{G}$, which is rotated by the pitch angle with respect to $\mathcal{F}$. In order to map from $\mathcal{N}$ to $\mathcal{B}$, we multiply the three

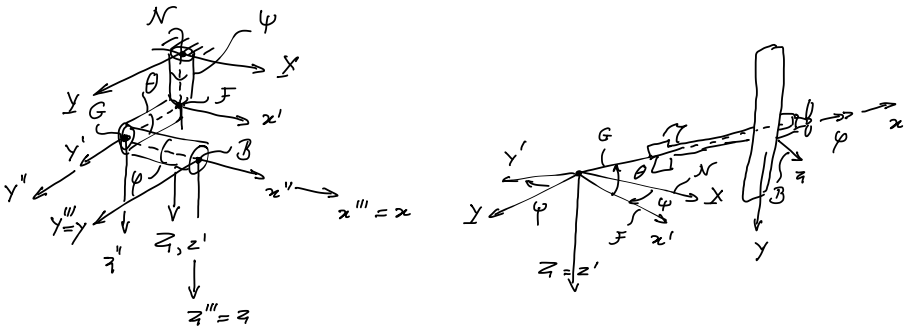


Figure 3.5: Cans-in-series for the airplane

individual rotation matrices in the correct order:

$$
\begin{aligned}
{}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} &= {}^{\mathcal{B}}\mathbf{C}_{\mathcal{G}}(\varphi)\,{}^{\mathcal{G}}\mathbf{C}_{\mathcal{F}}(\theta)\,{}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}}(\psi) && (3.23)\\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} && (3.24)
\end{aligned}
$$

and obtain the overall rotation matrix

$$
{}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = \begin{pmatrix} \cos\psi\cos\theta & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\varphi\sin\theta - \cos\varphi\sin\psi & \cos\varphi\cos\psi + \sin\varphi\sin\psi\sin\theta & \cos\theta\sin\varphi \\ \sin\varphi\sin\psi + \cos\varphi\cos\psi\sin\theta & \cos\varphi\sin\psi\sin\theta - \cos\psi\sin\varphi & \cos\varphi\cos\theta \end{pmatrix}. \tag{3.25}
$$

For the precession-nutation-spin convention, also called type II Euler angles or *classical Euler angles*, first rotation is *precession* by angle φ about the Z-axis, second is *nutation* by angle θ about the intermediate x'-axis, and third is *spin* by angle ψ about the body-fixed $z'' = z$-axis. The resulting rotation matrix for classical Euler angles is:

$$
{}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = \begin{pmatrix} \cos\varphi\cos\psi - \sin\varphi\cos\theta\sin\psi & \sin\varphi\cos\psi + \cos\varphi\cos\theta\sin\psi & \sin\theta\sin\psi \\ -\cos\varphi\sin\psi - \sin\varphi\cos\theta\cos\psi & -\sin\varphi\sin\psi + \cos\varphi\cos\theta\cos\psi & \sin\theta\cos\psi \\ \sin\varphi\sin\theta & -\cos\varphi\sin\theta & \cos\theta \end{pmatrix}. \tag{3.26}
$$

Problem 3.12 Show (3.26) by multiplication of the individual matrices.

This set of angles is often used for axially symmetric bodies that rotate mainly about their symmetry (spin) axis, such as tops and gyroscopes. It is also used to describe the rotation of our planet Earth. Type II Euler angles are usually restricted to the interval $[0, 2\pi)$ for precession and spin, and to $[0, \pi]$ for nutation.

A shortcoming of Euler angle sets is that all of them exhibit *singularities*. A singularity occurs whenever two of the rotation axes coincide. Then, it is not possible to reconstruct a unique set of Euler angles from a particular orientation of a body. For example, if a top is described by type II Euler angles and it is standing perfectly upright such that Z and z coincide, one cannot distinguish between precession and spin. One solution to this problem is to choose a set of Euler angles that exhibits no singularities during a particular movement of a body, or, if this is not possible, to switch to another set of Euler angles whenever a singularity is approached.

Example 3.6 A thin disk is rolling on the ground, as depicted in Figure 3.6. The disk has radius r, mass m and is subjected to a gravity with field strength g in the negative Z direction. The disk rolls without slipping on the XY-plane and contacts the ground in point C.

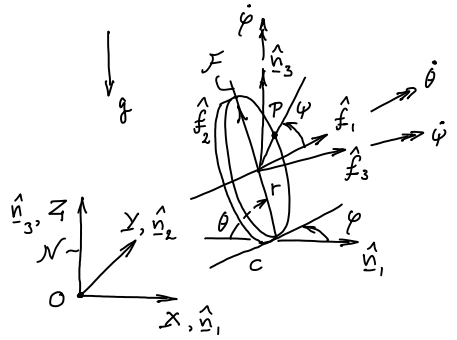


Figure 3.6: A free rolling disk.

The figure indicates an inertial XYZ coordinate system (associated with a triad $\mathcal{N}$), and a rotating triad $\mathcal{F}$, such that $\hat{\boldsymbol{f}}_3$ is orthogonal to the disk, and $\hat{\boldsymbol{f}}_1$ is horizontal. The orientation of the disk is described by the Euler angles φ,θ,ψ, as depicted in the figure. Point P is fixed to the disk.

What is the sequence of single-axis rotations for the disk, as depicted in Figure 3.6, with the given Euler angles φ,θ,ψ?

Exemplary solution

The figure shows the sequence Z-X-Z, because φ describes rotation about the inertial Z-axis (precession), θ describes rotation about the resulting new x'-axis (nutation), and ψ describes spin about the symmetry axis.

When visualizing mechanical system in three-dimensional space, it is often helpful to draw *orthographic projections* into intermediate triads, not just body-fixed or inertial triads.

Problem 3.13 The orientation of a cube (Figure 3.7) with respect to the world is described by yaw-pitch-roll Euler angles ψ, θ, φ. When all Euler angles are zero, the body-fixed triad $\mathcal{B}$ and the inertial triad $\mathcal{N}$ are the same.

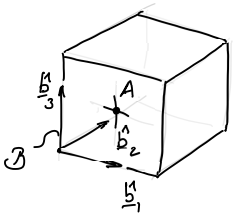


Figure 3.7: A cube with a point A marked on it.

The drawings in Figure 3.8 show different *orthographic projections* of the same cube. Two intermediate triads are defined: triad $\mathcal{F}$, which performs only yaw with respect to $\mathcal{N}$, and triad $\mathcal{G}$, which performs yaw and pitch with respect to $\mathcal{N}$.

Complete each one of these projections by drawing and labeling

1. all direction vectors of the two triads $\mathcal{N}$ and $\mathcal{B}$ that are contained within the paper plane or that are orthogonal to it. Do not draw or label any further direction vectors.
2. The spring attachment point A on the cube. You do not need to draw the point precisely, but it must be clearly visible whether it is to the left, to the right, or *on* any given line in the projection.

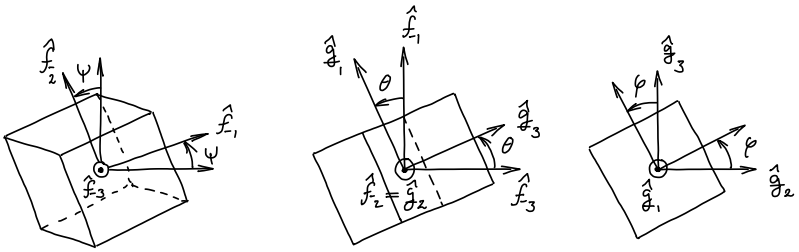


Figure 3.8: Cartoon template showing three different views of the cube.

▶

Problem 3.14 Figure 3.9 shows the unit vectors of a triad $\mathcal{N}$ (leftmost picture) and a triad $\mathcal{B}$ (rightmost picture).

Find the type I Euler angles ψ, θ, φ that describe the three successive rotations from $\mathcal{N}$ to $\mathcal{B}$. Justify your response by completing the two middle pictures of the cartoon.

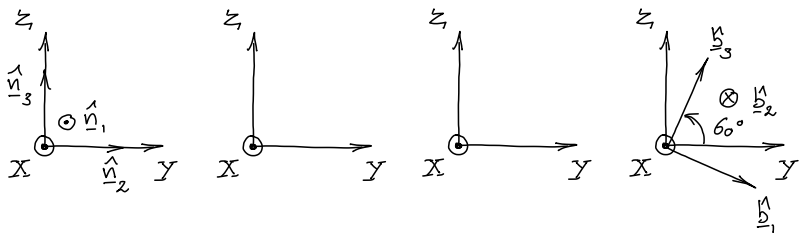


Figure 3.9: Cartoon template for the rotated triads.

▶

◯ **Problem 3.15** Write down the rotation matrices for the experiment in Problem 3.10 and generate the combined rotation matrix, via matrix multiplication. Check the expected orientation of the coordinate directions with the outcome of your experiment.

3.2.3 Euler's Rotation Theorem: Principal Axis of Rotation

Euler's rotation theorem states that the most general displacement of a rigid body *when a fixed point exists* is equivalent to a single rotation about some axis through that point [15].

So, all rotations of a body combined can be described as one single rotation around one axis by a unique angle. This means that the matrices we calculated in the previous section, via multiplications of individual rotation matrices, are rotation matrices themselves.

The axis and the angle of the overall rotation are called the *principal axis* and the *principal angle* of rotation, respectively. The direction of the principal axis of rotation is expressed by the unit vector $\hat{\boldsymbol{a}}$.

As the vector $\hat{\boldsymbol{a}}$ points in direction of the axis about which one triad $\mathcal{B}$ is rotated with respect to another triad $\mathcal{N}$, this vector itself does not change its direction during the rotation. This means that the vector components are identical in both triads, so

$$^{\mathcal{N}}\hat{\boldsymbol{a}} = {}^{\mathcal{B}}\hat{\boldsymbol{a}}. \tag{3.27}$$

This implies that

$$^{\mathcal{B}}\hat{\boldsymbol{a}} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}\,{}^{\mathcal{N}}\hat{\boldsymbol{a}} = {}^{\mathcal{N}}\hat{\boldsymbol{a}}. \tag{3.28}$$

We recall that a vector that does not change its direction when it is pre-multiplied by a matrix (that is not the identity matrix) is called an eigenvector of this matrix (see Appendix A.6). The change in length is given by the associated eigenvalue. In this case of a rotation matrix, the

vector does not change its length. Therefore, the direction unit vector $\hat{\boldsymbol{a}}$ has to be an eigenvector of the rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$, with eigenvalue 1.

To shortcut notation, rotation matrices are sometimes written just in terms of the axis and the angle about which is rotated, so if we rotate about the $\hat{\boldsymbol{a}}$-axis with an angle φ in positive direction, the rotation matrix is denoted as $\mathbf{C}(\hat{\boldsymbol{a}}, \varphi)$.

Problem 3.16 This Problem is about the two triads of Problem 3.3 on page 77. For the rotation matrix ${}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}}$ in (3.18P), determine the components of a unit vector ${}^{\mathcal{N}}\hat{\boldsymbol{a}}$ in direction of the principal axis of rotation, and the principal angle of rotation φ_a. State only numerical values. ▷

Problem 3.17 A nice visualization of Euler's rotation theorem is the *football theorem* [17]: In a soccer match, at the beginning of each half time, the ball is always placed on the center spot with its center at exactly the same location. How many points on the ball's surface are then certainly at the exact same location at the beginning of the second half time as they were at the beginning of the first? ▷►

Problem 3.18 Show that the principal angle of rotation can be calculated from the trace $\mathrm{tr}(\mathbf{C})$ of a rotation matrix $\mathbf{C}$ as

$$1 + 2\cos(\varphi) = \mathrm{tr}(\mathbf{C}). \tag{3.29P}$$

Problem 3.19 One of the eigenvalues of a general rotation matrix is 1, with the eigenvector the axis of rotation. What are the two other eigenvalues and eigenvectors? Interpret that result.

3.2.4 Screw Axis

Euler's Theorem is a special case of a more general kinematic theorem, which is often referred to as *Chasles' Theorem* and which treats displacements of a rigid body *without* a fixed point.

Michel Chasles (1793-1880) stated in 1830 (originally in French) that "one can always transport a rigid body from one configuration to any arbitrary other configuration by a continuous movement of a screw to which the body would be rigidly attached" [11]. In other words, whichever movement a rigid body undergoes from one configuration to another, one can always find a straight line or *screw axis* that is rigidly attached to the rigid body and which has the same location in the initial and in the final configuration [11]. This line or axis is sometimes also called *Mozzi's axis*, after Giulio Mozzi (1730-1813), who came to the same conclusion already in 1763 [9, 47].

So, even without a fixed point, the net displacement of an arbitrary rigid body can still be represented by a single rotation about a specific axis, only we have to add a translation along this axis.

Euler's Theorem is a special case of Mozzi-Chasles's Theorem where the translation along the axis is zero.

Rotation matrices or Euler angles only account for rotations of axis directions, they neglect translations of coordinate systems. This does not matter for the purpose of mapping vector components, because the projections of vector components in particular directions of a coordinate system are independent of a possible shift of origin.

3.3 Checking Plausibility and Frequent Mistakes

To check one's calculation of a specific rotation matrix, one can make use of Section 1.4, but also of the special properties of this class of square matrices. For example, one can:

- check orthogonality of rows and columns.
- check that rows and columns all have unit length.
- check right-handedness of the triad defined by the rows/columns.
- for successive rotations, check special cases where some angles are zero.
- visualize the system (Euler angles, cans-in-series).
- imagine taking the perspective of the new triad.
- look at how special vectors look in one triad and another, e.g. direction vectors of the coordinate axes.
- attempt to map in reverse direction using the transpose.

Particularly frequent mistakes are

- adding vectors with components expressed in two different triads. This mistake is easy to prevent by always indicating which projections are considered by a top left letter.
- choosing an incorrect order for multiplications of rotation matrices. This can be remedied by always including explicit triad labels, and checking that during the multiplication of two matrices, the right subscript equals the left superscript next to it.

3.4 Summary

A coordinate system is defined by its axis directions and a location of its origin. We use only right-handed orthogonal coordinate systems. A triad is defined only by its directions.

Rotation matrices map the components of a vector $\boldsymbol{r}$ from one triad to another:

$$^{\mathcal{B}}\boldsymbol{r} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}\, {}^{\mathcal{N}}\boldsymbol{r}. \tag{3.30}$$

Note that $\boldsymbol{r}$ remains the same vector, only its components are expressed with respect to different coordinate directions. The rows of the rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$ contain the unit vectors of the $\mathcal{B}$-triad expressed in the $\mathcal{N}$-triad, and the columns contain the unit vectors of the $\mathcal{N}$-triad expressed in the $\mathcal{B}$-triad. The inverse of a rotation matrix is equal to its transpose.

Euler's rotation theorem states that whenever the movement of a rigid body exhibits one fixed point, the movement of this body can be described by a single rotation about an axis through that point. This axis is called the principal axis of rotation. Chasles' theorem extends this further: Any general movement of a rigid body can be described by a rotation about one single equivalent axis in combination with a translation along this axis.

Alternative to the principal axis of rotation, any rotation can also be represented by a sequence of three rotations by so-called Euler angles about three defined axis directions. Frequent conventions are Type I (Z-Y-X, associated with yaw-pitch-roll) and Type II (Z-X-Z, associated with precession-nutation-spin). A shortcoming of Euler angles is that they exhibit singularities.

Following Euler, an arbitrary sequence of rotations can always be represented by one single rotation about the principal axis of rotation. Therefore, the mapping found for successive rotations, which results from multiplication of individual rotation matrices, for example

$${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{G}} \cdot {}^{\mathcal{G}}\mathbf{C}_{\mathcal{F}} \cdot {}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}}, \tag{3.31}$$

is also again a rotation matrix. Note that the order of multiplications *is* relevant.

3.5 Problems

3.5.1 Guided Problems

Problem 3.20 We study the rotation of a book (Figure 3.10). The book is initially on its flat front side (so the back side is shown in the figure) on the music shelf of a grand piano. Then, the pianist picks up the book and places it on the music rack, upright with the book front facing him. The music rack encloses an angle of 60° with the music shelf.

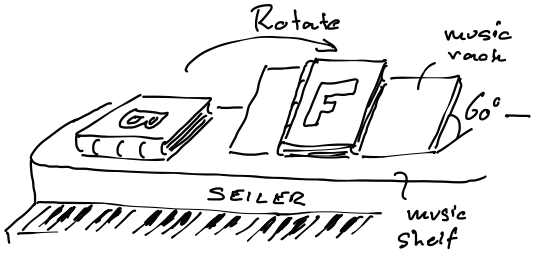


Figure 3.10: Book rotation.

a. Define axes directions for the book and for the grand piano.
The grand piano-fixed coordinate system should be denoted by XYZ coordinates, which define a triad $\mathcal{N}$. The book-fixed triad should be denoted by uvw coordinates, which define a triad $\mathcal{B}$. Choose the coordinate systems such that the book-fixed coordinate system coincides with the grand piano-fixed coordinate system in the initial configuration, so as long as the book is on the music shelf.
Draw both your coordinate systems onto Figure 3.10, when the book is on the rack. Also draw a 2-D (orthographic) projection of these coordinate systems and the book on the rack, in such a way that all axis directions are either contained in the paper plane or orthogonal to it. Ignore the translation of the book's center.

b. Use your orthographic drawing and trigonometry to determine the unit axis direction vectors ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_1, {}^{\mathcal{N}}\hat{\boldsymbol{b}}_2, {}^{\mathcal{N}}\hat{\boldsymbol{b}}_3$ of the $\mathcal{B}$-triad, with components expressed in the $\mathcal{N}$-triad, when the book is on the rack.

c. Use your orthographic drawing and trigonometry to determine the unit axis direction vectors ${}^{\mathcal{B}}\hat{\boldsymbol{n}}_1, {}^{\mathcal{B}}\hat{\boldsymbol{n}}_2, {}^{\mathcal{B}}\hat{\boldsymbol{n}}_3$ of the $\mathcal{N}$-triad, with components expressed in the $\mathcal{B}$-triad, when the book is on the rack.

d. Determine the rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$ that describes the transformation from XYZ coordinates to uvw coordinates. Do this by using the answers to parts b and c. Note that you can construct the matrix in two different ways. Do both, and check consistency of your results.

e. Determine a sequence of yaw-pitch-roll rotations which describes the

rotation of the book from shelf to rack. Draw a cans-in-series representation in initial and final configuration, and give the sequence of rotations in terms of axes, and the value of the angle for each rotation. Define names for all intermediate triads. (Optionally, challenge yourself by making sure you do not exceed the usual angle ranges.)

f. Draw the unit vectors of the coordinate axes before and after each of the successive rotations by the yaw-pitch-roll Euler angles (the leftmost picture contains the coordinate directions of the grand piano, the second picture shows only yaw, etc., until the rightmost picture contains the final coordinate directions of the $\mathcal{B}$-triad when the book is on the rack).
Figure 3.11 shows an (incomplete) example of how this could look. Create a similar cartoon, but with axes directions following your own orthographic projection, and complete it.

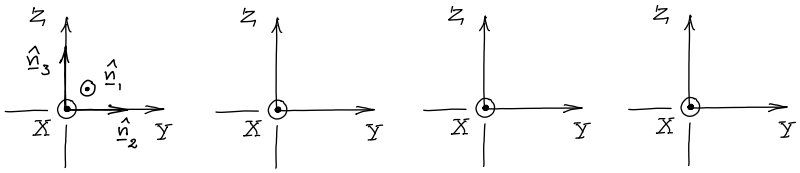


Figure 3.11: Incomplete example for Euler angle cartoon.

g. Using your answer to part e, determine the rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$. Do this by multiplying the individual rotation matrices for the intermediate rotations. For each of these intermediate rotation matrices, include appropriate sub- and superscripts using your triad names. Compare your result with the matrix you calculated in d.

h. Determine a sequence of precession-nutation-spin rotations which describes the rotation of the book from shelf to rack. Draw a cans-in-series representation in initial and final configuration, and give the sequence of rotations in terms of axes, and the value of the angle for each rotation. Define names for all intermediate triads. (Optionally, challenge yourself by making sure you do not exceed the usual angle ranges.)

i. Draw again the unit vectors of the coordinate axes before and after each of the successive rotations by the three Euler angles (the leftmost picture contains the coordinate directions of the grand piano, the second picture shows only precession, etc., until the rightmost picture contains the final coordinate directions of the $\mathcal{B}$-triad when the book is on the rack).

j. Using your answer to part h, determine again the rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$. Do this by multiplying the individual rotation matrices for the intermediate rotations. For each of these intermediate rotation

matrices, include appropriate sub- and superscripts using your triad names. Compare your result with the matrices you calculated in parts d and g.

k. According to Euler's rotation theorem (see Section 3.2.3), any rotation can be described as one single rotation around one axis by a unique angle. Determine a unit vector in direction of this axis, as well as the angle for this rotation. Use a real book and test your answer.

Problem 3.21 A free-body diagram of a top is depicted in Figure 3.12.

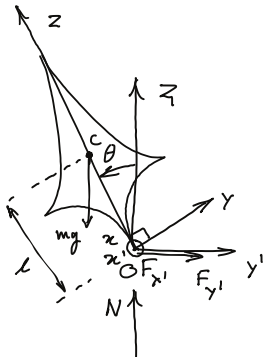


Figure 3.12: Top.

The top's motion is described by classical Euler angles (type II Euler angles). We define an inertial triad $\mathcal{N}$ associated with an XYZ coordinate system (note that only Z is drawn). A body-fixed triad $\mathcal{B}$ (not drawn) rotates along with the top and follows all its angles. An intermediate triad $\mathcal{F}$ associated with coordinate system rst rotates solely with the precession of the top. A second intermediate triad $\mathcal{G}$ associated with coordinate system uvw (shown in the figure) precesses and nutates, but does not spin.

a. Provide the components of the gravitational force vector ${}^{\mathcal{N}}\boldsymbol{F}_g$, expressed in the $\mathcal{N}$-triad.
b. Provide the components of the vector ${}^{\mathcal{G}}\boldsymbol{r}_{C/O}$, which defines the position of point C with respect to the ground contact point O, expressed in the $\mathcal{G}$-triad.
c. Calculate the elements of the rotation matrix ${}^{\mathcal{G}}\mathbf{C}_{\mathcal{N}}$ as functions of the Euler angles.
d. Use your above results to calculate the vector ${}^{\mathcal{G}}\boldsymbol{F}_g$, expressed in the $\mathcal{G}$-triad.
e. Use your above results to calculate the moment vector $\sum {}^{\mathcal{G}}\boldsymbol{M}_O$.
f. Conduct at least three different checks for plausibility and consistency of your above result.[a]

g. At time t_1, the Euler angles are $\varphi = \pi$ (precession), $\theta = \pi/6$ (nutation), $\psi = \pi/2$ (spin). Draw the unit vectors of the $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{B}$-triad into the three empty coordinate systems (so the rightmost picture contains the coordinate directions of the $\mathcal{B}$-triad at time t_1) in the template in Figure 3.13.

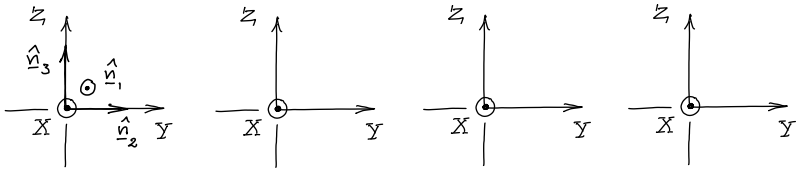


Figure 3.13: Template for drawing axes directions.

Use the drawing to further check plausibility of your results to the previous questions (draw the requested vectors and determine their projections on the axes).

h. Calculate the elements of ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$ as functions of classical Euler angles.

i. Implement a MATLAB$^{\text{TM}}$ function that returns ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$ as a function of the Euler angles.

j. Check your implementation by plotting the top in different orientations. You can do this by simplifying the top to a set of points, like in Figure 1.4, which have constant coordinates in a top-fixed coordinate system with origin on the ground. Then apply ${}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}}$ to these points. Use `plot3` to plot the points and lines connecting them.

[a] *Hint*: Some ideas: Start with special cases, such as perfectly upright spinning. Which moment do you expect gravity to create? Also check that each vector has the same length in either triad. Further, check the sign of components, such as the Z-component of the position vector. Does this match your intuition? Which influence does each Euler angle have on triad $\mathcal{G}$, and what does this tell you about the dependence of the rotation matrix on either of these angles? Use your rotation matrix to map and check other special vectors. Etc.

Problem 3.22 For a planned docking operation we study the attitude of a satellite $\mathcal{B}$ and a satellite $\mathcal{F}$ (see Figure 3.14). Here we are only interested in the attitude or orientation of the satellites, we do not consider the positions or translations.

Consider an inertial, right-handed, orthogonal reference coordinate system XYZ (associated with the triad $\mathcal{N}$). Two additional, local coordinate systems are given for each satellite: xyz (this triad is denoted by $\mathcal{B}$) and uvw (this triad is denoted by $\mathcal{F}$).

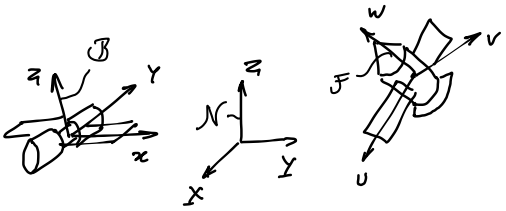


Figure 3.14: Two satellites in space.

Expressed in the $\mathcal{N}$-triad, the unit vectors of the $\mathcal{F}$-triad are:

$$ {}^{\mathcal{N}}\hat{\boldsymbol{f}}_1 = \frac{1}{4}\begin{pmatrix} 3 \\ -2 \\ \sqrt{3} \end{pmatrix}, \quad {}^{\mathcal{N}}\hat{\boldsymbol{f}}_2 = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -\sqrt{3} \end{pmatrix}, \quad {}^{\mathcal{N}}\hat{\boldsymbol{f}}_3 = \frac{1}{4}\begin{pmatrix} \sqrt{3} \\ 2\sqrt{3} \\ 1 \end{pmatrix}. \tag{3.32P} $$

Expressed in the $\mathcal{B}$-triad, the unit vectors of the $\mathcal{F}$-triad are:

$$ {}^{\mathcal{B}}\hat{\boldsymbol{f}}_1 = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \quad {}^{\mathcal{B}}\hat{\boldsymbol{f}}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad {}^{\mathcal{B}}\hat{\boldsymbol{f}}_3 = \frac{1}{5}\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}. \tag{3.33P} $$

a. Check that $\mathcal{F}$ forms a right-handed orthonormal basis.
b. Find the rotation matrix ${}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}}$ that maps vectors with components expressed in the $\mathcal{N}$-triad to their representations in the $\mathcal{F}$-triad.
c. Find the rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}$ that maps vectors with components expressed in the $\mathcal{F}$-triad to their representations in the $\mathcal{B}$-triad.
d. Use matrix multiplication to find the combined rotation ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$ that maps vectors ${}^{\mathcal{N}}\boldsymbol{r}$ with components expressed in $\mathcal{N}$ to their representation ${}^{\mathcal{B}}\boldsymbol{r}$ with components expressed in $\mathcal{B}$.
e. Determine the principal axis and angle of rotation representation of the rotation ${}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}}$ from the $\mathcal{N}$-triad to the $\mathcal{F}$-triad.

Problem 3.23 A model plane of mass m is suspended from a room's ceiling by a spring with linear characteristics and spring constant k. When no forces act on it, the spring has length l_0.

The attachment point O at the ceiling is the origin of the fixed XYZ coordinate system, which also defines triad $\mathcal{N}$.

The center of mass C of the plane is the origin of the plane's local coordinate system xyz, which also defines triad $\mathcal{B}$. Point C has coordinates C_X, C_Y, and C_Z in XYZ.

Point P, the attachment point of the spring on the plane, is given with respect to C by the relative position vector ${}^{\mathcal{B}}\boldsymbol{r}_{P/C} = (p_x, p_y, p_z)^{\mathrm{T}}$.

Gravity points in positive Z-direction, with constant acceleration of gravity g. The plane has no motors and air friction may be neglected.

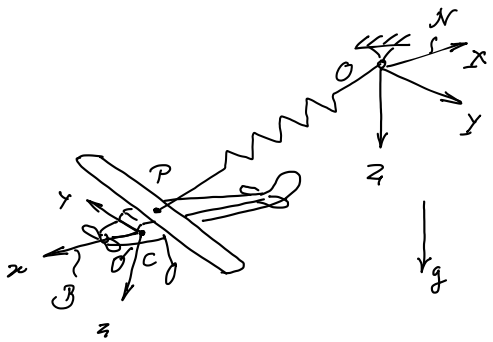


Figure 3.15: A model plane suspended from the ceiling.

a. First, draw a FBD of the plane, in generic position and orientation. Show multiple views if needed.
   Second, based on this FBD, calculate the resultant external force vector ${}^{\mathcal{N}}\boldsymbol{F}$ as well as the moment vector ${}^{\mathcal{B}}\boldsymbol{M}$ that the external forces create about the plane's center of mass. Note that the components of the vectors are expressed in different triads. Express both vectors as functions of appropriately selected rotation matrices, (relative) position vectors, and triad unit direction vectors, and of the constants m, k, l_0, and g.
b. Calculate the potential energy stored in the spring as a function of its parameters k, l_0, the relative position vector ${}^{\mathcal{B}}\boldsymbol{r}_{P/C}$, and the plane's current kinematic configuration given by the rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$ and the position $\boldsymbol{r}_{C/O}$.
c. Check your previous result for a few special cases that are easy to visualize, for example where the Euler angles are all zero or all equal to $\pi/2$, and/or where point P coincides with point C. Draw projections and calculate the spring length using trigonometry.
d. Draw a cans-in-series representation for the plane's successive rotations. Do this for a North-East-Down (NED) convention. In NED, the local triad of the plane is oriented such that if all Euler angles are zero, the plane is flying straight North and its local z-axis is directed downwards.
e. Implement a function that returns the rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$ as a function of yaw-pitch-roll Euler angles.
f. Check your implementation by plotting the plane in different orientations.

Problem 3.24 We like to study the dynamics of a sailboat and start by looking at the forces acting on the system, which consists of a hull, a sail and a rudder.

The sailboat is shown in Figure 3.16 together with the forces acting on it. Note that in the top and side view the boat is drawn with a zero roll and pitch angle, opposed to the back view, where we have a finite roll angle. Note that these are all incomplete FBDs.

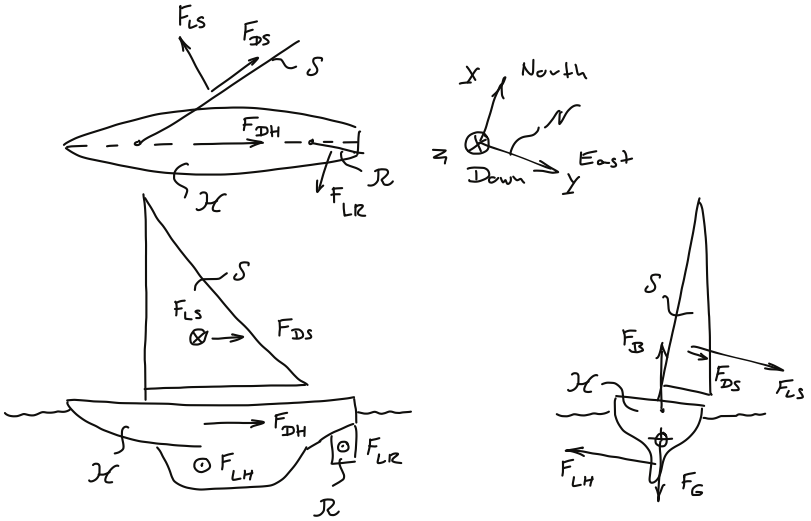


Figure 3.16: Forces on a sailboat.

The following seven force components are acting on the system:

Sail: Wind-induced drag F_{DS} and lift F_{LS}.

Hull: Water-induced buoyancy F_{B}, drag F_{DH} and lift F_{LH} on the boat's keel, and gravity F_{G}.

Rudder: A lift force F_{LR}.

The directions of all force components (so also F_{DS} and F_{DH}) are modeled as fixed with respect to the body they act on.

We define an inertial, right-handed, orthogonal reference coordinate system XYZ, with directions of the triad $\mathcal{N}$ (Nort–East–Down, NED), as well as three additional, local triads: the sail $\mathcal{S}$, the hull $\mathcal{H}$, and the rudder $\mathcal{R}$.

a. Complete the FBDs showing the three views of the boat. When doing so, add the axes directions of all four triads to each view, as well as missing dimensions (such as angles), and missing force components in any of the views. Re-draw the boat's sideview to show a finite positive pitch angle θ.

b. Write down 3×1-force vectors for all seven forces listed above. Choose the triads in such a way that each vector has only one nonzero component[a].
c. Calculate the rotation matrices ${}^{\mathcal{N}}\mathbf{C}_{\mathcal{S}}$, ${}^{\mathcal{N}}\mathbf{C}_{\mathcal{H}}$, and ${}^{\mathcal{N}}\mathbf{C}_{\mathcal{R}}$ that map vectors expressed in any of the three triads to the $\mathcal{N}$-triad. Provide the entries of the matrices as functions of Euler angles and any additional angles you defined in the FBDs. You may re-use previous knowledge on yaw-pitch-roll angles, and any previously implemented (MATLAB$^{\text{TM}}$) functions. Take care with signs.
d. Use your results to the previous two question parts to calculate the sum of forces acting on the complete system, expressed in the inertial triad $\mathcal{N}$, as functions of the provided force components, the Euler angles, and any further dimensions introduced in your FBDs.
e. Implement the rotation matrices as functions in MATLAB$^{\text{TM}}$ and test your results above with some special vectors (such as the direction of the boat's mast), and special values for the different angles.

[a] *Hint*: For example, the drag on the sail should be expressed in the sail triad (so find ${}^{\mathcal{S}}\boldsymbol{F}_{DS}$), but gravity in the inertial triad (so find ${}^{\mathcal{N}}\boldsymbol{F}_{G}$).

3.5.2 Practice Problems

Problem 3.25 This Problem is about the suspended model plane of Problem 3.23 on page 92. The plane's orientation with respect to the fixed $\mathcal{N}$- triad is described by type I Euler angles. At time t_b, these angles are $\psi = \pi/2$ (yaw), $\theta = \pi/2$ (pitch), $\varphi = \pi/2$ (roll).

Draw the unit vectors of the coordinate axes before (leftmost picture) and after each of the successive rotations by the three Euler angles in Figure 3.17 (So the rightmost picture should show the coordinate directions of the $\mathcal{B}$ triad at time t_b).

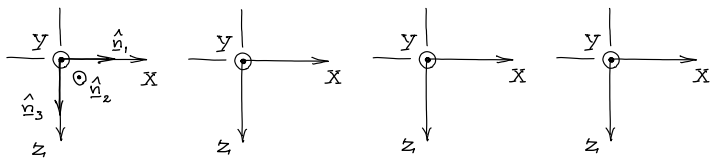


Figure 3.17: Cartoon template for the rotated coordinate axes of the model plane. ▶

Problem 3.26

A person is riding a hoverboard (Figure 3.18), which is a board with two independently rotating wheels. The hoverboard reacts to the tilt angle γ (see right picture in Figure 3.18) by varying the angular rate of the wheels.

The person is modeled as a single rigid body of mass m_p and center of

mass S, with inertia tensor $\mathbf{I}_S$. The distance between S and C is h.

The wheels have centers A and B. Each wheel is a ring with radius R and mass m_w. Rotations of the wheels about their axles are denoted by angles α and β, respectively, measured with respect to the vertical. The wheels only roll; there is no slip.

The board is massless, rigidly connected to the person, and has center C. The wheelbase (the distance between wheel centers) is given by d.

A triad $\mathcal{B}$ is fixed to the person, where $\hat{\boldsymbol{b}}_1$ points *anterior* (in direction of the nose) and $\hat{\boldsymbol{b}}_2$ points towards the person's left, along the wheels' axles.

The XYZ-coordinate system has origin O and directions of triad $\mathcal{N}$.

An intermediate triad $\mathcal{G}$ is defined by $\hat{\boldsymbol{g}}_2 = \hat{\boldsymbol{b}}_2$ and $\hat{\boldsymbol{g}}_3 = \hat{\boldsymbol{n}}_3$.

To describe the kinematic configuration, we choose C_X and C_Y, which are the X- and Y-coordinates of point C, the angle θ between $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{g}}_1$, which describes the board's orientation, the angle γ, which is the tilt of the person with respect to the vertical, and the wheels' angles α and β.

Gravity with acceleration constant g acts in negative Z-direction. The ground is flat.

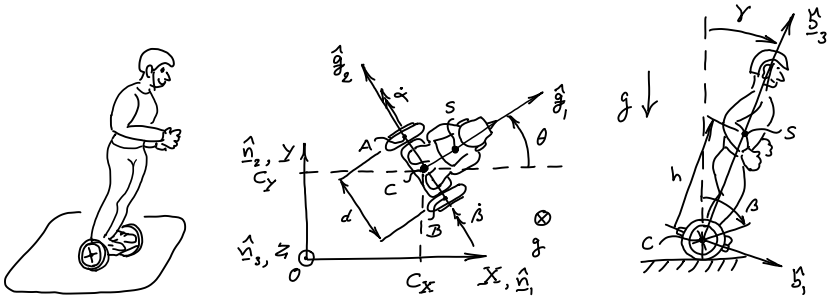


Figure 3.18: A person riding a hoverboard.

Which of the following statements is/are always true (Check all that apply)?

A. $\hat{\boldsymbol{b}}_2 \times \hat{\boldsymbol{b}}_1 = -\hat{\boldsymbol{b}}_3$
B. $\hat{\boldsymbol{n}}_1 \times \hat{\boldsymbol{b}}_1 = \sin\theta$
C. $\hat{\boldsymbol{n}}_3 \cdot \hat{\boldsymbol{b}}_3 = \cos\gamma$
D. None of the above

▷

◯ **Problem 3.27** This Problem is about the accelerometer of Problem 1.26 on page 41. State the relative position vector ${}^{\mathcal{C}}\boldsymbol{r}_{D/A}$ as a function of ${}^{\mathcal{C}}\boldsymbol{r}_{S/A}$, the rotation matrix ${}^{\mathcal{K}}\mathbf{C}_{\mathcal{C}}$, and the radius R. ▷

Problem 3.28 Which of the below plausibility checks could be applied to a rotation matrix?

A. The matrix must be symmetric.
B. The third column of the matrix must be the result of the cross product of the first and the second column.
C. Each column must contain a vector of length 1.
D. Multiplication of the matrix with its transpose must result in the identity matrix.
E. None of the above

▷

Problem 3.29 This Problem is about the skateboard of Problem 1.24 on page 39. Calculate the rotation matrix ${}^{\mathcal{N}}\boldsymbol{C}_{\mathcal{B}}$ between the body-fixed triad $\mathcal{B}$ and the inertial triad $\mathcal{N}$, as a function of ψ, γ.

▷

Problem 3.30 A 3-D accelerometer like the one in Problem 1.26 on page 41 is mounted on the torso of a person. Model the torso as a rigid body under the influence of gravity. We want to use the data of the accelerometer to calculate the orientation of the torso with respect to the world. We use a yaw-pitch-roll description, where the torso-fixed uvw-coordinate system has its w-axis pointing up during ideal upright stance, and u points forward (*anterior* in anatomical terms). The person assumes a static posture and holds it during the measurement.

Which of the following statements is/are true (Check all that apply)?

A. One can calculate the roll angle of the torso from the accelerometer data.
B. One can calculate the pitch angle of the torso from the accelerometer data.
C. One can calculate the yaw angle of the torso from the accelerometer data.
D. None of the above

▷

Problem 3.31 A thin disk of radius r is supported by a stiff arm via a ball bearing B, and it rotates around this arm with relative angular velocity of constant magnitude $\dot{\psi}$ (Figure 3.19).

The arm rotates around the Z-axis by means of ball bearing A, with constant angular velocity of magnitude $\dot{\varphi}$.

Neither of the two ball bearings blocks movement in axial direction.

The disk is momentarily contacting the ground in point P, which is at a distance R from the Z-axis.

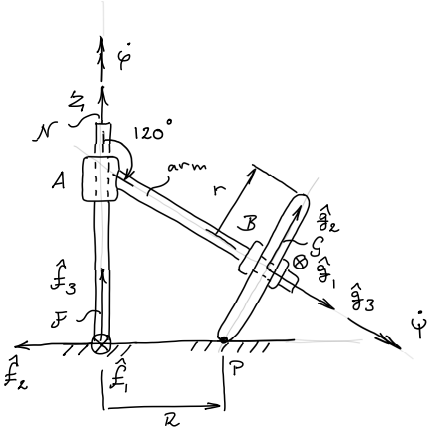


Figure 3.19: Rolling thin disk supported by a stiff arm.

The directions X, Y, and Z define an inertial triad $\mathcal{N}$, with Z pointing vertical. The directions of the unit vectors $\hat{\boldsymbol{b}}_1$, $\hat{\boldsymbol{b}}_2$, and $\hat{\boldsymbol{b}}_3$ of a triad $\mathcal{B}$ are fixed to the disk. An intermediate triad $\mathcal{F}$ rotates with the arm, whereby $\hat{\boldsymbol{f}}_3 = \hat{\boldsymbol{n}}_3$. A second intermediate triad $\mathcal{G}$ rotates with the arm but not with the disk, such that always $\hat{\boldsymbol{g}}_3 = \hat{\boldsymbol{b}}_3$ and $\hat{\boldsymbol{g}}_1 = \hat{\boldsymbol{f}}_1$.

The angle φ describes precession, and ψ describes spin of the disk with respect to $\mathcal{N}$. Provide the value of the nutation angle θ. Also provide a unit vector ${}^{\mathcal{B}}\hat{\boldsymbol{a}}$ for the principal axis of rotation that describes the rotation from the $\mathcal{N}$- to the $\mathcal{B}$-triad, with components expressed in the $\mathcal{B}$-triad, for the special case where $\varphi = \psi = 0$. ▶

Problem 3.32 We study the 3-D pendulum shown in Figure 3.20, which is a slender rod. The joint suspending the pendulum restricts translation in all directions and rotation around the rod's longitudinal axis.

The inertial coordinate system XYZ (associated with a triad $\mathcal{N}$) is defined such that X and Y are in the horizontal plane, and Z points down. The joint's axes of rotation intersect in the origin O. The xyz coordinate system (associated with a triad $\mathcal{B}$) is fixed to the rod, such that z is the longitudinal axis, and x is the axis of the depicted hinge.

The rotation of $\mathcal{B}$ with respect to $\mathcal{N}$ is described by two successive rotations: The first rotation by an angle α around the inertial Z-axis, the second rotation by an angle β around the body-fixed x-axis, such that $\mathcal{B}$ coincides with $\mathcal{N}$ when $\alpha = 0$ and $\beta = 0$.

The rotation matrix ${}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}}$ maps direction vectors with components expressed in the body-fixed triad $\mathcal{B}$ to their representations in the inertial triad $\mathcal{N}$.

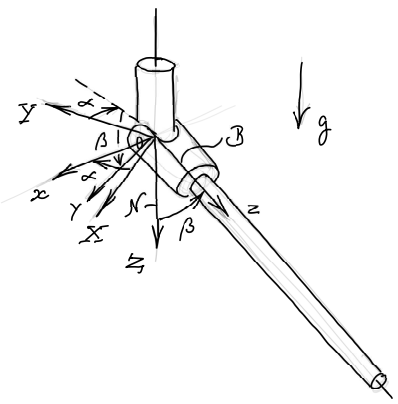


Figure 3.20: 3-D pendulum with a rod.

Prove with successive rotations that this rotation matrix ${}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}}$ equals

$$ {}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha)\cos(\beta) & \sin(\alpha)\sin(\beta) \\ \sin(\alpha) & \cos(\alpha)\cos(\beta) & -\cos(\alpha)\sin(\beta) \\ 0 & \sin(\beta) & \cos(\beta) \end{pmatrix}. \tag{3.34P} $$

▶

Problem 3.33 Two ships (Figure 3.21) are connected by a rope, which acts as a spring with unstressed length $l_0 = 89\,\mathrm{m}$ and spring stiffness $k = 9000\,\mathrm{N/m}$. An inertial coordinate system XYZ with associated triad $\mathcal{N}$ is indicated in the figure as well.

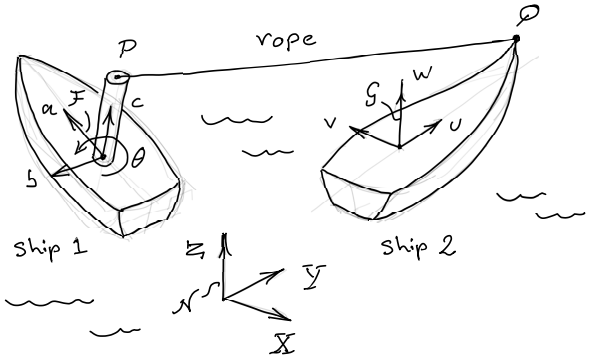


Figure 3.21: Ships

The first ship has a body-fixed coordinate system *abc* with associated triad $\mathcal{F}$. This ship is rotated by an angle of $\theta = \pi$ around the principal

axis ${}^{\mathcal{N}}\boldsymbol{p} = (0,\, 0,\, 1)$, as expressed in the $\mathcal{N}$-triad, see figure. The attachment point P of the rope on the first ship is given by its local position vector ${}^{\mathcal{F}}\boldsymbol{l}_1 = (0,\, 0,\, 5)^{\mathrm{T}}\mathrm{m}$, expressed in *abc*-coordinates.

The second ship has a body-fixed coordinate system *uvw* with associated triad $\mathcal{G}$. The directions of the aft-front (u) axis and the vertical (w) axis of the second ship, as expressed in the $\mathcal{N}$ triad, are given as the unit vectors ${}^{\mathcal{N}}\hat{\boldsymbol{g}}_1 = (-\frac{1}{2},\, \frac{3}{4},\, \frac{\sqrt{3}}{4})^{\mathrm{T}}$ and ${}^{\mathcal{N}}\hat{\boldsymbol{g}}_3 = (0,\, -\frac{1}{2},\, \frac{\sqrt{3}}{2})^{\mathrm{T}}$ respectively. The attachment point Q of the rope has the local position vector ${}^{\mathcal{G}}\boldsymbol{l}_2 = (12,\, 0,\, 4)^{\mathrm{T}}$ m in the *uvw*-coordinate system.

The vector $\boldsymbol{l}_{12}$ from the origin of coordinate system *abc* to the origin of coordinate system *uvw* is given in the $\mathcal{N}$-triad as ${}^{\mathcal{N}}\boldsymbol{l}_{12} = (46,\, 73,\, 15 - 5\sqrt{3})^{\mathrm{T}}$ m.

The rotation matrix ${}^{\mathcal{F}}\mathbf{C}_{\mathcal{G}}$ transforms direction vectors with components expressed in triad $\mathcal{G}$ into their representation in triad $\mathcal{F}$. Calculate this matrix. ▶

Problem 3.34 This Problem is about the ships connected by a rope of Problem 3.33 on page 99.

Calculate the force vector ${}^{\mathcal{N}}\boldsymbol{F}_{\mathrm{s}}$ the spring applies to ship 1. Give the numerical values of the vector's components in the $\mathcal{N}$-triad. Make sure to treat units correctly throughout all calculations (see also Section 1.1.4). ▶

Problem 3.35 A quadcopter has landed with its center of mass C on the tip of a pyramidal structure (Figure 3.22). Point C does not translate.

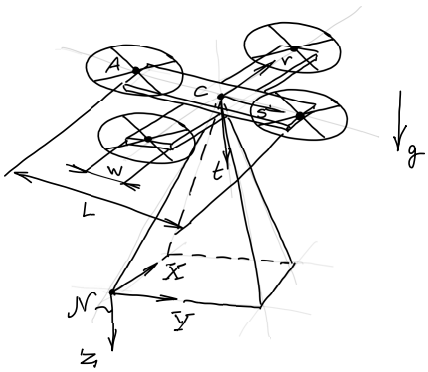


Figure 3.22: Quadcopter landed on a pyramid.

An inertial coordinate system XYZ is associated with triad $\mathcal{N}$. The coordinate system *rst*, associated with triad $\mathcal{B}$, is attached to the quadcopter's skeleton.

Euler angles type I (yaw ψ, pitch β, and roll γ) are used to describe the quadcopter's orientation. An intermediate triad $\mathcal{F}$ rotates with yaw and pitch, but not with roll.

Given the rotation matrix ${}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, calculate the elements of the rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$ as functions of γ. ▶

Problem 3.36 Earth is orbiting the Sun (Figure 3.23). We model Earth as a perfect rigid sphere with radius r and mass m, and define three coordinate systems with associated triads:

1. a *heliocentric ecliptic* coordinate system XYZ with associated triad $\mathcal{N}$. The origin is in the Sun S, and the Z-axis is orthogonal to the plane that contains the orbit of Earth around the Sun. This coordinate system does not rotate with Earth and is considered inertial.
2. a conventional *Earth-centered, Earth-fixed* coordinate system xyz with associated triad $\mathcal{T}$, which has its origin in the center of Earth E, its z-axis pointing towards the North pole, and its x-axis pointing in direction of the prime meridian.
3. a local *East-North-Up* coordinate system uvw with an associated triad $\mathcal{L}$. The origin D is in Delft, the w-axis always points outward from the center of Earth and is also orthogonal to the surface of Earth, and the u-axis always points towards the East point of the horizon.

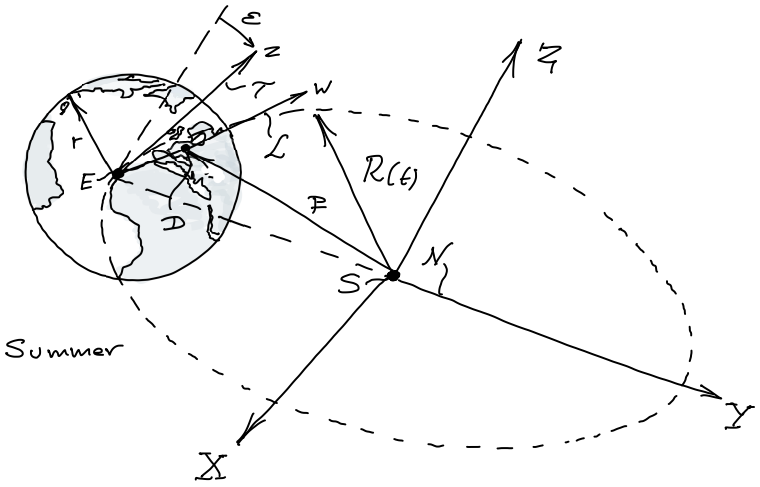


Figure 3.23: Earth orbit.

Given are two rotation matrices ${}^{\mathcal{L}}\mathbf{C}_{\mathcal{N}}$ and ${}^{\mathcal{L}}\mathbf{C}_{\mathcal{T}}$. The matrix ${}^{\mathcal{L}}\mathbf{C}_{\mathcal{N}}$ maps vectors with components expressed in the $\mathcal{N}$-triad to their representation with components expressed in the $\mathcal{L}$-triad. The matrix ${}^{\mathcal{L}}\mathbf{C}_{\mathcal{T}}$ maps vectors

with components expressed in the $\mathcal{T}$-triad to their representation with components expressed in the $\mathcal{L}$-triad.

Calculate the rotation matrix ${}^{\mathcal{T}}\mathbf{C}_{\mathcal{N}}$ that maps vectors with components expressed in the $\mathcal{N}$-triad to their representation with components expressed in the $\mathcal{T}$-triad. Provide your answer as a function of ${}^{\mathcal{L}}\mathbf{C}_{\mathcal{N}}$ and ${}^{\mathcal{L}}\mathbf{C}_{\mathcal{T}}$. ▷

Problem 3.37 This Problem is about the Earth orbiting the Sun of Problem 3.36 on page 101. At the beginning of summer, point E is located on the negative Y-axis and the z-axis is contained in the YZ-plane, as depicted in the figure. Also, the current rotation matrix ${}^{\mathcal{L}}\mathbf{C}_{\mathcal{N}}$ is known. Calculate the position vector ${}^{\mathcal{N}}\boldsymbol{D}$ of Delft in the XYZ-coordinate system in summer, with components expressed in the associated $\mathcal{N}$-triad, as a function of ${}^{\mathcal{L}}\mathbf{C}_{\mathcal{N}}$, R, and r. ▷▶

4 Derivatives in Moving Coordinate Systems

In this chapter, angular velocity of a triad or rigid body will be introduced and applied to find velocities and accelerations of points in moving coordinate systems. A fundamental theorem is the transport theorem, which is useful whenever we want to calculate derivatives of vectors in rotating triads. As special cases, several special coordinate systems will be reviewed, such as cylindrical and spherical coordinates. Then, general relative and absolute motion will be discussed, in order to be able to describe movement of a point relative to a fixed coordinate system, given that we know how this point moves with respect to a moving coordinate system.

4.1 Angular Velocity

4.1.1 Infinitesimal Rotations

Consider a body-fixed triad $\mathcal{B}$ and an inertial triad $\mathcal{N}$, which initially coincide. Then, the $\mathcal{B}$-triad performs an infinitesimal rotation $\mathrm{d}\theta_x$ about the X-axis, as described by the rotation matrix $\mathbf{C}_X$ in (3.16E):

$$ {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = \mathbf{C}_X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mathrm{d}\theta_x \\ 0 & -\mathrm{d}\theta_x & 1 \end{pmatrix}. \tag{4.1} $$

Here, we made use of the small-angle approximation, which means that linearization of the trigonometric functions is allowed, in particular:

$$ \begin{aligned} \cos \mathrm{d}\theta &\approx 1, \\ \sin \mathrm{d}\theta &\approx \mathrm{d}\theta. \end{aligned} \tag{4.2} $$

If we proceed and let the triad $\mathcal{B}$ rotate further, namely by $\mathrm{d}\theta_y$ about the new y'-axis and $\mathrm{d}\theta_z$ about the z''-axis, we obtain a combined rotation. Generating the matrices for infinitesimal y and z rotations, we

can find a combined rotation matrix:

$$ {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = \mathbf{C}_Z\mathbf{C}_Y\mathbf{C}_X = \begin{pmatrix} 1 & \mathrm{d}\theta_z & -\mathrm{d}\theta_y \\ -\mathrm{d}\theta_z & 1 & \mathrm{d}\theta_x \\ \mathrm{d}\theta_y & -\mathrm{d}\theta_x & 1 \end{pmatrix}. \tag{4.3} $$

Now, we are interested in the rotation matrix if we choose a different sequence of rotations. If we do this, what we find is that the overall rotation matrix of the infinitesimal rotations, as given by the product of the three individual rotation matrices $\mathbf{C}_X$, $\mathbf{C}_Y$ and $\mathbf{C}_Z$, will be identical regardless of the order of rotations (see Problem 4.2). So, for example:

$$ \mathbf{C}_Z\mathbf{C}_Y\mathbf{C}_X = \mathbf{C}_Y\mathbf{C}_X\mathbf{C}_Z. \tag{4.4} $$

This means that in the case of infinitesimal rotations, the order of matrices in the product of (4.3) is not important, in contrast to what we found earlier for finite rotations.

Problem 4.1 Use MATLAB$^{\mathrm{TM}}$ or another program of your liking to plot $y = \sin(\theta)$ and zoom in to $-0.1\,\mathrm{rad} < \theta < 0.1\,\mathrm{rad}$. At this small angle θ, how can you approximate y? Do the same for $y = \cos(\theta)$.

Problem 4.2 Calculate a rotation matrix (in analogy to (4.3)) for successive infinitesimal rotations about the Y, z', and x'' axes (in this order). Make use of the small-angle approximation. Then, calculate also the matrix for successive infinitesimal rotations about the Z, y', and x'' axes (in this order). You should find that the two matrices are the same.

4.1.2 The Instantaneous Axis of Rotation

With the rotation matrix of (4.3), a unit vector in the direction of the principal axis of this infinitesimal rotation, $\hat{\boldsymbol{a}}_i$, can be calculated as an eigenvector of the rotation matrix:

$$ \hat{\boldsymbol{a}}_i = \frac{1}{\sqrt{\mathrm{d}\theta_x^2 + \mathrm{d}\theta_y^2 + \mathrm{d}\theta_z^2}} \begin{pmatrix} \mathrm{d}\theta_x \\ \mathrm{d}\theta_y \\ \mathrm{d}\theta_z \end{pmatrix}. \tag{4.5} $$

Problem 4.3 Show that (4.5) is an eigenvector of (4.3). What is the value of the corresponding eigenvalue?

The principal axis of rotation of this instantaneous, infinitesimally small rotation is called the *instantaneous axis of rotation*.

Using the definition of an infinitesimally small angle increment $\mathrm{d}\varphi$,

$$\mathrm{d}\varphi := \sqrt{\mathrm{d}\theta_x^2 + \mathrm{d}\theta_y^2 + \mathrm{d}\theta_z^2}, \tag{4.6}$$

and of the vector $\mathrm{d}\boldsymbol{\theta}$, with[1]

$$\mathrm{d}\boldsymbol{\theta} := \begin{pmatrix} \mathrm{d}\theta_x & \mathrm{d}\theta_y & \mathrm{d}\theta_z \end{pmatrix}^{\mathrm{T}}, \tag{4.7}$$

we can write the direction vector of the instantaneous axis of rotation in (4.5) shorter as:

$$\hat{\boldsymbol{a}}_i = \frac{1}{\mathrm{d}\varphi}\mathrm{d}\boldsymbol{\theta}. \tag{4.8}$$

As the rotation matrix is the same for any order of three infinitesimal rotations, also the instantaneous axis of rotation (4.8) does not depend on the order in which we perform the rotations. That means we can express any infinitesimal rotation as a summation of vectors in directions of the original axes:

$${}^{\mathcal{B}}\mathrm{d}\boldsymbol{\theta} = {}^{\mathcal{N}}\mathrm{d}\boldsymbol{\theta} = \mathrm{d}\theta_x\hat{\boldsymbol{n}}_1 + \mathrm{d}\theta_y\hat{\boldsymbol{n}}_2 + \mathrm{d}\theta_z\hat{\boldsymbol{n}}_3. \tag{4.9}$$

So, we see that a sequence of several infinitesimal rotations is equivalent to their vector sum, so for example $\mathrm{d}\boldsymbol{\theta} = \mathrm{d}\boldsymbol{\theta}_1 + \mathrm{d}\boldsymbol{\theta}_2$.

4.1.3 Definition of Angular Velocity

The angular velocity $\boldsymbol{\omega}$ of a rigid body is defined as the infinitesimal vector $\mathrm{d}\boldsymbol{\theta}$ of (4.7) divided by the corresponding infinitesimally short time interval $\mathrm{d}t$:

$$\boldsymbol{\omega} := \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}. \tag{4.10}$$

As angular velocity is just a scalar multiple of $\mathrm{d}\boldsymbol{\theta}$, just like $\mathrm{d}\boldsymbol{\theta}$ in (4.9) we can write the angular velocity $\boldsymbol{\omega}$ in terms of its projections on the original axis directions (which, given that $\mathrm{d}t$ is infinitesimally short, coincide with the new axis directions):

$$\boldsymbol{\omega} = \omega_{n1}\hat{\boldsymbol{n}}_1 + \omega_{n2}\hat{\boldsymbol{n}}_2 + \omega_{n3}\hat{\boldsymbol{n}}_3. \tag{4.11}$$

As angular velocity is a vector, also multiple angular velocity vectors can be combined by simple vector addition, where the order of addition

[1] Increments $\mathrm{d}\varphi$ and $\mathrm{d}\boldsymbol{\theta}$ are *only* defined for the limit case of infinitesimal $\mathrm{d}t$. There are *no* corresponding variables φ and $\boldsymbol{\theta}$, integration is not possible.

does not matter (addition is commutative). For three arbitrary triads $\mathcal{G}$, $\mathcal{F}$, $\mathcal{B}$ that rotate with respect to each other, we can write

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{G}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{F}} + \boldsymbol{\omega}_{\mathcal{F}/\mathcal{G}}, \tag{4.12}$$

where the notation $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{G}}$ refers to the relative angular velocity of triad $\mathcal{B}$ with respect to triad $\mathcal{G}$, similar to the notation of Section 1.1.2. Using the simple rules of vector addition, (4.12) can be expanded to relate any number of triads and their relative motion.

By comparing (4.8) and (4.10), we recognize the important property:

$$\boldsymbol{\omega} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\hat{\boldsymbol{a}}_\mathrm{i}. \tag{4.13}$$

So, the angular velocity vector's direction defines the instantaneous axis of rotation, and its magnitude defines the current rate of change of the rotation angle with respect to time.

In many cases, we calculate angular velocity using vector addition, using for example Euler angles and their derivatives, or other successive single-axis rotations that describe the kinematics of a particular mechanical system.

Signs are again crucial for angular velocity. In Section 1.3.3, we already specified in rule 1 how angles must be labeled.

To determine the angular velocity vector $\boldsymbol{\omega}_{\mathcal{F}/\mathcal{G}}$ for rotation about a single axis by an angle φ, including correct sign, we apply (4.13) following these steps:

1. Specify a unit vector $\hat{\boldsymbol{u}}_\mathrm{i}$ in direction of the instantaneous axis of rotation of triad $\mathcal{F}$ with respect to $\mathcal{G}$. Here, the sign does not yet matter, although it is mostly convenient to choose unit vectors that point in positive axis directions, if possible.
2. Visualize in which direction triad $\mathcal{F}$ rotates if $\dot{\varphi}$ is positive, meaning that φ increases.
3. Use the right-hand rule: The thumb points in direction of the chosen unit vector $\hat{\boldsymbol{u}}_\mathrm{i}$. If the curl indicates the same rotation as that of triad $\mathcal{F}$ when the angle increases, then $\hat{\boldsymbol{a}}_\mathrm{i} = \hat{\boldsymbol{u}}_\mathrm{i}$. If not, then $\hat{\boldsymbol{a}}_\mathrm{i} = -\hat{\boldsymbol{u}}_\mathrm{i}$.

Example 4.1 A person of mass m_p sits on a desk chair. The desk chair can rotate without friction about the vertical axis with respect to the fixed ground. The person uses both arms to hold the axle of a spinning wheel of mass m_w, see Figure 4.1.

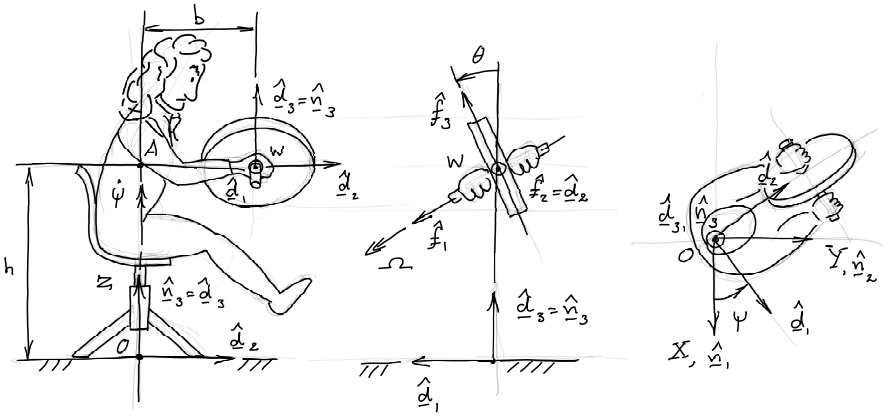


Figure 4.1: Rotating chair with person holding a rotating wheel. Left: Side view of the system. Center: Front view of the wheel. Right: Top view.

An inertial coordinate system XYZ is attached to the ground. It has origin O and is associated with triad $\mathcal{N}$. Triad $\mathcal{D}$ rotates with the chair about the vertical axis Z, described by the angle ψ. A triad $\mathcal{F}$ rotates with respect to $\mathcal{D}$ by an angle θ about the $\hat{\boldsymbol{d}}_2$-axis direction. A fourth triad $\mathcal{B}$ is fixed to the spinning wheel.

With respect to $\mathcal{F}$, the wheel spins with relative angular velocity of magnitude Ω about its axle, which points in the direction of $\hat{\boldsymbol{f}}_1$. The person is modeled as a rigid body. The constant distance from shoulder A to the wheel's center of mass W is b, and the constant distance from A to O is h.

Express the components of the relative angular velocities ${}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}}$ and ${}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{D}}$ as functions of $\psi, \theta, \dot{\psi}$ and $\dot{\theta}$.

Exemplary solution

We start with the angular velocity vector ${}^{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}}$. The instantaneous axis of rotation is the Z-axis. So, a unit vector in direction of the instantaneous axis is $\hat{\boldsymbol{n}}_3$. For a positive $\dot{\psi}$, meaning that angle ψ increases, the person will be rotating counter-clockwise as seen from above. The right-hand rule shows that with the thumb pointing in direction of $\hat{\boldsymbol{n}}_3$, the curl of the fingers follows the rotation of the person for positive $\dot{\psi}$. That means the angular velocity has a positive sign:

$$ {}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} = \dot{\psi}\,{}^{\mathcal{N}}\hat{\boldsymbol{n}}_3 = \begin{pmatrix} 0 & 0 & \dot{\psi} \end{pmatrix}^{\mathrm{T}} \tag{4.14E} $$

For the relative angular velocity ${}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{D}}$, it is easier to first express the vector in triad $\mathcal{D}$, and then to map its components to triad $\mathcal{N}$ using a rotation matrix. This is because the instantaneous axis of rotation is given by the vector $\hat{\boldsymbol{d}}_2$, which is easy to express in $\mathcal{D}$. The angular rate is $\dot{\theta}$.

We apply again the right-hand rule to determine the sign: Increasing θ leads to relative rotation of triad $\mathcal{F}$ in the same direction as the curl of the fingers when the thumb points in direction of $\hat{\boldsymbol{d}}_2$. So, also here the sign is positive:

$$^{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{D}} = \dot{\theta}\,{}^{\mathcal{D}}\hat{\boldsymbol{d}}_2 = \begin{pmatrix} 0 & \dot{\theta} & 0 \end{pmatrix}^{\mathrm{T}}. \tag{4.15E}$$

Finally, we express the vector in $\mathcal{N}$:

$$^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{D}} = {}^{\mathcal{N}}\mathbf{C}_{\mathcal{D}}\,{}^{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{D}} = \begin{pmatrix} -\dot{\theta}\sin\psi & \dot{\theta}\cos\psi & 0 \end{pmatrix}. \tag{4.16E}$$

Problem 4.4 This Problem is about the rotor of Problem 1.22 on page 37. A rotor-fixed triad $\mathcal{B}$ associated with the xyz-axes as drawn in Figure 1.21 is introduced. Give a unit vector ${}^{\mathcal{B}}\hat{\boldsymbol{u}}$ that points in the direction of the instantaneous axis of rotation of the combined rigid body (consisting of rotor, axle, and added weight), with components expressed in the $\mathcal{B}$-triad. The sign does not matter. ▶

Problem 4.5 This Problem is about the person on a chair of Example 4.1 on page 106. Given the relative angular velocities $\boldsymbol{\omega}_{\mathcal{F}/\mathcal{D}}$ and $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{F}}$, calculate $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{D}}$ as a function of these vectors. ▶

Problem 4.6 This Problem is about the rotating chair with a person of Example 4.1 on page 106. Given the angular velocity $\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$, calculate $\boldsymbol{\omega}_{\mathcal{N}/\mathcal{F}}$ as a function of this variable. ▷▶

4.1.4 Point Rotating about a Fixed Axis

In the following, we are interested in calculating the velocity vector $\boldsymbol{v}$ of a point P that is located on a rotating rigid body or part of a rotating coordinate system.

We start with a special case, namely that there exists at least one fixed point during this movement (which point can be located inside or outside the body, but is rigidly connected to it).

Example 4.2 If a body is attached to a spherical joint that is fixed in space, the center of this joint is fixed. Another example could be that of a body that is rotating about an axis that is fixed in space (e.g. a hinge joint) and cannot slide along this axis; then all points on this axis are fixed. On the contrary, if the same body rotates around an axis and also slides along this very axis, one cannot find a single point that maintains its position during this movement.

In the special case where there is one fixed point throughout the entire movement, Euler's theorem tells us that the movement of point

P between two instants of time can always be described by one single equivalent rotation only (so no translation) about one principal axis of rotation.

The above example of a hinge is quite restrictive, as for the purpose of calculating velocities, it does not matter whether this axis itself is fixed over a longer period of time. All that is relevant is that *instantaneously*, the movement can be described by an *infinitesimal* rotation about an *instantaneous* axis of rotation only.

To find the velocity vector of point P in this special case, we first need the point's position vector $\boldsymbol{r}$ with respect to a fixed point, so a point on the axis of rotation. See Figure 4.2 for a visualization of the vectors used. The distance R is the shortest distance between point P and the fixed axis. We choose a fixed coordinate system XYZ where the Z-axis coincides with the instantaneous axis of rotation, such that the origin is on this axis (Figure 4.2).

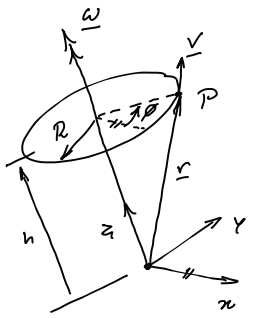


Figure 4.2: A visualization of angular velocity and the important vectors to calculate velocity.

In that coordinate system, the position vector $\boldsymbol{r}$ of point P is:

$$\boldsymbol{r} = \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ h \end{pmatrix}, \tag{4.17}$$

with constant h. By taking the derivative with respect to time, we find the velocity vector of P in the coordinate system XYZ:

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \begin{pmatrix} -R\dot{\varphi}\sin\varphi \\ R\dot{\varphi}\cos\varphi \\ 0 \end{pmatrix}. \tag{4.18}$$

Note that height h is constant as we restrict the movement of P to be a pure rotation about the axis at that instant.

Concerning the direction of this vector, we can see from Figure 4.2 that it will always be normal to the plane spanned by the vectors $\boldsymbol{\omega}$ and $\boldsymbol{r}$, and that this direction follows from the right-hand rule: If the thumb points in the direction of the angular velocity vector $\boldsymbol{\omega}$ (so in the direction of the instantaneous axis of rotation) and the index finger in the direction of $\boldsymbol{r}$ (a vector from the axis of rotation to the respective point), then the velocity vector $\boldsymbol{v}$ of the point is given in the direction of the middle finger. Concerning the magnitude of the velocity vector, we can calculate from (4.18) that it is given as $R\dot{\varphi}$. This equals the area of the parallelogram spanned by the vectors $\boldsymbol{\omega}$ and $\boldsymbol{r}$. Recalling the definition from Section A.3, it becomes evident that $\boldsymbol{v}$ can be obtained by taking the cross product of $\boldsymbol{\omega}$ and $\boldsymbol{r}$:

$$\boldsymbol{v} = \tilde{\boldsymbol{\omega}}\boldsymbol{r} = \boldsymbol{\omega} \times \boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix} \times \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ h \end{pmatrix} = \begin{pmatrix} -R\dot{\varphi}\sin\varphi \\ R\dot{\varphi}\cos\varphi \\ 0 \end{pmatrix}. \tag{4.19}$$

Just like when calculating moments of forces earlier in Section 1.2.1, the choice of h is arbitrary, as it has no influence on the final result. That means one may take an arbitrary vector $\boldsymbol{r}$ from the axis of rotation to point P to calculate the velocity.

A cross product calculation of velocity due to rotation is preferable to a manual calculation with scalars, because it is less susceptible to errors. However, it is important to remember that the order in which one takes two vectors in a cross product does make a difference.

Problem 4.7 This Problem is about the rotating chair with a person of Example 4.1 on page 106. Calculate the velocity vector ${}^{\mathcal{D}}\boldsymbol{v}_W$ of point W with components expressed in the $\mathcal{D}$-triad as functions of ψ, $\dot{\psi}$, θ, $\dot{\theta}$, Ω and the given constants h and b (Remark: None of these terms has to necessarily appear in the requested function). ▶

Problem 4.8 Use the tilde matrix notation (A.5) of angular velocity and the definition of a vector derivative (B.1) to prove (4.19), departing from infinitesimal rotations as in (4.3).

4.1.5 Instantaneous Movement of a Rigid Body

It is not always possible to describe the instantaneous movement of a rigid body in 3-D by a pure rotation about a fixed axis.

However, the instantaneous change in orientation or attitude of a rigid body is fully described by the body's angular velocity. Also, the complete instantaneous movement, including change of location, can always be depicted as a combination of this rotation around any axis aligned with the angular velocity and the simultaneous translation of a point on this axis. It is often convenient to choose that point to be the center of mass of the body.

By shifting the axis of rotation, infinitely many representations of the same movement of the body can be found. When shifting the axis of rotation, the angular velocity stays the same, but the associated additional linear velocity component changes, such that the net result of all velocities of all points in the rigid body is identical.

◯ **Problem 4.9** Consider the instantaneous relative movement of the moon with respect to Earth. One can describe this movement as a rotation around an axis through the Earth's center. However, one can alternatively describe it as a rotation around an axis through the moon's *own* center, in combination with a linear velocity that is tangential to the moon's orbit.

Visualize both options. Does this thought experiment require shifting the moon's axis of rotation? Does it change the magnitude or direction of angular velocity? ▷

This equivalence is similar to the principle of equipollence that we saw for force system resultants in Section 1.2.2, so for a pair of resultant force and resultant moment: The line of action of the resultant force vector always remains parallel and the force magnitude stayed identical, but when the line is shifted in space, the moment component changes. Analogously, the direction of the instantaneous axis of rotation and the magnitude of angular velocity of a rigid body remain the same when switching to an equivalent representation.

We also saw in Section 1.2.2 that we can find one location for the line of action such that the moment component (if existent) is aligned with the force vector. Analogously for a rigid body, we can find one instantaneous axis of rotation such that the remaining linear velocity (if existent) is aligned with it. This has already been shown in Mozzi-Chasles' Theorem (Section 3.2.4). This still holds for the case of infinitesimally small displacements. In those cases, as Chasles puts it, the screw axis "slides on itself while the body rotates around this line, so that the movement of the body is none other than that of a screw in its nut" [11].

If, in a special case, the body continues to rotate around this axis for an extended amount of time and if also the translational movement

component in the direction of the axis has a constant speed, any particular point on the body performs a helical movement, as depicted in Figure 4.3. Therefore, we can intuitively understand Mozzi-Chasles'

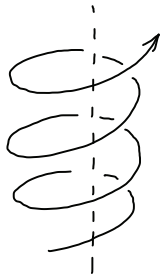


Figure 4.3: A visualization of the screw displacement described by Mozzi-Chasles' Theorem.

theorem as saying that any arbitrary movement of a rigid body can be described by an instantaneous screw movement.

4.1.6 Instantaneous Center of Rotation

When we try to apply Mozzi-Chasles' Theorem for a 2-D-case, we recognize that it greatly simplifies:

In 2-D or *planar movement*, any rotation can only take place about an axis that is perpendicular to the plane, which means that a body's angular velocity vector $\boldsymbol{\omega}$ is perpendicular to all vectors in the plane and can be represented by its scalar signed magnitude only. So, an arbitrary 2-D movement is equivalent to viewing the "screw" of Figure 4.3 from above. This means the instantaneous axis of rotation degenerates to a single point, and no translation along the axis is possible. So, in 2-D, we have an instantaneous *center* of rotation I instead of an *axis*. In other words, Mozzi-Chasles' theorem reduces to saying that the most general infinitesimal displacement of a rigid body *in 2-D* is equivalent to an infinitesimal rotation about a fixed point. This is in fact the same conclusion as what we get from Euler's theorem, namely that any movement where a fixed point exists can be represented by a single rotation.

Figure 4.4 shows a rigid body performing general movement in 2-D. As we know that a fixed center of rotation I must exist, the velocity $\boldsymbol{v}_A$ of point A results from the cross product as in (4.19). In 2-D, the body's angular velocity vector $\boldsymbol{\omega}$ is directed out of the paper and $\boldsymbol{r}_{A/I}$

is contained within the plane of the paper. Therefore, in 2-D, $\boldsymbol{\omega}$ is orthogonal to $\boldsymbol{r} := \boldsymbol{r}_{A/I}$. We can therefore calculate the point's speed v_A using just the signed magnitude ω of the vector $\boldsymbol{\omega}$, combined with r, the signed magnitude of $\boldsymbol{r}$:

$$v_A = \omega r, \tag{4.20}$$

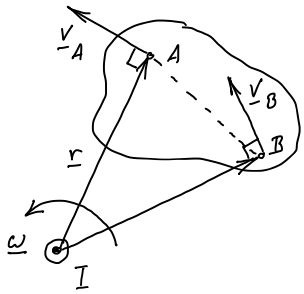


Figure 4.4: The instantaneous center of rotation of a rigid body in 2-D.

The orthogonality of the vectors $\boldsymbol{r}_{P/I}$ and $\boldsymbol{v}_P$ for any point P on the body also provides a possibility to reconstruct the instantaneous center of rotation of a rigid body in 2-D from just two points A and B with known velocity vectors (or at least known directions of these velocities): Lines through these points that are perpendicular to the respective velocities (see Figure 4.4) must intersect in the instantaneous center of rotation I.

Similarly, if we know the angular velocity and a single velocity vector in terms of both direction and magnitude, we can locate the instantaneous center of rotation by orthogonality and calculating r from (4.20).

Problem 4.10 Use the definition of the cross product in (A.2) in Appendix A to show that in this special case in 2-D, $\boldsymbol{\omega} \times \boldsymbol{r}_{A/I} = |\boldsymbol{\omega}||\boldsymbol{r}_{A/I}|$.

4.2 Vector Derivatives in Rotating Triads

4.2.1 Time Derivative of a Unit Vector

We recall that the velocity of a point P on a rigid body rotating with angular velocity $\boldsymbol{\omega}$ following a pure rotation about an instantaneous axis is given by (see also Section 4.1.4):

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r}. \tag{4.21}$$

Note again that this was derived for a special case, namely where the point is only rotating, not translating along the axis.

Instead of a position vector $\boldsymbol{r}$, we can equally consider a unit vector $\hat{\boldsymbol{e}}_i$ (see Figure 4.5) of a rotating triad $\mathcal{E}$. So, the derivative of this unit vector, if the triad rotates with angular velocity $\boldsymbol{\omega}$, is also given by the cross product of $\boldsymbol{\omega}$ with the unit vector itself:

$$\dot{\hat{\boldsymbol{e}}}_i = \boldsymbol{\omega} \times \hat{\boldsymbol{e}}_i. \tag{4.22}$$

This holds for any unit vector. Given that a unit direction vector holds no information on absolute position (in contrast to a position vector), translation does not change the vector. Therefore, (4.22) holds in general for any rotating triad.

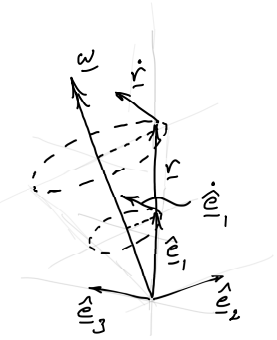


Figure 4.5: Analogy between a unit vector derivative and a position vector derivative.

4.2.2 Transport Theorem

We define two coordinate systems: XYZ, with axis directions of triad $\mathcal{N}$, and xyz, with axis directions of triad $\mathcal{F}$. Triad $\mathcal{F}$ rotates with respect to triad $\mathcal{N}$ with angular velocity $\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$. We want to calculate the derivative of a vector $\boldsymbol{A}$ relative to the $\mathcal{N}$-triad. The physical meaning of the vector is not important, it could be position of a point, velocity of a point, angular momentum of a system of particles, etc.

This vector $\boldsymbol{A}$ can be expressed in terms of its components in the $\mathcal{F}$-triad:

$$\boldsymbol{A} = A_1 \hat{\boldsymbol{f}}_1 + A_2 \hat{\boldsymbol{f}}_2 + A_3 \hat{\boldsymbol{f}}_3. \tag{4.23}$$

As the unit vectors of this triad rotate, they change direction over time, so their derivatives are not zero. Therefore, the product rule should be

applied to calculate the derivatives:

$$(\dot{\boldsymbol{A}})_{\mathcal{N}} = \underbrace{\left(\dot{A}_1\hat{\boldsymbol{f}}_1 + \dot{A}_2\hat{\boldsymbol{f}}_2 + \dot{A}_3\hat{\boldsymbol{f}}_3\right)}_{\text{change relative to } \mathcal{F}\text{-triad}} + \underbrace{\left(A_1\dot{\hat{\boldsymbol{f}}}_1 + A_2\dot{\hat{\boldsymbol{f}}}_2 + A_3\dot{\hat{\boldsymbol{f}}}_3\right)}_{\text{rotation relative to } \mathcal{N}\text{-triad}}. \quad (4.24)$$

The first term on the right-side in (4.24) represents the change of $\boldsymbol{A}$ relative to the $\mathcal{F}$-triad. We introduce a short notation:

$$\dot{A}_1\hat{\boldsymbol{f}}_1 + \dot{A}_2\hat{\boldsymbol{f}}_2 + \dot{A}_3\hat{\boldsymbol{f}}_3 =: (\dot{\boldsymbol{A}})_{\mathcal{F}} \;\Rightarrow\; {}^{\mathcal{F}}(\dot{\boldsymbol{A}})_{\mathcal{F}} = \begin{pmatrix}\dot{A}_1\\ \dot{A}_2\\ \dot{A}_3\end{pmatrix}. \quad (4.25)$$

The round brackets in combination with the right subscript $\mathcal{F}$ means "relative to the $\mathcal{F}$-triad". So if triad $\mathcal{F}$ did not rotate at all, $(\dot{\boldsymbol{A}})_{\mathcal{N}}$ and $(\dot{\boldsymbol{A}})_{\mathcal{F}}$ would be the same.

The second term on the right-side in (4.24) is the change of $\boldsymbol{A}$ that is caused by the angular velocity of triad $\mathcal{F}$ relative to triad $\mathcal{N}$. The derivative of a unit vector is already known from Section 4.2.1: From (4.22), we know that $\dot{\hat{\boldsymbol{f}}}_i = \boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times \hat{\boldsymbol{f}}_i$. Substituting this in (4.24), the second term becomes:

$$\begin{aligned} &A_1\dot{\hat{\boldsymbol{f}}}_1 + A_2\dot{\hat{\boldsymbol{f}}}_2 + A_3\dot{\hat{\boldsymbol{f}}}_3 \\ &\quad = \underbrace{A_1(\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times \hat{\boldsymbol{f}}_1) + A_2(\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times \hat{\boldsymbol{f}}_2) + A_3(\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times \hat{\boldsymbol{f}}_3)}_{\text{rotation relative to } \mathcal{N}\text{-triad}}. \end{aligned} \quad (4.26)$$

Simplification of (4.26) results in:

$$\begin{aligned} &A_1\dot{\hat{\boldsymbol{f}}}_1 + A_2\dot{\hat{\boldsymbol{f}}}_2 + A_3\dot{\hat{\boldsymbol{f}}}_3 \\ &\quad = \underbrace{\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times (A_1\hat{\boldsymbol{f}}_1 + A_2\hat{\boldsymbol{f}}_2 + A_3\hat{\boldsymbol{f}}_3) = \boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times \boldsymbol{A}}_{\text{rotation relative to } \mathcal{N}\text{-triad}}. \end{aligned} \quad (4.27)$$

Combination of (4.25) and (4.27) yields

$$(\dot{\boldsymbol{A}})_{\mathcal{N}} = (\dot{\boldsymbol{A}})_{\mathcal{F}} + \boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times \boldsymbol{A}. \quad (4.28)$$

This fundamental equation is called the *transport theorem*. It allows calculating the change of a vector with respect to a particular triad given two pieces of information: how the vector changes with respect to a second triad, and how the second triad rotates with respect to the first.

Note that triad $\mathcal{N}$ does not need to be fixed. Theoretically, it can move as well. So, the transport theorem holds for arbitrary triads, irrespective of how they move.

◯ **Example 4.3** If we define two new triads $\mathcal{G}$ and $\mathcal{D}$ and we know that $\boldsymbol{\Omega}$ is the relative angular velocity of $\mathcal{G}$ with respect to $\mathcal{D}$, we can calculate the change of a vector $\boldsymbol{A}$ with respect to $\mathcal{D}$ by

$$(\dot{\boldsymbol{A}})_{\mathcal{D}} = (\dot{\boldsymbol{A}})_{\mathcal{G}} + \boldsymbol{\Omega} \times \boldsymbol{A}. \tag{4.29E}$$

It does not matter in which triad the vector components are expressed. All triads yield the correct solution, only differently projected. The vectors do need to be expressed in consistent triads though.

◯ **Example 4.4** An equation like the following is correct, because all vectors in this equation are expressed in the same $\mathcal{N}$-triad:

$${}^{\mathcal{N}}(\dot{\boldsymbol{A}})_{\mathcal{N}} = {}^{\mathcal{N}}(\dot{\boldsymbol{A}})_{\mathcal{B}} + {}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times {}^{\mathcal{N}}\boldsymbol{A} \tag{4.30E}$$

and, although expressed in another triad $\mathcal{B}$, the following is also correct, because the vectors are still all expressed in one same triad:

$${}^{\mathcal{B}}(\dot{\boldsymbol{A}})_{\mathcal{N}} = {}^{\mathcal{B}}(\dot{\boldsymbol{A}})_{\mathcal{B}} + {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times {}^{\mathcal{B}}\boldsymbol{A}. \tag{4.31E}$$

Often it is beneficial to denote all vector components in the local triad, and not in the inertial triad. Then, ${}^{\mathcal{B}}(\dot{\boldsymbol{A}})_{\mathcal{B}}$ consists directly of the derivatives of the components of ${}^{\mathcal{B}}\boldsymbol{A}$.

When dealing with derivatives with respect to an inertial $\mathcal{N}$-triad, we often drop the subscript $\mathcal{N}$ in the derivative and call this the *absolute derivative*.

◎ **Example 4.5** This Example is about the governor of Example 2.2 on page 48. Use the transport theorem to show that the velocity of the ball on the left, which has position vector ${}^{\mathcal{F}}\boldsymbol{r}_1 = (0,\, -2L\sin\theta,\, 2L(1-\cos\theta))^{\mathrm{T}}$ is given by the vector ${}^{\mathcal{F}}\dot{\boldsymbol{r}}_1 = (2L\dot{\psi}\sin\theta,\, -2L\dot{\theta}\cos\theta,\, 2L\dot{\theta}\sin\theta)^{\mathrm{T}}$.

Exemplary solution

We define an inertial triad $\mathcal{N}$ such that triad $\mathcal{F}$ coincides with $\mathcal{N}$ for $\psi = 0$. The transport theorem relating relative derivatives of the position vector in these two triads is:

$${}^{\mathcal{F}}(\dot{\boldsymbol{r}}_1)_{\mathcal{N}} = {}^{\mathcal{F}}(\dot{\boldsymbol{r}}_1)_{\mathcal{F}} + {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times {}^{\mathcal{F}}\boldsymbol{r}_1. \tag{4.32E}$$

Substituting the given ${}^{\mathcal{F}}\dot{\boldsymbol{r}}_1$ and the vectors

$${}^{\mathcal{F}}(\dot{\boldsymbol{r}}_1)_{\mathcal{F}} = \begin{pmatrix} 0 \\ -2L\dot{\theta}\cos\theta \\ 2L\dot{\theta}\sin\theta \end{pmatrix}, \quad {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} \tag{4.33E}$$

resolves to the expected velocity vector.

Problem 4.11 A coordinate system xyz, associated with triad $\mathcal{F}$ rotates with an angle ψ about the Z-axis of an inertial coordinate system XYZ, which is associated with triad $\mathcal{N}$. The variables a, b and c are the components of the unit direction vector $\hat{\boldsymbol{f}}_1$ of the rotating triad, expressed in the $\mathcal{F}$-triad, so ${}^{\mathcal{F}}\hat{\boldsymbol{f}}_1 = (a,\, b,\, c)^{\mathrm{T}}$. What are the values of a, b, c? Explain *in words* why in general ${}^{\mathcal{F}}\dot{\hat{\boldsymbol{f}}}_1 \neq (\dot{a},\, \dot{b},\, \dot{c})^{\mathrm{T}}$.

Problem 4.12 Show that if a triad $\mathcal{F}$ rotates with angular velocity $\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$ with respect to triad $\mathcal{N}$, then $(\dot{\boldsymbol{\omega}}_{\mathcal{F}/\mathcal{N}})_{\mathcal{N}} = (\dot{\boldsymbol{\omega}}_{\mathcal{F}/\mathcal{N}})_{\mathcal{F}}$.

If second derivatives are to be calculated, the notation becomes trickier, because it is possible that only the first or only the second or both differentiation steps are relative. To achieve clear notation, for a second derivative where the two reference triads are not identical, we write both triads as successive subscripts from left to right. If $\mathcal{N}$ is a fixed triad and $\mathcal{F}$ is a moving triad, then

$$\frac{\mathrm{d}}{\mathrm{d}t}((\dot{\boldsymbol{A}})_{\mathcal{F}}) = (\ddot{\boldsymbol{A}})_{\mathcal{F}\mathcal{N}}. \tag{4.34}$$

In case both differentiation steps are with respect to the same triad, we only write the letter once, for brevity: $(\ddot{\boldsymbol{A}})_{\mathcal{F}} = (\ddot{\boldsymbol{A}})_{\mathcal{F}\mathcal{F}}$. With these definitions, applying the transport theorem to (4.34) gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}((\dot{\boldsymbol{A}})_{\mathcal{F}}) = (\ddot{\boldsymbol{A}})_{\mathcal{F}} + \boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times (\dot{\boldsymbol{A}})_{\mathcal{F}}. \tag{4.35}$$

Example 4.6 The position vector of a fly in a jar is measured in a jar-fixed triad $\mathcal{J}$ as a vector function of time t: ${}^{\mathcal{J}}\boldsymbol{r} = c\left(0,\, 0,\, t^2\right)^{\mathrm{T}}$, with constant c. Assuming that the jar rotates with constant angular velocity vector ${}^{\mathcal{J}}\boldsymbol{\omega}_{\mathcal{J}/\mathcal{N}} = (\omega_x,\, 0,\, 0)^{\mathrm{T}}$ with respect to an inertial triad $\mathcal{N}$, determine the fly's absolute acceleration.

Exemplary solution

The absolute velocity vector of the fly is

$${}^{\mathcal{J}}(\dot{\boldsymbol{r}})_{\mathcal{N}} = {}^{\mathcal{J}}(\dot{\boldsymbol{r}})_{\mathcal{J}} + {}^{\mathcal{J}}\boldsymbol{\omega}_{\mathcal{J}/\mathcal{N}} \times {}^{\mathcal{J}}\boldsymbol{r} \tag{4.36E}$$

$$= c\begin{pmatrix} 0 \\ 0 \\ 2t \end{pmatrix} + ct^2\omega_x \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -ct^2\omega_x \\ 2ct \end{pmatrix}. \tag{4.37E}$$

The absolute acceleration vector of the fly is

$$
\begin{aligned}
{}^{\mathcal{J}}(\ddot{\boldsymbol{r}})_{\mathcal{N}} &= {}^{\mathcal{J}}(\ddot{\boldsymbol{r}})_{\mathcal{N}\mathcal{J}} + {}^{\mathcal{J}}\boldsymbol{\omega}_{\mathcal{J}/\mathcal{N}} \times {}^{\mathcal{J}}(\dot{\boldsymbol{r}})_{\mathcal{N}} && \text{(4.38E)}\\
&= \begin{pmatrix} 0 \\ -2ct\omega_x \\ 2c \end{pmatrix} + \begin{pmatrix} 0 \\ -2ct\omega_x \\ -ct^2\omega_x^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -4ct\omega_x \\ 2c - ct^2\omega_x^2 \end{pmatrix}. && \text{(4.39E)}
\end{aligned}
$$

4.3 Special Coordinate Systems

The transport theorem enables calculating the velocity of a point with the help of different moving coordinate systems. Several commonly employed moving coordinate systems will be reviewed in the following.

4.3.1 Cylindrical Coordinates

The location of a point P can be uniquely defined in multiple ways. We already saw Cartesian coordinates, for example in an inertial XYZ coordinate system. *Cylindrical coordinates* offer an alternative description via three coordinates: The distance ρ from the Z-axis, the angle φ enclosed between the X-axis and the point's position vector projection onto the XY-plane, and the Z-coordinate Z_P of the point, see Figure 4.6.

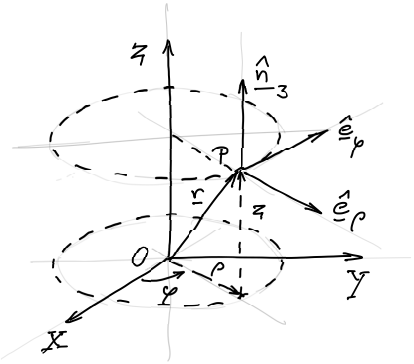


Figure 4.6: Cylindrical Coordinates

We can also define triads based on these coordinates: While the XYZ-coordinate system is associated with a fixed $\mathcal{N}$-triad, the $\rho\varphi Z_P$-coordinates define the rotating $\mathcal{B}$-triad, where the first unit direction vector $\hat{\boldsymbol{b}}_1 = \hat{\boldsymbol{e}}_\rho$ points in the direction of the projection ρ, the second

vector $\hat{\boldsymbol{b}}_2 = \hat{\boldsymbol{e}}_\varphi$ is perpendicular to $\hat{\boldsymbol{e}}_\rho$ in the XY-plane, and the third vector always points in Z-direction, $\hat{\boldsymbol{b}}_3 = \hat{\boldsymbol{n}}_3$.

The position vector $\boldsymbol{r}$ of point P results from vector addition (see Figure 4.6):

$$\boldsymbol{r} = \rho\hat{\boldsymbol{e}}_\rho + Z_P\hat{\boldsymbol{n}}_3. \tag{4.40}$$

The $\mathcal{B}$-triad is rotating with angular velocity

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \dot{\varphi}\hat{\boldsymbol{n}}_3. \tag{4.41}$$

The velocity of point P relative to the fixed XYZ-coordinate system results from applying the transport theorem on its position vector:

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = (\dot{\boldsymbol{r}})_\mathcal{N} = (\dot{\boldsymbol{r}})_\mathcal{B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}. \tag{4.42}$$

Expressing all components in the $\mathcal{B}$-triad leads to

$${}^\mathcal{B}\boldsymbol{v} = {}^\mathcal{B}\dot{\boldsymbol{r}} = \begin{pmatrix} \dot{\rho} \\ 0 \\ \dot{Z}_P \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix} \times \begin{pmatrix} \rho \\ 0 \\ Z_P \end{pmatrix}, \tag{4.43}$$

and finally

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{\rho}\hat{\boldsymbol{e}}_\rho + \dot{\varphi}\rho\hat{\boldsymbol{e}}_\varphi + \dot{Z}_P\hat{\boldsymbol{n}}_3. \tag{4.44}$$

Problem 4.13 Apply the transport theorem again to (4.44) to derive the acceleration components in cylindrical coordinates, so in the $\mathcal{B}$-triad. ►

4.3.2 Spherical Coordinates

The current position of a point P in a *spherical coordinate system* is specified by coordinates $\{r, \theta, \varphi\}$, see Figure 4.7.

We define a rotating triad $\mathcal{B}$-triad, where the unit vector $\hat{\boldsymbol{b}}_1 = \hat{\boldsymbol{e}}_r$ points in the direction of $\boldsymbol{r}$, the unit vector $\hat{\boldsymbol{b}}_3 = \hat{\boldsymbol{e}}_\varphi$ points horizontally and perpendicular to $\hat{\boldsymbol{e}}_r$, so in the direction of increasing φ, and the unit vector $\hat{\boldsymbol{b}}_2 = \hat{\boldsymbol{e}}_\theta$ is perpendicular to both $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_\varphi$ and points in the direction of increasing θ.

With these definitions, the position vector can be expressed as (See Figure 4.7):

$$\boldsymbol{r} = r\hat{\boldsymbol{e}}_r. \tag{4.45}$$

The XYZ-coordinate system defines a fixed $\mathcal{N}$-triad. Triad $\mathcal{B}$ is rotating both about the Z-axis and an axis given by the direction of $\hat{\boldsymbol{e}}_\varphi$. As

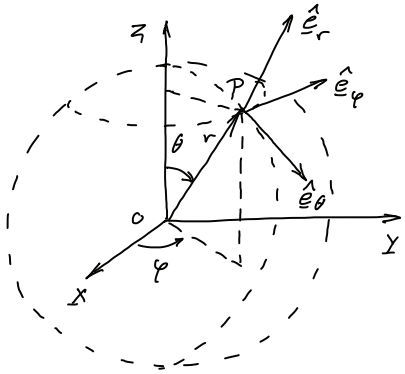


Figure 4.7: Spherical Coordinates

angular velocity vectors can be added, the total angular velocity $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ of the triad is:

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \dot{\varphi}\hat{\boldsymbol{n}}_3 + \dot{\theta}\hat{\boldsymbol{e}}_\varphi = \dot{\varphi}\cos(\theta)\hat{\boldsymbol{e}}_r - \dot{\varphi}\sin(\theta)\hat{\boldsymbol{e}}_\theta + \dot{\theta}\hat{\boldsymbol{e}}_\varphi. \tag{4.46}$$

With this information, combined with the knowledge how the point's coordinates change in the $\mathcal{B}$-triad, the transport theorem provides the absolute velocity of the point with respect to the $\mathcal{N}$-triad:

$$(\dot{\boldsymbol{r}})_\mathcal{N} = (\dot{\boldsymbol{r}})_\mathcal{B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}. \tag{4.47}$$

Choosing to express all components in the $\mathcal{B}$ triad, we obtain

$${}^\mathcal{B}(\dot{\boldsymbol{r}})_\mathcal{N} = \begin{pmatrix} \dot{r} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \dot{\varphi}\cos(\theta) \\ -\dot{\varphi}\sin(\theta) \\ \dot{\theta} \end{pmatrix} \times \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}, \tag{4.48}$$

and finally

$$(\dot{\boldsymbol{r}})_\mathcal{N} = \dot{r}\hat{\boldsymbol{e}}_r + r\dot{\theta}\hat{\boldsymbol{e}}_\theta + r\dot{\varphi}\sin(\theta)\hat{\boldsymbol{e}}_\varphi. \tag{4.49}$$

Problem 4.14 Show that $\hat{\boldsymbol{n}}_3 = \cos(\theta)\hat{\boldsymbol{e}}_r - \sin(\theta)\hat{\boldsymbol{e}}_\theta$. Use two different approaches, once by establishing and using the rotation matrix ${}^\mathcal{B}\mathbf{C}_\mathcal{N}$, and once using vector addition and Figure 4.7.

Problem 4.15 Apply the transport theorem to derive the acceleration components of the point expressed in triad $\mathcal{B}$. ▷

4.3.3 Natural Coordinates

In *natural coordinates* (Figure 4.8), the point P is described in terms of its scalar position coordinate s along a given path from A to B. Hereby, s is the total distance the point has traveled since starting in A (so the length of the path from A to P).

In any instant, the movement of the point in an infinitesimally small time interval can be approximated by a circular, 2-D movement. The circle is called the *osculating circle*, because it *kisses* (*osculare* in Latin) the path of the point in the neighbourhood of the considered location.

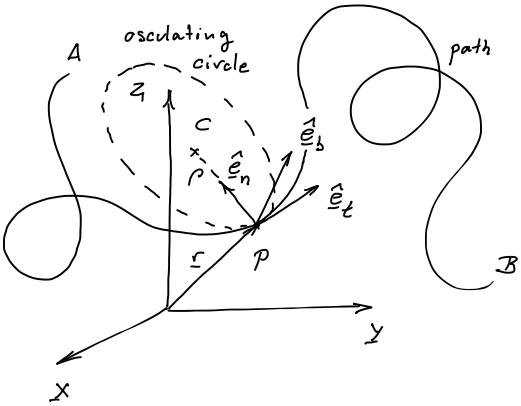


Figure 4.8: Natural Coordinates

We use this osculating circle to define a rotating triad: The unit vector $\hat{\boldsymbol{e}}_\mathrm{t}$ is defined to be tangent to the path, and pointing in the direction of increasing s. The unit vector $\hat{\boldsymbol{e}}_\mathrm{n}$ is normal to the path and points to the center C of the osculating circle. The unit vector $\hat{\boldsymbol{e}}_\mathrm{b}$ is called the binormal unit vector, and it is defined by $\hat{\boldsymbol{e}}_\mathrm{b} := \hat{\boldsymbol{e}}_\mathrm{t} \times \hat{\boldsymbol{e}}_\mathrm{n}$. That means the vector is orthogonal to the osculating circle.

The velocity vector of point P is, by the above definition, equal to the speed $\dot{s}$ of P along its path multiplied by $\hat{\boldsymbol{e}}_\mathrm{t}$, so

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{s}\hat{\boldsymbol{e}}_\mathrm{t}. \tag{4.50}$$

The acceleration results from the transport theorem, with the definition of ρ being the radius of the osculating circle (see Problem 4.17):

$$\ddot{\boldsymbol{r}} = \ddot{s}\hat{\boldsymbol{e}}_\mathrm{t} + \frac{\dot{s}^2}{\rho}\hat{\boldsymbol{e}}_\mathrm{n}. \tag{4.51}$$

Problem 4.16 This Problem is about the Earth orbiting the Sun of Problem 3.36 on page 101. At a certain instant, the position vector ${}^{\mathcal{N}}\boldsymbol{D}$ of point D (Delft), as well as its current velocity ${}^{\mathcal{N}}\dot{\boldsymbol{D}}$ and acceleration ${}^{\mathcal{N}}\ddot{\boldsymbol{D}}$ are given with components expressed in the $\mathcal{N}$-triad. For this instant, calculate the tangential and the binormal unit vectors ${}^{\mathcal{N}}\hat{\boldsymbol{e}}_{\mathrm{t}}$ and ${}^{\mathcal{N}}\hat{\boldsymbol{e}}_{\mathrm{b}}$ of a natural coordinate triad associated with the path of point D, with components expressed in the $\mathcal{N}$-triad, as vector functions of ${}^{\mathcal{N}}\boldsymbol{D}$, ${}^{\mathcal{N}}\dot{\boldsymbol{D}}$ and ${}^{\mathcal{N}}\ddot{\boldsymbol{D}}$. ▷►

Problem 4.17 Try deriving (4.51) yourself.[a]

[a] *Hint*: In order to derive this equation, you need to recognize that angular velocity $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ of the moving triad is a vector that is orthogonal to the circular movement, so orthogonal to the plane that contains the osculating circle. A relationship between the magnitude of $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$, the speed $\dot{s}$, and the circle's radius ρ can be found by knowing that the center of the circle is the instantaneous center of rotation of point P.

4.4 Relative and Absolute Motion

4.4.1 Relative and Absolute Velocity

After the above special cases, we now investigate the general case of calculating velocities and accelerations of a point relative to different coordinate systems. Again, we consider a fixed coordinate system XYZ with axis directions of triad $\mathcal{N}$, and a moving (rotating and translating) coordinate system xyz with axis directions of triad $\mathcal{F}$.

With the transport theorem (4.28) of Section 4.2, we are able to describe the derivative of position vector $\boldsymbol{r}_{P/O}$ as a function of the derivatives of the relative position vector $\boldsymbol{r}_{P/Q}$ with respect to the xyz-coordinate system, and of the position vector $\boldsymbol{r}_{Q/O}$ of the origin of this moving coordinate system.

To abbreviate notations, we use the shorthand notations

$$\boldsymbol{Q} := \boldsymbol{r}_{Q/O}, \quad \boldsymbol{P} := \boldsymbol{r}_{P/O}, \quad \boldsymbol{p} := \boldsymbol{r}_{P/Q}. \tag{4.52}$$

Figure 4.9 visualizes these vectors.

As the absolute position can be found from

$$\boldsymbol{P} = \boldsymbol{Q} + \boldsymbol{p}, \tag{4.53}$$

also the velocities can be found from addition:

$$\dot{\boldsymbol{P}} = \dot{\boldsymbol{Q}} + \dot{\boldsymbol{p}}. \tag{4.54}$$

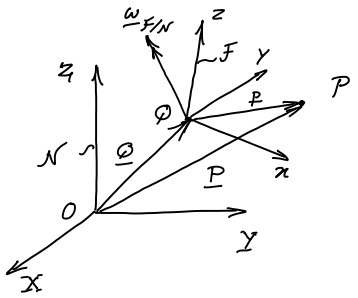


Figure 4.9: Visualization of vectors used to describe relative velocity and acceleration

The derivative $\dot{\boldsymbol{P}} = \dot{\boldsymbol{r}}_{P/O}$ is the *absolute velocity* of point P. The derivative $\dot{\boldsymbol{p}} = \dot{\boldsymbol{r}}_{P/Q}$ is the *relative velocity* of P with respect to Q.

Often, we know the derivative of vector $\boldsymbol{Q}$ as measured in the fixed XYZ coordinate system, as it is the velocity of the origin of the rotating coordinate system.

We also often know the relative rate of change of $\boldsymbol{p}$, as measured with respect to the moving triad $\mathcal{F}$. The transport theorem allows calculating the absolute rate of change in $\boldsymbol{p}$ with respect to triad $\mathcal{N}$:

$$(\dot{\boldsymbol{p}})_{\mathcal{N}} = (\dot{\boldsymbol{p}})_{\mathcal{F}} + \boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times \boldsymbol{p}, \tag{4.55}$$

where $\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$ is the angular velocity of the $\mathcal{F}$-triad.

Therefore, the velocity of point P is

$$\boldsymbol{v}_P = \dot{\boldsymbol{P}} = \underbrace{\dot{\boldsymbol{Q}}}_{\text{velocity of } Q} + \underbrace{(\dot{\boldsymbol{p}})_{\mathcal{F}} + \boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times \boldsymbol{p}}_{\text{velocity of } P \text{ relative to } Q}. \tag{4.56}$$

This can also be written as

$$\boldsymbol{v}_P = \underbrace{\boldsymbol{v}_Q}_{\text{velocity of } Q} + \underbrace{\boldsymbol{v}_{P/Q}}_{\text{velocity of } P \text{ relative to } Q}. \tag{4.57}$$

Problem 4.18 Use (4.56) to confirm the velocity components in cylindrical and spherical coordinate systems.

Example 4.7 Consider a falling smartphone, as in Figure 4.10. The owner of the phone realizes it is falling, and tries to catch it. He manages to get his hand under the phone, but unfortunately cannot catch it cleanly.

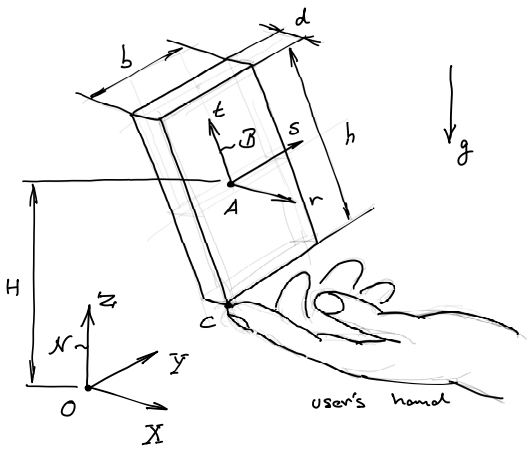


Figure 4.10: Falling smartphone

The figure indicates an inertial XYZ coordinate system (associated with a triad $\mathcal{N}$), and a rotating triad $\mathcal{B}$ associated with an rst coordinate system, which is fixed to the phone and has its origin A in the centroid of the phone.

At a certain instant t_i, the phones is in contact with the user's hand, and the rotation matrix of the phone's body-fixed $\mathcal{B}$-triad with respect to the $\mathcal{N}$-triad is

$$ {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{3} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix}. \tag{4.58E} $$

The angular velocity vector of the phone at the instant t_i, with components expressed in the $\mathcal{B}$-triad, is ${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathrm{p}} = \frac{\omega_{\mathrm{p}}}{2}\sqrt{2}(-1,\, 1,\, 0)^{\mathrm{T}}$, with given magnitude ω_{p}. Point A is on the Z-axis, at a height H above the XY-plane, and it has zero velocity at this instant.

Determine the components of the velocity vector ${}^{\mathcal{B}}\boldsymbol{v}_C$ of point C at instant t_i, expressed in the $\mathcal{B}$-triad, as functions of $\omega_{\mathrm{p}}, b, d, h, H$ (Remark: None of these terms has to necessarily appear in the requested functions).

Exemplary solution

We can find the velocity of point C by summing the velocity of point A and the relative velocity of C with respect to A:

$$ {}^{\mathcal{B}}\boldsymbol{v}_C = {}^{\mathcal{B}}\boldsymbol{v}_A + {}^{\mathcal{B}}\boldsymbol{v}_{C/A}. \tag{4.59E} $$

Points A and C are both part of the same rigid body, such that the relative movement of C can be described as a pure rotation about point A. Also, point A has zero velocity (so it is a point on the body's instantaneous axis

of rotation). Filling in the values, we find:

$$^{\mathcal{B}}\boldsymbol{v}_C = \mathbf{0} + {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathrm{p}} \times {}^{\mathcal{B}}\boldsymbol{r}_{C/A} = \frac{\omega_{\mathrm{p}}}{2}\sqrt{2}\begin{pmatrix}-1\\1\\0\end{pmatrix} \times \frac{1}{2}\begin{pmatrix}d\\-b\\-h\end{pmatrix}, \tag{4.60E}$$

resulting in

$$^{\mathcal{B}}\boldsymbol{v}_C = \frac{\omega_{\mathrm{p}}}{4}\sqrt{2}\begin{pmatrix}-h\\-h\\b-d\end{pmatrix}. \tag{4.61E}$$

Example 4.8 This Example is about the rolling disk on an arm of Problem 3.31 on page 97.

Find $\dot{\psi}$ as a function of $\dot{\varphi}$, R, and r (Remark: It is possible that not all of these terms appear in the requested function), such that the disk rolls on the ground without slipping in any direction.

Exemplary solution

Following the transport theorem, the velocity of point P is a function of the velocity of point B, the angular velocity $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ of the disk, and the relative position of P with respect to B:

$$\boldsymbol{v}_P = \boldsymbol{v}_B + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{P/B}. \tag{4.62E}$$

It simplifies equations if a point D on the Z-axis is introduced that lies at the same height as point B, as shown in Figure 4.11.

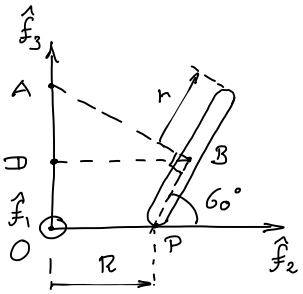


Figure 4.11: Point D on the Z-axis of the thin disk on an arm.

The velocity of B can be expressed as a combined movement of a rotation about the Z-axis and unknown translations v_{By} and v_{Fz} in y and z-directions of the $\mathcal{F}$-triad, which are caused by possible axial movement in the bearings:

$$^{\mathcal{F}}\boldsymbol{v}_B = {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times {}^{\mathcal{F}}\boldsymbol{r}_{B/D} + \begin{pmatrix}0 & v_{Fy} & v_{Fz}\end{pmatrix}^{\mathrm{T}}. \tag{4.63E}$$

Substituting the values, we obtain

$$ {}^{\mathcal{F}}\boldsymbol{v}_B = \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix} \times \begin{pmatrix} 0 \\ R + r\cos 60^\circ \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_{Fy} \\ v_{Fz} \end{pmatrix} = \begin{pmatrix} -\dot{\varphi}\left(R + \frac{r}{2}\right) \\ v_{Fy} \\ v_{Fz} \end{pmatrix}. \quad (4.64\text{E}) $$

For the total angular velocity and the position vector, we obtain

$$ {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} + {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{F}} = \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2}\sqrt{3}\dot{\psi} \\ -\frac{1}{2}\dot{\psi} \end{pmatrix}, \quad (4.65\text{E}) $$

$$ {}^{\mathcal{F}}\boldsymbol{r}_{P/B} = \begin{pmatrix} 0 \\ -\frac{r}{2} \\ -\frac{1}{2}\sqrt{3}r \end{pmatrix}. \quad (4.66\text{E}) $$

By substituting in (4.62E), we find that point P has the velocity

$$ {}^{\mathcal{F}}\boldsymbol{v}_P = \begin{pmatrix} -\dot{\varphi}\left(R + \frac{r}{2}\right) - \frac{3}{4}r\dot{\psi} + \frac{r}{2}\left(\dot{\varphi} - \frac{1}{2}\dot{\psi}\right) \\ v_{Fy} \\ v_{Fz} \end{pmatrix} = \begin{pmatrix} -R\dot{\varphi} - r\dot{\psi} \\ v_{Fy} \\ v_{Fz} \end{pmatrix}. \quad (4.67\text{E}) $$

Since the disk rolls on the ground without slip, $\boldsymbol{v}_P = \boldsymbol{0}$, which leads to the final answer:

$$ \dot{\psi} = -\frac{R}{r}\dot{\varphi}. \quad (4.68\text{E}) $$

4.4.2 Relative and Absolute Acceleration

Finding the acceleration $\boldsymbol{a}_P$ requires differentiation of $\boldsymbol{v}_P$ with respect to time again. After some simplifications, we obtain from (4.56):

$$ \boldsymbol{a}_P = \ddot{\boldsymbol{P}} = \underbrace{\ddot{\boldsymbol{Q}}}_{1.} + \underbrace{(\ddot{\boldsymbol{p}})_{\mathcal{F}}}_{2.} + \underbrace{\dot{\boldsymbol{\omega}} \times \boldsymbol{p}}_{3.} + \underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{p})}_{4.} + \underbrace{2\boldsymbol{\omega} \times (\dot{\boldsymbol{p}})_{\mathcal{F}}}_{5.}, \quad (4.69) $$

where $\boldsymbol{\omega}$ is a shorthand notation for $\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$. These terms represent the following:

1. The acceleration of Q, the origin of the xyz-coordinate system, relative to the fixed XYZ coordinate system.
2. The acceleration of P relative to the xyz-coordinate system. So this is the second derivative of the relative position vector components, as measured inside the $\mathcal{F}$-triad.

3. Acceleration caused by angular acceleration, so it is a change in tangential velocity $\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times \boldsymbol{p}$ due to changing angular velocity $\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$. This term is also called *Euler acceleration*.
4. The *centripetal acceleration*, and it is directed towards an axis that is parallel to the instantaneous axis of rotation of the xyz coordinate system and passing through point Q.
5. The *Coriolis acceleration*. It occurs if the point has a velocity relative to the moving xyz-coordinate system.

Note that many of these acceleration components only occur because we are looking at a moving coordinate system, and we are also describing movements with respect to this moving coordinate system.

Example 4.9 This Example is about the rotor of Problem 1.22 on page 37. The rotor has not been statically balanced properly, and the center of mass of the combined body is still not on the axle. The vector from the origin O to the combined body's center of mass is ${}^{\mathcal{B}}\boldsymbol{r}_C = (r_x,\, r_y,\, r_z)^{\mathrm{T}}$ in the xyz coordinate system with associated rotor-fixed triad $\mathcal{B}$. The rotor is spinning about the x-axis with constant angular rate ω_x. Show algebraically that the acceleration vector of the center of mass of the combined body is given by ${}^{\mathcal{B}}\ddot{\boldsymbol{r}}_C = (0,\, -\omega_x^2 r_y,\, -\omega_x^2 r_z)^{\mathrm{T}}$.

Exemplary solution

Acceleration in the rotating triad can be calculated by (4.69), where we choose $\mathcal{B}$ to take the role of $\mathcal{F}$, the point O to be the fixed point O, and C to take the role of P. With these choices, we obtain:

- $\ddot{\boldsymbol{Q}} = \boldsymbol{0}$, because the origin of the xyz-coordinate system is fixed.
- ${}^{\mathcal{B}}\boldsymbol{p} = {}^{\mathcal{B}}\boldsymbol{r}_C$ is constant and therefore $(\dot{\boldsymbol{p}})_{\mathcal{B}} = \boldsymbol{0}$ and $(\ddot{\boldsymbol{p}})_{\mathcal{B}} = \boldsymbol{0}$.
- $\dot{\boldsymbol{\omega}} = \boldsymbol{0}$, because the angular velocity $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ is constant.

This means that terms 1. 2. 3. and 5. all equal zero and thus (4.69) simplifies to:

$$\ddot{\boldsymbol{r}}_P = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{p}). \tag{4.70E}$$

With all vectors expressed in $\mathcal{B}$, we obtain:

$$\begin{aligned} {}^{\mathcal{B}}\ddot{\boldsymbol{r}}_C &= {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times ({}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times {}^{\mathcal{B}}\boldsymbol{p}) && \text{(4.71E)} \\ &= \begin{pmatrix} \omega_x \\ 0 \\ 0 \end{pmatrix} \times \left(\begin{pmatrix} \omega_x \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} \right) = \begin{pmatrix} 0 \\ -\omega_x^2 r_y \\ -\omega_x^2 r_z \end{pmatrix}. && \text{(4.72E)} \end{aligned}$$

Example 4.10 One of the best known elements of a modern roller coaster is the vertical loop (Figure 4.12). Interestingly, such loops are shaped as "reversed teardrops", and not as perfect circles.

This shape can be explained by considering centripetal acceleration of the passengers, because a human body can only handle forces that are equivalent to about $5g$ along its longitudinal (spinal) axis [73].

The acceleration profile depends on the shape of the vertical loops. Early roller coasters indeed incorporated circular loops, but induced such high normal accelerations that it injured or even killed passengers. Reversed teardrop shapes allow for smaller normal accelerations. To reveal why some shapes are preferred over others, we model the roller coaster and examine the dynamics.

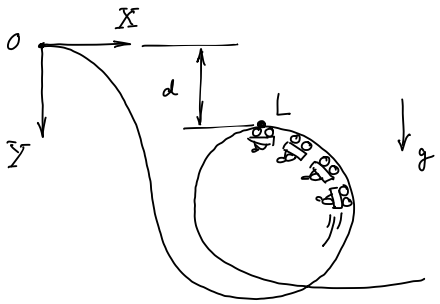


Figure 4.12: Roller coaster with vertical loop.

We model the track as frictionless and consider only a single wagon. The wagon, including its passengers, is modeled as a particle of mass m. The origin O of an XY-coordinate system is chosen as the highest point of the roller coaster, where the wagon is (momentarily) exactly at rest. The wagon's height is described by the coordinate y_w. The vertical distance between the highest point of the roller coaster and the highest point L of the loop is d. The gravitational acceleration has magnitude g and points in positive Y-direction.

We take the origin O as reference height for zero potential energy and find from conservation of energy for the speed v of the wagon at any location that

$$0 = \tfrac{1}{2}mv^2 - mgy_\mathrm{w}, \tag{4.73E}$$

so

$$v = \sqrt{2gy_\mathrm{w}}. \tag{4.74E}$$

The centripetal acceleration component a_c at a height where the track's osculating circle has the radius r follows as (see Problem 4.19):

$$a_\mathrm{c} = \frac{v^2}{r} = \frac{2gy_\mathrm{w}}{r}. \tag{4.75E}$$

Now, we consider a circular loop with constant radius R. In that case the wagon experiences a centripetal acceleration at the top of

$$a_{cL} = \frac{2gd}{R}.$$

At the bottom, $y_w = d + 2R$, so the centripetal acceleration is

$$a_{cB} = \frac{2g(d+2R)}{R} = a_{cL} + 4g.$$

In the "best" case in terms of load on the body (though thrill-seekers might disagree, as they do not ride a roller coaster to feel the least "g-forces") the wagon is almost at rest at the top of the loop. Then, centripetal accelerations are lowest, with $a_{cL} = 0$ and $a_{cB} = 4g$. Including forces due to gravity, the passengers will experience a total force equivalent to $5g$ at the bottom of a circular loop. This is close to the health threshold of humans (Note that this is not the acceleration itself, which still remains $4g$).

For a more realistic limiting case of "weightlessness" at the top (meaning that the normal force between particle and track is zero, see Problem 4.20), the drop distance needs to be $d = R/2$, such that $a_{cL} = g$ and the centripetal acceleration equals gravitational acceleration. In that case, the force at the bottom is equivalent to $6g$, which is above the general human threshold.

To reduce the acceleration, the shape of the loop has to be altered. A "reversed teardrop" shape has a smaller radius at the top than it has at the bottom. This way, the radius is larger at higher speeds and thus the centripetal acceleration is kept within limits. This teardrop shape can be designed in different ways [55].

Problem 4.19 This Problem is about the roller coaster of Example 4.10 on page 127. The explanation made implicit assumptions on triads and reference points. Make these definitions explicit and bring them into the form of Section 4.4, so draw the reference point Q and the axis directions of triad $\mathcal{F}$, as well as the vectors $\boldsymbol{Q}$ and $\boldsymbol{p}$, as defined in Figure 4.9, into Figure 4.12. Note that the osculating circle was defined in Section 4.3.3. Do this for a general position in the loop, and do not assume that the loop is a perfect circle. Also explain how the angular velocity vector $\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$ is defined and can be calculated. Then determine and explain which of the five acceleration components in (4.69) are zero and which are not.

Problem 4.20 This Problem is about the roller coaster of Example 4.10 on page 127. What is the magnitude and direction of the normal force the track exerts on the wagon at the top, at the bottom, and at a third location of your choice in the loop if $d = 0$ and the track is circular with radius R? Start by drawing free-body diagrams of the wagon/passenger unit and clearly label all forces.

Problem 4.21 A ship is installing a monopile for an offshore wind turbine (see Figure 4.13). The monopile is suspended from a crane of arm length d and height $3d$ by a massless cable of length l, connecting points D and G. The crane is rigid and fixed to the ship.

The following coordinate systems with associated triads are defined:

XYZ: ENU coordinate system, X points East and Z points vertically up, origin O at a lighthouse, triad $\mathcal{N}$.

rst: Ship-fixed, r points to bow, t up along the crane, crane arm is always parallel to the s-axis, origin K on bottom of mast, triad $\mathcal{S}$.

uvw: Monopile-fixed, w points longitudinally, origin P in monopile center of mass, triad $\mathcal{M}$.

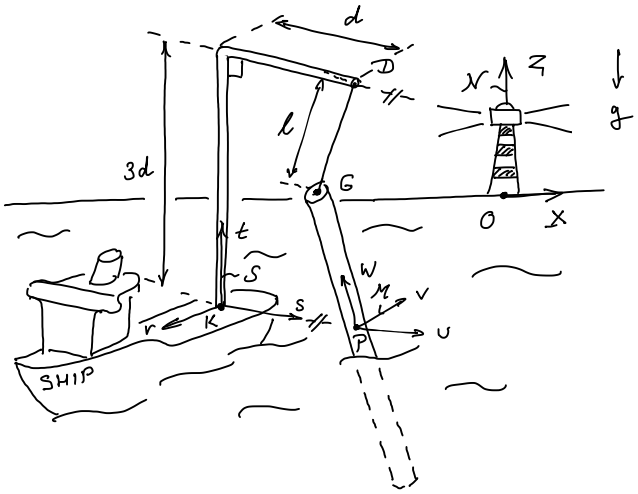


Figure 4.13: Ship installing a monopile.

The cable is assumed to be always in tension and of constant length l. The monopile is a rigid body with height h, mass m, and negligible diameter. The gravitational field is directed in negative Z-direction and has the constant g.

Given is the acceleration of point D obtained with the transport theorem:

$$
\begin{aligned}
{}^{\mathcal{S}}\ddot{\boldsymbol{r}}_D = {}^{\mathcal{S}}\mathbf{C}_{\mathcal{N}}\,{}^{\mathcal{N}}\boldsymbol{a}_K + {}^{\mathcal{S}}(\ddot{\boldsymbol{p}})_{\mathcal{S}} + {}^{\mathcal{S}}\dot{\boldsymbol{\omega}}_{\mathrm{s}} \times {}^{\mathcal{S}}\boldsymbol{p} \\
+ \underbrace{{}^{\mathcal{S}}\boldsymbol{\omega}_{\mathrm{s}} \times \left({}^{\mathcal{S}}\boldsymbol{\omega}_{\mathrm{s}} \times {}^{\mathcal{S}}\boldsymbol{p}\right)}_{1.} + \underbrace{2\,{}^{\mathcal{S}}\boldsymbol{\omega}_{\mathrm{s}} \times {}^{\mathcal{S}}(\dot{\boldsymbol{p}})_{\mathcal{S}}}_{2.},
\end{aligned}
\tag{4.76P}
$$

with a relative position vector $\boldsymbol{p}$, acceleration ${}^{\mathcal{N}}\boldsymbol{a}_K$ of point K, the ship's angular velocity ${}^{\mathcal{S}}\boldsymbol{\omega}_{\mathrm{s}}$ and angular acceleration ${}^{\mathcal{S}}\dot{\boldsymbol{\omega}}_{\mathrm{s}}$, and rotation matrix ${}^{\mathcal{S}}\mathbf{C}_{\mathcal{N}}$.

State the names of the terms 1. and 2. in (4.76P). ▷

Problem 4.22 This Problem is about the ship installing a monopile of Problem 4.21 on page 130. State the components of the vectors ${}^{\mathcal{S}}\boldsymbol{p}$, ${}^{\mathcal{S}}(\dot{\boldsymbol{p}})_{\mathcal{S}}$, and ${}^{\mathcal{S}}(\ddot{\boldsymbol{p}})_{\mathcal{S}}$ as functions of l, h, d such that (4.76P) holds. ▷

Problem 4.23 Use (4.69) to calculate the acceleration components in cylindrical and spherical coordinate systems.

Problem 4.24 Derive (4.69) from (4.56). In order not to forget terms, be aware of (4.35).

Problem 4.25 Figure 4.14 shows a biker riding in circles with diameter D about the Z-axis of the inertial coordinate system XYZ with origin O, which is associated with triad $\mathcal{N}$.

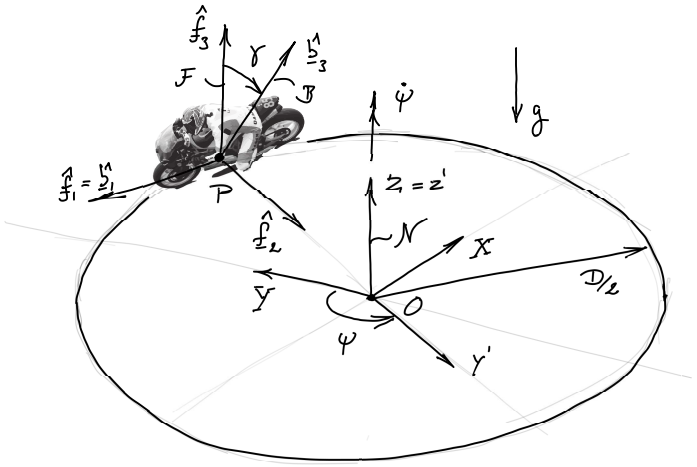


Figure 4.14: Motorbike [Photo NovaBike02, courtesy of Roy Mulder, NovaBike Racing Team, TU Delft].

Triad $\mathcal{F}$ is associated with coordinate system $x'y'z'$, which is chosen such that the x'-axis always points in the tangential direction of the bike's path and the local z'-axis coincides with the inertial Z-axis. The origin of this coordinate system always coincides with O and the angle between the Y-axis and the y'-axis is ψ.

A third coordinate system xyz has axis directions of triad $\mathcal{B}$, and its origin P lies in $-\frac{D}{2}\hat{\boldsymbol{f}}_2$, with respect to O. Triad $\mathcal{B}$ is rotated with respect to triad $\mathcal{F}$, by an angle γ about the *negative* x'-axis.

The biker and his bike are simplified as one rigid body having mass m. This body's center of mass is located at point C, which has the relative position vector $\boldsymbol{p}_C = h\hat{\boldsymbol{b}}_3$ with respect to point P, with constant h. Gravity with acceleration constant g acts in negative Z-direction.

Calculate the Coriolis acceleration of point C that is apparent in the $x'y'z'$

coordinate system. Express all components in the $\mathcal{F}$-triad, as functions of $\psi, \gamma, \dot{\psi}, \dot{\gamma}, \ddot{\psi}, \ddot{\gamma}$ and the given constants (Remark: None of these terms has to necessarily appear in the requested functions). ▶

Problem 4.26 Compare (4.69) with (4.51). Which terms correspond? Explain why all other terms in (4.69) are zero. Start your explanation by drawing the vectors $\boldsymbol{Q}$ and $\boldsymbol{p}$ into Figure 4.8. Also remember that the osculating circle is a second-order approximation of the particle's path at the considered location.

4.5 Checking Plausibility and Frequent Mistakes

When calculating velocities and accelerations ((4.56) or (4.69)) of a point using rotating triads, useful plausibility checks, in addition to Section 1.4, are to:

- use multiple different rotating and translating coordinate systems. The result should always be the same.
- express the position of a point in Cartesian coordinates of a fixed coordinate system, and to take the double derivative (possibly with the help of a tool like the MATLAB$^{\text{TM}}$ Symbolic toolbox, to avoid lengthy calculations).
- choose special coordinate systems such as cylindrical and natural coordinates, and to use both the special equations as well as the more general ones.
- check units, while keeping in mind that $1\,\text{rad/s} = 1\,\text{s}^{-1}$.

Particularly frequent mistakes are

- adding vectors with components expressed in two different triads. This mistake is easy to prevent by always indicating which projections are considered by a top left letter.
- not being clear about which angular velocity is considered (of a body, of a triad, relative or absolute...).
- applying scalar equations from planar (2-D) dynamics. There, some terms may not exist.
- filling in specific, or even numerical values for positions or velocities at certain time instants *before* taking derivatives, such that the values appear constant and their derivatives zero.

It is also vital to remember that a rigid body has only *one* absolute angular velocity *vector*.

4.6 Summary

The movement of any rigid body can instantaneously be described by the velocity of its center of mass and an angular velocity, which describes the change in the body's orientation. Angular velocity is related to the instantaneous axis of rotation via:

$$\boldsymbol{\omega} = \dot{\varphi}\hat{\boldsymbol{a}}_{\mathrm{i}}, \tag{4.77}$$

This axis is the principal axis of rotation for an infinitesimally small rotation. In the case of a 2-D movement, the instantaneous axis of rotation becomes an instantaneous center of rotation.

For infinitesimal rotations, the order of rotations does not matter. Therefore, angular velocity vectors of several rotations may be added, for example:

$$\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{F}/\mathcal{G}} + \boldsymbol{\omega}_{\mathcal{G}/\mathcal{N}}. \tag{4.78}$$

A fundamental equation to calculate derivatives of vectors with respect to time is the transport theorem:

$$(\dot{\boldsymbol{A}})_{\mathcal{N}} = (\dot{\boldsymbol{A}})_{\mathcal{F}} + \boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times \boldsymbol{A}. \tag{4.79}$$

It breaks the calculation of a derivative down to two components:

1. Change of the vector's projections onto the directions of the rotating triad $\mathcal{F}$, so $(\dot{\boldsymbol{A}})_{\mathcal{F}}$,
2. Change of the vector caused by rotation of the triad $\mathcal{F}$ relative to the triad $\mathcal{N}$ with angular velocity vector $\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$.

Frequent examples that require such rotating triads are cylindrical, spherical, or natural coordinate systems.

The generic calculation of velocity and acceleration in translating and rotating coordinate systems is enabled by the equations

$$\boldsymbol{v}_P = \dot{\boldsymbol{P}} = \dot{\boldsymbol{Q}} + (\dot{\boldsymbol{p}})_{\mathcal{F}} + \boldsymbol{\omega} \times \boldsymbol{p} \tag{4.80}$$

and

$$\boldsymbol{a}_P = \ddot{\boldsymbol{P}} = \ddot{\boldsymbol{Q}} + (\ddot{\boldsymbol{p}})_{\mathcal{F}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{p} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{p}) + 2\boldsymbol{\omega} \times (\dot{\boldsymbol{p}})_{\mathcal{F}}, \tag{4.81}$$

where $\boldsymbol{\omega}$ is short for $\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$.

4.7 Problems

4.7.1 Guided Problems

Problem 4.27 The tippe top from Problem 1.16 on page 32 (see Figure 4.15). The tippe top's spherical bottom has radius R and the geometric center S. The tippe top's rotation is described via precession, nutation, and spin, as in Problem 1.16.

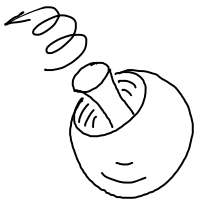


Figure 4.15: Tippe Top.

a. Draw *kinematic diagrams* of the tippe top. These should show the Euler angles and associated angular velocity vectors (with scalar labels), as well as a minimum number of further coordinates needed to fully describe the kinematic state of the system. You will need multiple views and multiple coordinate systems and/or triads for this purpose, because you may only draw vectors that are contained within or orthogonal to the paper plane.
Include four coordinate systems/triads:
   - Inertial coordinate system XYZ with origin O. The axes directions also define triad $\mathcal{N}$.
   - A tippe-top-fixed triad $\mathcal{B}$ with unit vectors $\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3$
   - An intermediate triad $\mathcal{A}$ with unit vectors $\hat{\boldsymbol{a}}_1, \hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_3$. This triad performs only precession; it does not follow the nutation or spin rotation of the tippe top.
   - An intermediate triad $\mathcal{G}$ with unit vectors $\hat{\boldsymbol{g}}_1, \hat{\boldsymbol{g}}_2, \hat{\boldsymbol{g}}_3$. This triad performs precession and nutation, but it does not follow spin of the tippe top about its symmetry axis.

   You may refer to the drawings provided with Problem 1.16 for inspiration regarding such projections. However, be aware that XYZ was not inertial in Problem 1.16.

b. Calculate the angular velocity vectors ${}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{A}/\mathcal{N}}$, ${}^{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{G}/\mathcal{A}}$, and ${}^{\mathcal{G}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{G}}$, with components as a function of the precession, nutation and spin angles φ, θ, ψ, respectively, and angular speeds $\dot{\varphi}$, $\dot{\theta}$, and $\dot{\psi}$.

c. Use your result from part b, in combination with vector addition, to calculate the tippe top's angular velocity ${}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$, with components

as functions of φ, θ, ψ, $\dot{\varphi}$, $\dot{\theta}$, and $\dot{\psi}$. Do not forget to apply rotation matrices where needed.

d. We will now use the transport theorem, in the form of (4.56), with the associated Figure 4.9.
The goal is to calculate the velocity vector of point K, which is the point on the tippe top that is currently in contact with the ground, as a function of the previously defined kinematic variables.
We will use four different associations. First, use the below table with associations between the definitions in the book and this example. Using these associations, state which vectors the variables ${}^{\mathcal{F}}\boldsymbol{Q}$, ${}^{\mathcal{F}}\boldsymbol{p}$ and ${}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$ correspond to, using the index notation of this course. [a].
Then, calculate the components of ${}^{\mathcal{F}}\boldsymbol{Q}$, ${}^{\mathcal{F}}\boldsymbol{p}$ and ${}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$, as functions of φ, θ, ψ, $\dot{\varphi}$, $\dot{\theta}$, and $\dot{\psi}$, the radius R, and the minimal number of further variables you defined in your kinematic diagrams.

Theory (4.56)	**This example**
point P	point K
origin of XYZ	point O
point Q	point S
$\mathcal{F}$	$\mathcal{G}$
$\mathcal{N}$	$\mathcal{N}$

e. Use these associations between the definitions in the book and this example:

Theory (4.56)	**This example**
point P	point K
origin of XYZ	point O
point Q	point S
$\mathcal{F}$	$\mathcal{N}$
$\mathcal{N}$	$\mathcal{N}$

Using these associations, state the vectors ${}^{\mathcal{N}}\boldsymbol{Q}$, ${}^{\mathcal{N}}\boldsymbol{p}$, and ${}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$ and calculate their components.

f. Use these associations between the definitions in the book and this example:

Theory (4.56)	**This example**
point P	point K
origin of XYZ	point O
point Q	point S
$\mathcal{F}$	$\mathcal{B}$
$\mathcal{N}$	$\mathcal{N}$

Using these associations, state the vectors ${}^{\mathcal{F}}\boldsymbol{Q}$, ${}^{\mathcal{F}}\boldsymbol{p}$, and ${}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$ and calculate their components.

g. Use these associations between the definitions in the book and this example:

Theory (4.56)	**This example**
point P	point K
origin of XYZ	point O
point Q	point O
$\mathcal{F}$	$\mathcal{A}$
$\mathcal{N}$	$\mathcal{N}$

Using these associations, state the vectors ${}^{\mathcal{N}}\boldsymbol{Q}$, ${}^{\mathcal{N}}\boldsymbol{p}$, and ${}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$ and calculate their components.

h. Use the transport theorem and the vectors found in the previous parts to calculate, separately for two associations of your choice, the missing elements in (4.56), so ${}^{\mathcal{F}}(\dot{\boldsymbol{p}})_{\mathcal{F}}$ and ${}^{\mathcal{F}}\dot{\boldsymbol{Q}}$, with components expressed in the respective triad $\mathcal{F}$. Make clear each time which association you chose. [b]

Then, again separately for the two associations, substitute all variables in (4.56). Express the resulting velocity vectors in the same triad (for example $\mathcal{A}$), and compare results.

Optional: Challenge yourself by repeating this for the remaining two associations.

i. Calculate the acceleration vector of the center of mass C of the tippe top (located at a distance d below S when the top is upright), with components expressed in a clearly defined triad of your choice, as functions of the Euler angles, your further defined minimal set of coordinates, first and second derivatives of these variables, and the given constants. Perform at least three independent plausibility checks.

j. Establish constraint equations that link Euler angles, their derivatives, the radius R, and variables you defined, if the tippe top does not slip. [c]

[a] *Hint*: For example, the first vector would be ${}^{\mathcal{S}}\boldsymbol{r}_{S/O} = {}^{\mathcal{S}}\boldsymbol{\delta}_{\mathcal{S}}$

[b] *Hint*: tippe top, so it is not always on the ground. Point K is fixed to the

[c] *Hint*: Be careful when calculating velocities: Point K on the tippe top that is in contact with the ground if there is no slip?

What is the velocity vector of the point K on the tippe top that is in contact with the ground if there is no slip?

Problem 4.28 *Four-bar linkages* are often used as *polycentric* knee joints in prosthetic legs. Such joints help keep the leg extended when the heel strikes the ground, and they also increase ground clearance during swing. Both features increase safety.

For the situation in Figure 4.16, we want to determine the instantaneous center of rotation (ICR) I of the shank (member $B-C$) if the thigh (member $A-D$) is fixed in space. Note that the shank is rigidly connected to bar BC.

a. Graphically find the ICR of the shank for a configuration where the leg is completely extended, such as is diplayed in Figure 4.16. [a]

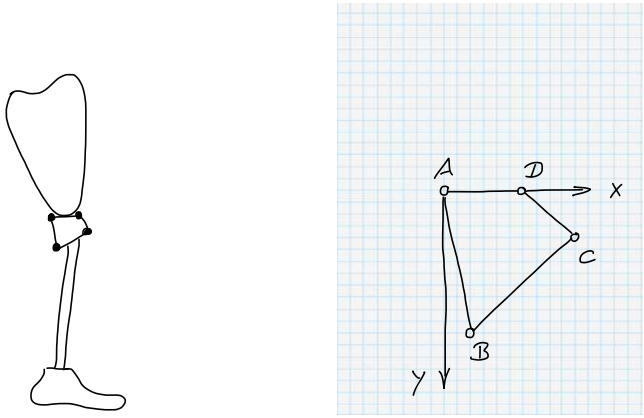


Figure 4.16: Polycentric prosthetic knee joint

b. Set up the system of equations that allows calculating the position vector $\boldsymbol{r}_{I/A}$ of the ICR of bar BC algebraically, provided coordinates of all points in the figure.
You can do this in the following way:
1. Express the velocity of point B as a vector function of the angular velocity vector $\boldsymbol{\omega}_{BC}$ of the bar and of the (yet unknown) location of its ICR.
2. Set up a scalar equation that ensures this velocity is orthogonal to $\boldsymbol{r}_{B/A}$ (Why?).
3. Perform analogous steps 1 to 2 for point C.
4. Check whether you have enough (independent) equations compared to the number of unknowns[b].

c. Using the grid in the figure, fill in the components of the following vectors, all expressed in the $\mathcal{N}$-triad: ${}^{\mathcal{N}}\boldsymbol{r}_{B/A} = k(...)^{\mathrm{T}}$, ${}^{\mathcal{N}}\boldsymbol{r}_{C/D} = k(...)^{\mathrm{T}}$, ${}^{\mathcal{N}}\boldsymbol{r}_{C/B} = k(...)^{\mathrm{T}}$, where k is a given scaling factor.
Substitute the values into your system of equations and solve for the coordinates of the ICR in the depicted configuration (as functions of k)[c]. Compare your results to the geometric solution of the problem.

d. Now, we want to determine the instantaneous axis of rotation of a platform in 3-D, as depicted in Figure 4.17.
The platform is suspended on five arms (links). Given are the position vectors ${}^{\mathcal{N}}\boldsymbol{R}_i$, $i = 1 \dots 5$ of the five anchor points R_i on the ground, as well as the relative position vectors ${}^{\mathcal{N}}\boldsymbol{\rho}_i = {}^{\mathcal{N}}\boldsymbol{r}_{P_i/C}$ of the five points on the platform with respect to some point C on the platform, all with components expressed in the inertial $\mathcal{N}$-triad.
Set up the system of equations that allows analytically calculating the location and the direction of the instantaneous axis of rotation of the platform (i.e. XYZ coordinates of one point on the axis, and a unit

direction vector along the axis).
Check your results for plausibility by investigating special cases[d].

e. Explain why you have less equations than unknowns in both the 2-D and the 3-D case, such that there is ambiguity in your results for some of the unknowns (for which?). For 2-D and 3-D, explain whether the equation system could be solved for the location of the instantaneous axis/center if one of the links was cut, or if another link was added.
f. Using MATLAB^{TM} or another program of your choice, implement a function that returns location and direction of the instantaneous axis of rotation of the platform when provided the locations of all endpoints of the links[e]. Use this function to further check plausibility of your results for special cases.

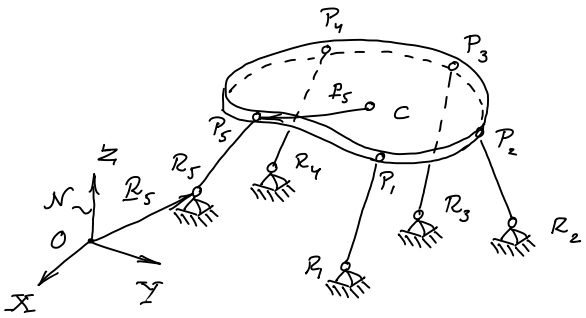


Figure 4.17: Platform

[a] *Hint*: Note that the choice of point C is arbitrary, so you can for example choose any of the link endpoints, or the mean of all points

[e] *Hint*: A special case where you would know the instantaneous axis of rotation: Imagine the platform is just a bar. Three links are connected to the left endpoint, and two links are connected to the right endpoint. Make sure the anchors are chosen well on the ground, such that the system can indeed only rotate about one axis.

[d] *Hint*: You may wish to use the tilde matrix.

[c] *Hint*: If not, check whether you can use vector addition to express some unknowns as functions of others, and of given geometry.

[b] *Hint*: Assume the leg will start flexing and then determine the *directions* of the velocity vectors of points C and B (Note that points A and D are connected to the thigh, which does not move.).

Problem 4.29 A milling machine uses a rotating cutter to shape an object by removing material from it. The milling cutter's position vector $\boldsymbol{r}(t)$ with respect to a fixed point O is given as a function of time t. The components in a fixed XYZ coordinate system with origin O and associated

triad $\mathcal{N}$ are given as

$$\boldsymbol{r}(t) = \begin{pmatrix} kt\cos(\Omega t) \\ kt\sin(\Omega t) \\ kt \end{pmatrix}, \tag{4.82P}$$

with the constant parameters k and Ω.

a. Visualize the cutter's path by drawing one projection in the XY plane and one in the XZ plane. What kind of path is the cutter following?
b. Calculate the absolute derivative ${}^{\mathcal{N}}\dot{\boldsymbol{r}}$.

Cylindrical coordinates

c. We want to describe the cutter's location using cylindrical coordinates and the same origin O. Triad $\mathcal{B}$ is defined by the $\hat{\boldsymbol{b}}_1 = \hat{\boldsymbol{e}}_\rho(t)$, $\hat{\boldsymbol{b}}_2 = \hat{\boldsymbol{e}}_\varphi(t)$, and $\hat{\boldsymbol{b}}_3 = \hat{\boldsymbol{n}}_3(t)$ directions.Calculate the components $\rho(t)$ and $z(t)$, as well as the associated direction vectors ${}^{\mathcal{N}}\hat{\boldsymbol{e}}_\rho(t)$ and ${}^{\mathcal{N}}\hat{\boldsymbol{n}}_3(t)$ for such a coordinate system and the given cutter path, as functions of time. Write out the position vector ${}^{\mathcal{B}}\boldsymbol{r}(t)$.
d. Again for cylindrical coordinates, calculate the angle $\varphi(t)$ that ${}^{\mathcal{N}}\boldsymbol{e}_\rho(t)$ encloses with the positive X-Axis, and calculate the rotation matrix ${}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}}(t)$. Verify your result using ${}^{\mathcal{N}}\boldsymbol{r}$ and ${}^{\mathcal{B}}\boldsymbol{r}$.
e. Calculate the components of the angular velocity vectors ${}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}(t)$ and ${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}(t)$.
f. Calculate the absolute derivative ${}^{\mathcal{B}}\dot{\boldsymbol{r}}$ in three different ways:
   a) using the transport theorem,
   b) using the product rule of differentiation and the derivatives of components $\rho(t)$, $z(t)$ and unit vectors ${}^{\mathcal{N}}\hat{\boldsymbol{e}}_\rho(t)$, ${}^{\mathcal{N}}\hat{\boldsymbol{e}}_z(t)$ separately,
   c) using ${}^{\mathcal{N}}\dot{\boldsymbol{r}}$ and the rotation matrix ${}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}}(t)$ found in previous parts of this problem.

   All three methods should give the same result.

Spherical coordinates

g. Now, we use spherical coordinates, associated with a triad $\mathcal{S}$ and the same origin O. Calculate $r(t)$ and $\hat{\boldsymbol{e}}_r(t)$ and write out ${}^{\mathcal{S}}\boldsymbol{r}(t)$.
h. Calculate $\varphi(t)$, $\theta(t)$, ${}^{\mathcal{N}}\hat{\boldsymbol{e}}_\varphi(t)$, and ${}^{\mathcal{N}}\hat{\boldsymbol{e}}_\theta(t)$ for spherical coordinates, as well as the rotation matrix ${}^{\mathcal{N}}\mathbf{C}_{\mathcal{S}}(t)$ [a].
i. Calculate the components of the angular velocity vectors ${}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}}(t)$ and ${}^{\mathcal{S}}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}}(t)$. Then use the transport theorem again to calculate the absolute derivative ${}^{\mathcal{S}}\dot{\boldsymbol{r}}$.

Natural coordinates

j. Now, we consider the cutter's natural coordinate system. Calculate the tangential unit vector $\hat{\boldsymbol{e}}_t$ and the speed $\dot{s}$ of the cutter[b].
k. Calculate the normal and binormal unit vectors $\hat{\boldsymbol{e}}_n(t)$ and $\hat{\boldsymbol{e}}_b(t)$[c].
l. Use a symbolic calculation program, e.g. MATLAB™'s symbolic toolbox, to re-do all above calculations.

[a] *Hint*: First calculate $\ddot{s}(t)$, ${}^{\mathcal{N}}\dot{\boldsymbol{r}}(t)$. Then use vector addition to calculate the normal component's direction and magnitude. You may be looking for $|\boldsymbol{r}|$. This is quicker to calculate from cylindrical components than from XYZ-components, so you can re-use your result for ${}^{\mathcal{B}}\dot{\boldsymbol{r}}$, or just use it to check.

[b] *Hint*: To calculate ${}^{\mathcal{N}}\dot{\hat{\boldsymbol{e}}}_\varphi(t)$, you may find it helpful to use the cross product and also re-use your result for ${}^{\mathcal{N}}\dot{\hat{\boldsymbol{e}}}_\rho(t)$ from cylindrical coordinates.

[c] *Hint*:

4.7.2 Practice Problems

Problem 4.30 This Problem is about the three-dimensional rod pendulum of Problem 3.32 on page 98. Show that the angular velocity vector ${}^{\mathcal{B}}\boldsymbol{\omega}$ of the pendulum, with components expressed in the body-fixed triad $\mathcal{B}$, is ${}^{\mathcal{B}}\boldsymbol{\omega} = (\dot{\beta},\ \dot{\alpha}\sin(\beta),\ \dot{\alpha}\cos(\beta))^{\mathrm{T}}$. Use the rotation matrix ${}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}}$ (3.34P) from Problem 3.32 in your calculation.

Problem 4.31 This Problem is about the skateboard of Problem 1.24 on page 39. A student wants to calculate absolute acceleration $\boldsymbol{a}_P$ of the rider's center of mass, using (4.69).

Which of the below statement(s) *must* be true for this equation to be correctly applied? Check all that apply.

A. $\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$
B. $\boldsymbol{Q} = \boldsymbol{r}_{A/O}$
C. $\boldsymbol{p} = \boldsymbol{r}_{P/O}$
D. $\boldsymbol{\omega} \cdot \boldsymbol{p} = 0$
E. None of the above

▷

Problem 4.32 This Problem is about the accelerometer of Problem 1.26 on page 41.

The 3-D accelerometer is first tested in outer space, when no gravity acts. There, it indicates the correct acceleration value in each direction.

This sensor is then mounted at the center of mass of a speedboat under the influence of gravity, which acts in negative Z-direction. The boat's orientation is always known, such that we can transform the accelerometer's measured acceleration vector to the $\mathcal{N}$-triad.

A student predicts what this sensor should indicate in different situations.

First, the boat is perfectly still. The student predicts that in this situation, the accelerometer will indicate an acceleration of $1g$ in positive Z-direction.

Later, the speedboat jumps out of the water. The student predicts that the accelerometer will indicate $0g$ in all directions while the boat is in the air.

Are these predictions correct (Check all that apply)[a]?

A. Yes
B. No, it should measure $1g$ in *negative* Z-direction when the boat is still.
C. No, it should measure $0g$ when the boat is still.
D. No, it should measure $1g$ in *negative* Z-direction when the boat is in the air.
E. No, the magnitude of the measured acceleration must be $1g$ at all times.
F. None of the above

▷

[a] *Hint*: Be aware that the sensor can only measure spring deflection. So, use your FBD to calculate spring deflection in the different cases.

Problem 4.33 This Problem is about the rolling thin disk on an arm of Problem 3.31 on page 97. Find the distance R from the Z-axis to point P as a function of $\dot{\varphi}$, $\dot{\psi}$, and r (Remark: It is possible that not all of these terms appear in the requested function), such that the *instantaneous* axis of rotation of the disk is parallel to the Z-axis. ▶

Problem 4.34 A person is standing on a disk of radius R_d that has its surface in the XY-plane of the space-fixed XYZ coordinate system (triad $\mathcal{N}$), as shown in Figure 4.18. The disk rotates at a constant angular speed ω about the Z-axis. The person wants to throw a ball of mass m. The $x'y'z'$ coordinate system (triad $\mathcal{B}$) is fixed to the disk and rotates with it, where the z'-axis coincides with the Z-axis, the x' and y' axes lie within the disk surface, and the y'-axis encloses a variable angle β with the Y-axis.

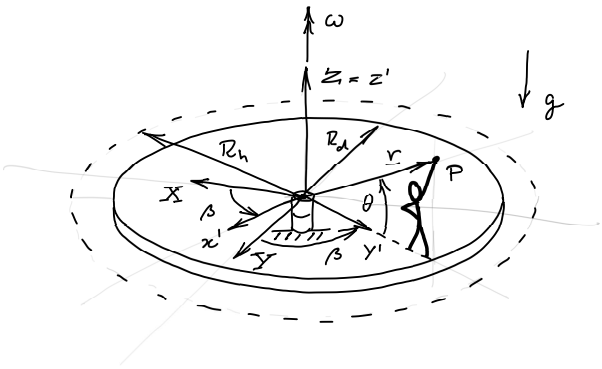


Figure 4.18: Person standing on a disk holding a ball.

The person will release the ball at point P, which is located at the distance R_d from the Z-axis and contained in the $y'z'$-plane. The *relative* velocity of the ball with respect to the disk will have a magnitude of v_0 and point in

the direction of the vector $\boldsymbol{r}$. The person can control the inclination θ within a range of 0 to $\pi/2$. Gravity points in negative Z-direction, with constant acceleration of gravity g.

Assuming the ball's *relative* velocity vector with respect to the disk is constant for a finite time interval before the release, find the velocity vector ${}^{\mathcal{B}}\dot{\boldsymbol{r}}_0$ and the acceleration vector ${}^{\mathcal{B}}\ddot{\boldsymbol{r}}_0$ of the ball immediately before release, with components expressed in the $x'y'z'$-coordinate system, as functions of θ, β, ω, R_d, and v_0 (Remark: It is possible that not all of these terms appear in the requested functions).

Problem 4.35 In an inertial coordinate system XYZ, a rigid, massless tube is attached to the fixed point O by means of two revolute joints, and it is rotating with a variable angular velocity $\boldsymbol{\omega}$ about O (Figure 4.19).

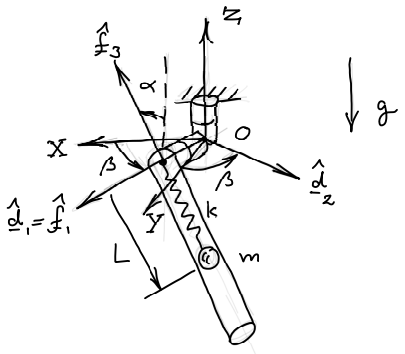


Figure 4.19: Particle in tube

Coordinate system $x'y'z'$ rotates with β with respect to inertial triad $\mathcal{N}$ and is associated with triad $\mathcal{G}$. Coordinate system $x''y''z''$ is associated with triad $\mathcal{F}$ and rotates with α about the x'-axis. The cans in series illustrate the *degrees of freedom* α and β; neglect dimensions and mass of these cans.

Constrained to slide frictionless within this tube, a particle of mass m is attached to O via a linear-elastic connection, having stiffness k and variable length L. Gravity with acceleration constant g acts in the negative Z-direction.

A right-handed coordinate system tnb is associated with the particle's natural triad $\mathcal{D}$, such that the t-axis is tangent to the path, the n-axis is normal to the path and points to the center of curvature C, and the b-axis (the binormal) is orthogonal to both t and n. The associated unit vectors are $\hat{\boldsymbol{e}}_t$, $\hat{\boldsymbol{e}}_n$ and $\hat{\boldsymbol{e}}_b$. The position of the particle with respect to O is given by position vector $\boldsymbol{r}$.

For which condition(s) listed in Table 4.1 do the points O and C have to coincide?

Table 4.1: Conditions for which points O and C might coincide.

A.	B.	C.	D.	E.
$\dot{\boldsymbol{\omega}} = \mathbf{0}$	$(\dot{\boldsymbol{r}})_{\mathcal{D}} = \mathbf{0}$	$\frac{\mathrm{d}}{\mathrm{d}t}(\lvert\boldsymbol{r}\rvert) = 0$	$\hat{\boldsymbol{e}}_{\mathrm{n}} \cdot \boldsymbol{r} = 0$	$\hat{\boldsymbol{e}}_{\mathrm{n}} \times \boldsymbol{r} = \mathbf{0}$

▷

Problem 4.36 This Problem is about the tube with a particle in it of Problem 4.35 on page 142.

Here, we only consider movement in the YZ-plane, with $\beta = 0$, $\dot{\beta} = 0$ and $\ddot{\beta} = 0$. We also consider an instant where the initially positive $\dot{\alpha}$ is decreasing, and the spring is extending with $\dot{L} > 0$ and $\ddot{L} > 0$.

The acceleration $\ddot{\boldsymbol{r}}$ of the particle can be calculated using the relative motion in triad $\mathcal{F}$:

$$\ddot{\boldsymbol{r}} = \underbrace{\ddot{\boldsymbol{Q}}}_{1.} + \underbrace{(\ddot{\boldsymbol{p}})_{\mathcal{F}}}_{2.} + \underbrace{\dot{\boldsymbol{\omega}} \times \boldsymbol{p}}_{3.} + \underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{p})}_{4.} + \underbrace{2\boldsymbol{\omega} \times (\dot{\boldsymbol{p}})_{\mathcal{F}}}_{5.}. \tag{4.83P}$$

Variables in the above equation are chosen such that term 1 is the zero vector.

For the individual terms 2 to 5, show whether their components are positive, negative or zero in the asked directions at the considered instant, by completing Table 4.2.

Table 4.2: Signs of direction vectors.

	$\hat{\boldsymbol{f}}_1$	$\hat{\boldsymbol{f}}_2$	$\hat{\boldsymbol{f}}_3$
1.	0	0	0
2.			
3.			
4.			
5.			

Start by identifying all variables in the equation, such that the equation holds. ▷

Problem 4.37 This Problem is about the tube with a particle in it of Problem 4.35 on page 142. Assume that the connection has infinite stiffness: $k \to \infty$. Use Newton's second law to prove that for steady motion ($\ddot{\alpha} = \ddot{\beta} = 0$) with $\alpha \neq 0$, the following equation holds:

$$\cos(\alpha) \;=\; \frac{g}{\dot{\beta}^2 L}. \tag{4.84P}$$

Start by drawing a FBD of the particle in the $y'z'$-plane for an angle α. Note that the massless tube cannot exert a force in the y''-direction. ▶

Problem 4.38 This Problem is about the disk in steady roll of Problem 1.9 on page 19. Calculate the angular velocity vector ${}^{\mathcal{N}}\boldsymbol{\omega}$ of the wheel as a function of v, r, R, and θ. ▷►

Problem 4.39 Someone is riding a bicycle on Earth on the equator from West to East and reading a constant speed of 30 km/h on the tachometer. A befriended second rider is making a trip from South to North, and reading the same speed. The riders are wondering about their respective absolute velocity and absolute acceleration vectors, taking into account rotation of the Earth. When answering this question, first make a choice of what is considered *absolute*. Do this by defining a "fixed point", which could be the Earth's center or the Sun's, and the directions of an "inertial" coordinate triad. Second, define one or more appropriate rotating triad(s). Third, apply the transport theorem.

Problem 4.40 A ball is rolling on a rotating disk. A woman is standing on the disk and observing the ball. The ball has nonzero relative velocity and acceleration components both radial and tangential to the disk, and the disk is currently accelerating. What kind of accelerations of the ball does the observer see in her disk-fixed coordinate system?

Problem 4.41 This Problem is about the person on a disk with a ball of Problem 4.34 on page 141. The person releases the ball at the instant when $\beta = 0$ and point P is at a height h above the ground. Independent of what you found in Problem 4.34, assume the velocity vector right before release is ${}^{\mathcal{N}}\dot{\boldsymbol{r}}_0 = (v_{X0},\, v_{Y0},\, v_{Z0})^{\mathrm{T}}$, with components $v_{X0} = 2\sqrt{hg}$, $v_{Y0} = \frac{1}{2}\sqrt{3hg}$, and $v_{Z0} = \frac{1}{2}\sqrt{hg}$ expressed in the $\mathcal{N}$ (XYZ) triad.

Find the radius R_{h} of the circle on which the ball will land (so the distance between the Z-axis and the point where the ball hits the ground), as a function of m, h, R_{d}, and g (Remark: None of these terms has to necessarily appear in the requested functions), where the ground is defined as the XY-plane. Include two projected FBDs of the ball during flight (so *after* release) in your argumentation, one projected into the YZ-plane and one projected into the XY-plane, for a generic position vector of the ball ${}^{\mathcal{N}}\boldsymbol{r}_{\mathrm{b}} = (X_{\mathrm{b}},\, Y_{\mathrm{b}},\, Z_{\mathrm{b}})^{\mathrm{T}}$ with components expressed in the $\mathcal{N}$ triad. ►

5 Work and Energy

This chapter addresses kinetic energy of a translating and rotating body, and how it is influenced by work of forces and moments. The *inertia tensor* of a rigid body will be introduced, which describes mass distribution of a rigid body. We will see that the inertia tensor becomes much less complex if the body is symmetric with respect to one or more planes. We will show how the inertia tensor changes when using shifted and rotated reference coordinate systems. With that knowledge, we will discover that there always is a triad in which the inertia tensor is diagonal, which simplifies calculations. Also, we will find a way to write kinetic energy as a product of scalars instead of matrices and vectors, using the instantaneous axis of rotation. At the end of this chapter, we will re-visit the principle of Work and energy in the context of rigid bodies.

5.1 Kinetic Energy of a Rigid Body

5.1.1 Kinetic Energy in Terms of the Inertia Tensor

In Section 2.2.2, we found that the kinetic energy T of a system of N particles is

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i |\dot{\boldsymbol{r}}_i|^2 = \sum_{i=1}^{N} \tfrac{1}{2} m_i \dot{\boldsymbol{r}}_i^{\mathrm{T}} \dot{\boldsymbol{r}}_i, \tag{5.1}$$

where the i-th particle has mass m_i and position vector $\boldsymbol{r}_i$ with respect to the origin O of an inertial coordinate system.

A rigid body consists of infinitely many mass elements. Therefore, our goal is to re-write the above equation for kinetic energy in a form that we do not need to calculate the speed of each single particle individually.

We let $N \to \infty$, such that the summation transforms to an integral over the rigid body, and the mass m_i of an individual particle becomes an infinitesimally small *mass element* $\mathrm{d}m$ inside the body.

We start by expressing the position vector of the mass element $\mathrm{d}m$ located at point P in the rigid body as a vector addition of the position

vector $\boldsymbol{r}_{C/O}$ of the total center of mass C of the system with respect to O, and the relative position vector $\boldsymbol{r}_{P/C}$ of the mass element with respect to C. A rotating coordinate system xyz is fixed to the rigid body, has its origin in C, and has directions of triad $\mathcal{B}$. The body and therefore also triad $\mathcal{B}$ rotate with angular velocity $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ with respect to the inertial XYZ coordinate system, which has directions of triad $\mathcal{N}$. All variables are illustrated in Figure 5.1. We use the shorthand notations

$$\boldsymbol{r}_C := \boldsymbol{r}_{C/O}, \quad \boldsymbol{r}_P := \boldsymbol{r}_{P/O}, \quad \boldsymbol{\rho} := \boldsymbol{r}_{P/C}, \quad \boldsymbol{\omega} := \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}. \tag{5.2}$$

As discussed in Section 1.2.4, the position vector of the center of mass $\boldsymbol{r}_C$ with respect to the origin of some coordinate system is as given in (1.11) on page 11.

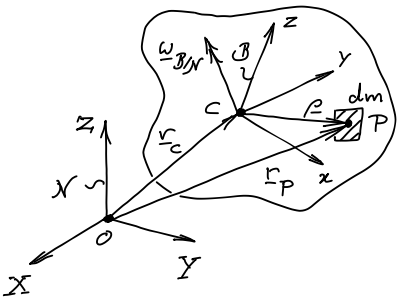


Figure 5.1: Vectors used to describe absolute and relative positions of a mass element within a rigid body that is rotating with angular velocity $\boldsymbol{\omega}$.

With these definitions and the knowledge on relative velocity from Section 4.4.1, we can re-write (5.1) as:

$$T = \int_m \frac{1}{2} |\dot{\boldsymbol{r}}_C + (\dot{\boldsymbol{\rho}})_{\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{\rho}|^2 \mathrm{d}m \tag{5.3}$$

The sum of masses m_i has been exchanged for the integral over the infinitesimally small mass increments $\mathrm{d}m$.

Problem 5.1 Derive (5.3) from (5.1) using your knowledge on absolute and relative motion.

In this special case, where the mass elements are all rigidly connected, their relative position ${}^{\mathcal{B}}\boldsymbol{\rho}$ with respect to C is always constant in the body-fixed coordinate system xyz. Therefore, the relative change $(\dot{\boldsymbol{\rho}})_{\mathcal{B}}$

is the zero vector. Writing out the squared magnitude as a scalar product in (5.3) and then separating the summands resolves to three terms:

$$T = \int_m \frac{1}{2}(\dot{\boldsymbol{r}}_C + \boldsymbol{\omega} \times \boldsymbol{\rho})^{\mathrm{T}}(\dot{\boldsymbol{r}}_C + \boldsymbol{\omega} \times \boldsymbol{\rho})\mathrm{d}m \tag{5.4}$$

$$= \underbrace{\int_m \frac{1}{2}\dot{\boldsymbol{r}}_C^{\mathrm{T}}\dot{\boldsymbol{r}}_C \mathrm{d}m}_{1.} + \underbrace{\int_m \dot{\boldsymbol{r}}_C^{\mathrm{T}}(\boldsymbol{\omega} \times \boldsymbol{\rho})\mathrm{d}m}_{2.} + \underbrace{\int_m \frac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{\rho})^{\mathrm{T}}(\boldsymbol{\omega} \times \boldsymbol{\rho})\mathrm{d}m}_{3.} \tag{5.5}$$

Note that in the simplification, we made use of the fact that the scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is commutative: $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{a}$.

Our goal is still to remove the need to calculate quantities for each single particle when calculating kinetic energy. We see that we can achieve this for the first term in (5.5), by factoring all those variables outside of the summation that do not change with the chosen mass element:

$$\int_m \frac{1}{2}|\dot{\boldsymbol{r}}_C|^2 \mathrm{d}m = \frac{1}{2}|\dot{\boldsymbol{r}}_C|^2 \int_m \mathrm{d}m = \frac{1}{2}m|\dot{\boldsymbol{r}}_C|^2. \tag{5.6}$$

Hereby, the mass m is the overall mass of the system. Therefore, we have successfully eliminated the need to do calculations for individual mass elements. However, the terms involving relative position vectors $\boldsymbol{\rho}$ in (5.5) still need to be re-written.

For the second term in (5.5), we try the same, namely factoring all terms that do not change with the mass element out of the summation:

$$\int_m \dot{\boldsymbol{r}}_C^{\mathrm{T}}(\boldsymbol{\omega} \times \boldsymbol{\rho})\mathrm{d}m = \dot{\boldsymbol{r}}_C^{\mathrm{T}}\left(\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}\mathrm{d}m\right). \tag{5.7}$$

If we fill in $\boldsymbol{\rho} = \boldsymbol{r}_{P/C}$ in the definition of the center of mass (1.12), we recognize that the term $\int_m \boldsymbol{\rho}\mathrm{d}m$ is proportional to the position vector of the center of mass of the system in this xyz coordinate system (with overall mass m as the proportionality constant). Since we have chosen the origin of xyz to be just at the center of mass C of the system, the coordinates of C must be zero in the chosen coordinate system:

$$\int_m \boldsymbol{\rho}\mathrm{d}m = \int_m \boldsymbol{r}_{P/C}\mathrm{d}m = \boldsymbol{r}_{C/C} \cdot m = \boldsymbol{0}. \tag{5.8}$$

We are only left with one single term to simplify, namely the third one in (5.5). Here, it is more complicated to factor all terms that do not depend on the chosen mass element out of the term, because the

vector $\boldsymbol{\omega}$ is contained in the scalar product of two cross products. The key to simplifying this term is to re-write the cross product via the tilde matrix notation (see Appendix A.3), such that we obtain matrix multiplications and can factor the two vectors $\boldsymbol{\omega}$ out of the summation.

The tilde matrix of vector $\boldsymbol{\rho} = \begin{pmatrix} x & y & z \end{pmatrix}^{\mathrm{T}}$ is

$$\tilde{\boldsymbol{\rho}} := \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}, \tag{5.9}$$

such that the third term in (5.5) can be written as:

$$\begin{aligned} \int_m \frac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{\rho})^{\mathrm{T}}(\boldsymbol{\omega} \times \boldsymbol{\rho})\mathrm{d}m &= \int_m \frac{1}{2}(-\tilde{\boldsymbol{\rho}}\boldsymbol{\omega})^{\mathrm{T}}(-\tilde{\boldsymbol{\rho}}\boldsymbol{\omega})\mathrm{d}m \\ &= \int_m \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\tilde{\boldsymbol{\rho}}^{\mathrm{T}}\tilde{\boldsymbol{\rho}}\boldsymbol{\omega}\mathrm{d}m. \end{aligned} \tag{5.10}$$

Now, it is possible to factor the vector $\boldsymbol{\omega}$ out of the summation, on the left and on the right. Combining (5.6), (5.8), and (5.10), we obtain for kinetic energy (5.5):

$$T = \frac{1}{2}m|\dot{\boldsymbol{r}}_C|^2 + \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\left(\int_m \tilde{\boldsymbol{\rho}}^{\mathrm{T}}\tilde{\boldsymbol{\rho}}\mathrm{d}m\right)\boldsymbol{\omega} \tag{5.11}$$

Expanding the product of the two tilde matrices $\tilde{\boldsymbol{\rho}}^{\mathrm{T}}\tilde{\boldsymbol{\rho}}$, and with the definition of the *inertia tensor* $\mathbf{I}_C$ of the rigid body with respect to its center of mass,

$$\mathbf{I}_C := \int_m \tilde{\boldsymbol{\rho}}^{\mathrm{T}}\tilde{\boldsymbol{\rho}}\,\mathrm{d}m \tag{5.12}$$

$$= \begin{pmatrix} \int_m \left(z^2+y^2\right)\mathrm{d}m & -\int_m xy\,\mathrm{d}m & -\int_m xz\,\mathrm{d}m \\ -\int_m xy\,\mathrm{d}m & \int_m \left(x^2+z^2\right)\mathrm{d}m & -\int_m yz\,\mathrm{d}m \\ -\int_m xz\,\mathrm{d}m & -\int_m yz\,\mathrm{d}m & \int_m \left(x^2+y^2\right)\mathrm{d}m \end{pmatrix}, \tag{5.13}$$

we can re-write (5.11) to obtain a fundamental equation for kinetic energy of a rigid body:

$$T = \frac{1}{2}m|\dot{\boldsymbol{r}}_C|^2 + \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\mathbf{I}_C\boldsymbol{\omega}. \tag{5.14}$$

So, we can express the kinetic energy of a rigid body as a function of the body's intrinsic properties, namely mass m and inertia tensor $\mathbf{I}_C$,

and the body's current movement, as described by the body's angular velocity $\boldsymbol{\omega}$ and the velocity $\dot{\boldsymbol{r}}_C$ of its center of mass.

An inertia tensor $\mathbf{I}_C$ has no meaning, like angular momentum, unless one also states the point of reference for the vector $\boldsymbol{\rho}$. So far, we only considered the case that this reference is the center of mass C, which means that x, y, z in (5.13) are the coordinates of the mass element with respect to a body-fixed coordinate system that has its origin in C.

Example 5.1 This Example is about the biker of Problem 4.25 on page 131. Given is the inertia tensor ${}^{\mathcal{F}}\mathbf{I}_C$ of the body with respect to point C, velocity vector ${}^{\mathcal{F}}\boldsymbol{v}_C$ of point C, angular velocity ${}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$ of triad $\mathcal{F}$ with respect to triad $\mathcal{N}$, and angular velocity ${}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{F}}$ of triad $\mathcal{B}$ with respect to triad $\mathcal{F}$. Calculate the kinetic energy T of the combined bike and rider, as a function of ${}^{\mathcal{F}}\mathbf{I}_C$, ${}^{\mathcal{F}}\boldsymbol{v}_C$, ${}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$, ${}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{F}}$ and the given constants.

Exemplary solution

The kinetic energy of the body is

$$T = \frac{1}{2}\left(m|\boldsymbol{v}_C|^2 + \boldsymbol{\omega}_{\mathrm{b}}^{\mathrm{T}}\mathbf{I}_C\boldsymbol{\omega}_{\mathrm{b}}\right), \tag{5.15E}$$

where $\boldsymbol{v}_C$ denotes velocity of the body's center of mass, and $\boldsymbol{\omega}_{\mathrm{b}}$ angular velocity of the body. This angular velocity is found via vector summation:

$$\boldsymbol{\omega}_{\mathrm{b}} = \boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{F}}. \tag{5.16E}$$

Substituting this in (5.15E) and expressing all variables in the $\mathcal{F}$-triad gives:

$$T = \frac{1}{2}\left(m|\,{}^{\mathcal{F}}\boldsymbol{v}_C|^2 + \left({}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} + {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{F}}\right)^{\mathrm{T}}{}^{\mathcal{F}}\mathbf{I}_C\left({}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} + {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{F}}\right)\right). \tag{5.17E}$$

Problem 5.2 Can the third term in (5.5) be negative for any arbitrary angular velocity vector $\boldsymbol{\omega}$? Also, recall that this third term resolves to the second term in (5.14), the rotational energy component, and make a statement about definiteness of the inertia tensor (see also Appendix A.5).

5.1.2 Moments and Products of Inertia

In the inertia tensor (5.13), the diagonal terms are called *moments of inertia*, the off-diagonal terms are *products of inertia*[1].

A moment of inertia is the integral over the mass elements times their squared distance from a particular axis. So, the moment of inertia

[1] Note that often products of inertia are defined *without* the minus sign, in contrast to the definition used here.

is larger the further out the mass is placed from this axis, but it is irrelevant how the mass is distributed *along* that axis.

Example 5.2 Calculate the moment of inertia I_a of a solid cone of homogeneous material about its symmetry axis, as a function of its base radius R, height H, and density ρ. Define all variables used in a drawing.

Exemplary solution

We choose the y-axis to be the cone's symmetry axis, and the origin O to be the vertex. We further parameterize the location of the volume element by radius r, angle φ, and h, which describes the location in axial direction (see Figure 5.2). So, the element's x- and z-coordinates are related by $x^2+z^2 = r^2$. So, we need to integrate the squared distances to the central axis of the infinitesimal volumes over the mass.

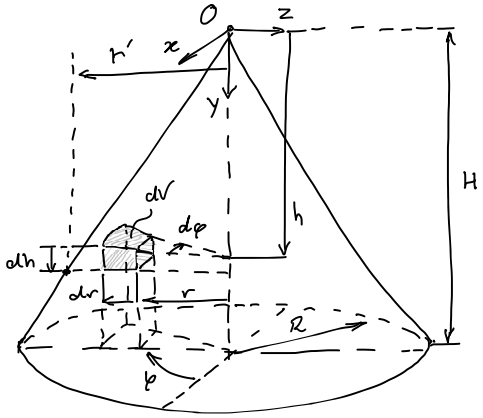


Figure 5.2: Cone with an infinitesimal volume element.

$$I_\mathrm{a} = \int_m (x^2 + z^2)\mathrm{d}m = \int_m r^2 \mathrm{d}m \tag{5.18E}$$

Given homogeneous density, mass and volume are related by ρ as proportionality constant, so

$$I_\mathrm{a} = \rho \int_V r^2 \mathrm{d}V. \tag{5.19E}$$

The volume of the depicted infinitesimal element is

$$\mathrm{d}V = \mathrm{d}r \cdot \mathrm{d}h \cdot r\mathrm{d}\varphi, \tag{5.20E}$$

so we can write the integral as:

$$I_\mathrm{a} = \rho \int_0^{\varphi'} \int_0^{H'} \int_0^{r'} r^2 \, \mathrm{d}r \cdot \mathrm{d}h \cdot r\mathrm{d}\varphi. \tag{5.21E}$$

Note that one could also integrate in a different order, for example first over φ, then over r, and then over h. The chosen order dictates the limits.

To determine the correct limits, a geometric interpretation of the figure helps: The chosen order of the integral implies that we first sum the moments of inertia of all volume elements in radial direction. Here, the limit is defined by the radius r', which is a function of the element's h. In this integration step, we obtain the moment of inertia of a body that looks like a thin and low piece of cake.

Then, we sum all moments of inertia in axial direction of the cone, which extends over a length H. This gives us the inertia of a thin triangular slice.

Finally, the moments of inertia of all resulting slices are added by integrating over the entire range of angles from 0 to 2π. So, the limits are:

$$r' = h\frac{R}{H}, \qquad H' = H, \qquad \varphi' = 2\pi. \tag{5.22E}$$

This means we need to calculate the integral

$$I_\mathrm{a} = \rho \int_0^{2\pi} \int_0^{H} \int_0^{\frac{R}{H}h} r^3 \,\mathrm{d}r \,\mathrm{d}h \,\mathrm{d}\varphi. \tag{5.23E}$$

Sequentially evaluating the integrals for these limits results in:

$$I_\mathrm{a} = \rho \int_0^{2\pi} \int_0^{H} \frac{1}{4}\left(h\frac{R}{H}\right)^4 \,\mathrm{d}h \,\mathrm{d}\varphi = \frac{\rho}{20}\frac{R^4}{H^4}\int_0^{2\pi} H^5 \,\mathrm{d}\varphi, = \frac{\pi\rho R^4 H}{10} \tag{5.24E}$$

One could now continue and express this result as a function of the cone's mass and geometry instead of density and geometry, to compare with values that are commonly found in tables.

Problem 5.3 Re-do Example 5.2 with another sequence of integration, namely first φ, then r, and then h. Describe again the shapes of the intermediate bodies. ▷

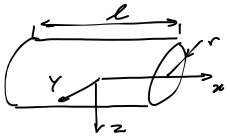
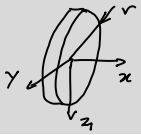
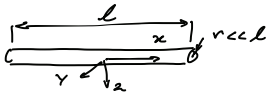
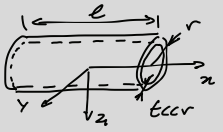
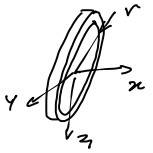
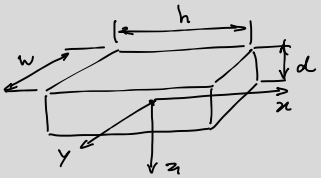
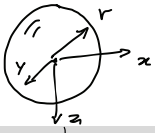
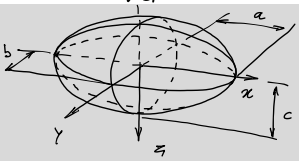
To simplify reference in the following, the individual terms of the inertia tensor are denoted by subscripts in the following way:

$$\mathbf{I}_C := \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}, \tag{5.25}$$

The components in (5.13) indicate another important property of the inertia tensor, namely its symmetry. For example, $I_{xy} = I_{yx}$. This means that $\mathbf{I}_C^\mathrm{T} = \mathbf{I}_C$.

Table 5.1 lists the moments of inertia of some standard bodies with respect to their centers of mass. All products of inertia are zero.

Table 5.1: Mass moments of inertia of several standard shaped objects.

cylinder solid		$I_{xx} = \frac{1}{2}mr^2$ $I_{yy} = I_{zz} = \frac{1}{12}m\left(3r^2 + l^2\right)$
cylinder disk $l << r$		$I_{xx} = \frac{1}{2}mr^2$ $I_{yy} = I_{zz} = \frac{1}{4}mr^2$
cylinder thin rod $r << l$		$I_{xx} = 0$ $I_{yy} = I_{zz} = \frac{1}{12}ml^2$
cylinder thin- walled $t << r$		$I_{xx} = mr^2$ $I_{yy} = I_{zz} = \frac{1}{12}m\left(6r^2 + l^2\right)$
cylinder ring $t << r$ $l << r$		$I_{xx} = mr^2$ $I_{yy} = I_{zz} = \frac{1}{2}mr^2$
cuboid solid		$I_{xx} = \frac{1}{12}m\left(w^2 + d^2\right)$ $I_{yy} = \frac{1}{12}m\left(d^2 + h^2\right)$ $I_{zz} = \frac{1}{12}m\left(w^2 + h^2\right)$
sphere solid		$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}mr^2$
ellipsoid solid		$I_{xx} = \frac{1}{5}m\left(b^2 + c^2\right)$ $I_{yy} = \frac{1}{5}m\left(c^2 + a^2\right)$ $I_{zz} = \frac{1}{5}m\left(a^2 + b^2\right)$

Problem 5.4 Explain (without formally performing the integration) why the moment of inertia I_{xx} of a thin-walled cylinder about its longitudinal axis is equal to the value given in Table 5.1.

Problem 5.5 Use integration to calculate the moment of inertia of a homogeneous solid cylinder about its longitudinal axis. Check with Table 5.1.

Problem 5.6 Calculate the moment of inertia of a disk and of a thin rod about their respective rotational symmetry axes. Do this by interpreting the bodies as special cases of a solid cylinder, as given in Table 5.1.

Problem 5.7 Use integration to calculate the moment of inertia for a homogeneous sphere of radius r and mass m about an arbitrary axis with respect to its center of mass. Compare your result with Table 5.1.

Problem 5.8 Use integration to calculate the moment of inertia of a thick-walled cylinder (so a hollow cylinder with mass *not* concentrated at one radius from its axle) about its symmetry axis. Use inner radius r_i, outer radius r_o, length l, and mass m. Check your result by filling in specific parameters to obtain some of the special cylinder cases in Table 5.1.

The integrals in the moments and products of inertia are sums over infinitesimally small mass elements. Sums (and integrals) may be broken into arbitrary pieces. So, moments and products of inertia of composite bodies result from adding the components' respective moments and products of inertia. The terms must use the same reference axes.

Example 5.3 An industrial washing machine (Figure 5.3) is loaded with two pieces of clothing at points A and B.

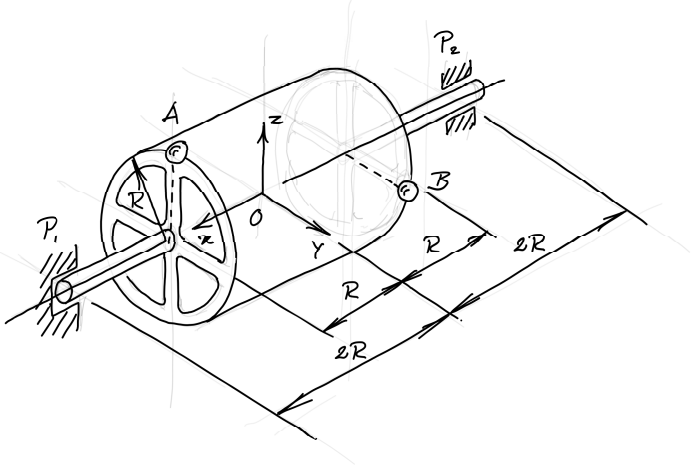


Figure 5.3: Industrial washing machine

A coordinate system xyz, with associated triad $\mathcal{B}$, is connected to the drum and rotates with it. Origin O is located in the geometric drum center. The washing machine is modeled as a thin-walled drum of homogenous material, with radius R, mass m, and width $2R$. Mass of the axle is neglected.

The pieces of clothing are modeled as particles of mass m_a and m_b, respectively, which are rigidly connected to the drum. The positions of points A and B with respect to O are given by the respective vectors $\boldsymbol{p}_a = (R,\, 0,\, R)^{\mathrm{T}}$ and $\boldsymbol{p}_b = (-R,\, R,\, 0)^{\mathrm{T}}$.

Calculate the first row $(I_{xx},\, I_{xy},\, I_{xz})$ of ${}^{\mathcal{B}}\mathbf{I}_O$, the inertia tensor of the combined body around origin O, with components expressed in triad $\mathcal{B}$. Express the three terms as functions of m, m_a, m_b, and R.

Exemplary solution

The complete inertia tensor $\mathbf{I}_O$ is the sum of the inertia tensors $\mathbf{I}_{\mathrm{d}O}$, $\mathbf{I}_{aO}$, and $\mathbf{I}_{bO}$ of the drum, the a-particle at A, and the b-particle at B, respectively:

$$\mathbf{I}_O = \mathbf{I}_{\mathrm{d}O} + \mathbf{I}_{aO} + \mathbf{I}_{bO}. \tag{5.26E}$$

The drum is a standard thin-walled cylinder, so its moment and products of inertia are:

$$I_{\mathrm{d},xx} = mR^2,\; I_{\mathrm{d},xy} = I_{\mathrm{d},xz} = 0. \tag{5.27E}$$

For particle a, the integrals reduce to:

$$I_{a,xx} = \int \left(y^2 + z^2\right) \mathrm{d}m = R^2 m_a, \tag{5.28E}$$

$$I_{a,xy} = -\int (xy)\, \mathrm{d}m = -x_a y_a m_a = 0, \tag{5.29E}$$

$$I_{a,xz} = -\int (xz)\, \mathrm{d}m = -x_a z_a m_a = -R^2 m_a. \tag{5.30E}$$

The same can be done analogously for particle b:

$$I_{b,xx} = R^2 m_b, \tag{5.31E}$$

$$I_{b,xy} = -x_b y_b m_b = R^2 m_b, \tag{5.32E}$$

$$I_{b,xz} = -x_b z_b m_b = 0. \tag{5.33E}$$

Adding the different terms results in the final answer:

$$I_{xx} = R^2(m + m_a + m_b), \tag{5.34E}$$

$$I_{xy} = R^2 m_b, \tag{5.35E}$$

$$I_{xz} = -R^2 m_a. \tag{5.36E}$$

5.1.3 Inertia Tensors for Symmetric Bodies

When a body exhibits symmetry, calculating its inertia tensor is greatly simplified. As we will see in the following, particular products of inertia will be zero for particular types of symmetry. Figure 5.4 shows a body that is symmetric to the xz-plane. To calculate the product of inertia I_{xy}, we can split the integral into two parts, namely the top and bottom half of this body. If we define the two relative position vectors, $\boldsymbol{\rho}_1 = (x_1,\, y_1,\, z_1)^{\mathrm{T}}$ for a mass element at the top, and $\boldsymbol{\rho}_2 = (x_2,\, y_2,\, z_2)^{\mathrm{T}}$ for a corresponding mirrored element at the bottom, we obtain:

$$I_{xy} = -\int_m xy \,\mathrm{d}m = -\int_{m_1} x_1 y_1 \,\mathrm{d}m_1 - \int_{m_2} x_2 y_2 \,\mathrm{d}m_2. \tag{5.37}$$

The two mass elements are mirrored versions of each other, so $x_2 = x_1$, $y_2 = -y_1$, and $\mathrm{d}m_2 = \mathrm{d}m_1$. Substitution simplifies (5.37):

$$I_{xy} = -\int_{m_1} x_1 y_1 \,\mathrm{d}m_1 + \int_{m_1} x_1 y_1 \,\mathrm{d}m_1 = 0. \tag{5.38}$$

This shows that for any body that is symmetric to the xz-plane, the product of inertia I_{xy} will be zero. The same way, we can show that also I_{yz} will be zero for such a body.

So, for any body that is symmetric to a particular plane, the products of inertia for all planes perpendicular to the plane of symmetry will be zero. This also implies that if a body is symmetric to at least two orthogonal planes, the inertia tensor will be diagonal for any coordinate system where the axis directions span these planes of symmetry.

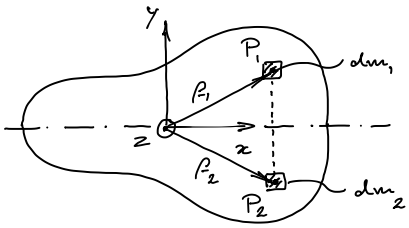


Figure 5.4: This body is symmetric to the xz-plane; for such a body the products of inertia I_{xy} and I_{yz} are zero.

Remark: As we will see later, even if a body is not symmetric, we can always find a specific triad in which the body's inertia tensor is diagonal. So, symmetry to two planes is not a necessary condition for a diagonal inertia tensor, it is only a sufficient condition.

Problem 5.9 Make a drawing and use a similar derivation as in (5.37) and (5.38) to show that for any body that is symmetric to the xy-plane, the product of inertia I_{xz} will be zero.

Problem 5.10 Show that if a body is rotationally symmetric to a particular axis, the inertia tensor will be diagonal for any coordinate system where one axis is aligned with the axis of symmetry.

Problem 5.11 This Problem is about the quadcopter on a pyramid of Problem 3.35 on page 100. Its skeleton can be modeled as two rigidly connected, overlapping, identical flat panels of length L, width W and mass M. Calculate the inertia tensor ${}^{\mathcal{B}}\mathbf{I}_C$ of this skeleton with respect to point C with elements expressed in the $\mathcal{B}$-triad as functions of M, L and W. ▶

Problem 5.12 This Problem is about the Earth orbiting the Sun of Problem 3.36 on page 101. The axial tilt of Earth is approximately $\varepsilon = 23°$. Assume the specific instant where Delft lies in the YZ-plane. For that configuration, state the inertia tensors ${}^{\mathcal{T}}\mathbf{I}_E$ and ${}^{\mathcal{N}}\mathbf{I}_E$ of Earth with respect to its center of mass E. Express all components as functions of m and r. ▷▶

5.1.4 Kinetic Energy in the Case of a Fixed Point

Using a very similar derivation as in Section 5.1.1, we can find that in the special case where a point O exists that has *zero absolute velocity and also does not move with respect to the body*, we can write kinetic energy even shorter than in (5.14):

$$T = \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\mathbf{I}_O\boldsymbol{\omega}. \tag{5.39}$$

Here, the inertia tensor $\mathbf{I}_O$ is defined as in (5.13); however, the vectors $\boldsymbol{\rho}$ of the mass increments are now defined with respect to point O and not with respect to the center of mass. Exemplary cases where such a point exists are those where the rigid body is connected to an inertial system via a hinge or a ball joint. Then, the joint can be chosen as fixed reference point O.

Often, we know the inertia tensor with respect to the center of mass C, for example because we are dealing with a special body like a cylinder or a sphere. In those cases, it is convenient to calculate moments of inertia with respect to another point not via integration, but via the parallel-axis or Steiner's theorem, which will be outlined next.

Problem 5.13 Prove (5.39).

5.2 Changing Inertia Tensor Reference Axes

5.2.1 Inertia with Shifted Reference Point

The *parallel-axis theorem* enables calculating the inertia tensor of a rigid body with respect to a different point than the body's center of mass.

This theorem is also often called *Steiner's theorem* and attributed to the Swiss mathematician Jakob Steiner (1796-1863) and his lectures [69]. However, already in 1673, the Dutch scientist Christiaan Huygens (1629-1695) stated a very similar proposition in his "Horologium Oscillatorium" (The pendulum clock) [32].

The theorem is particularly useful if a fixed point exists on the body, for example because of a spherical joint or hinge connecting to the body to the ground. In that case, Euler's second law can be applied using this fixed point as reference.

In Figure 5.5, a body-fixed coordinate system xyz has its origin in the center of mass C of a rigid body, and we are interested in the body's inertia tensor with respect to a shifted uvw coordinate system with origin in A. Both coordinate systems share the same axis directions.

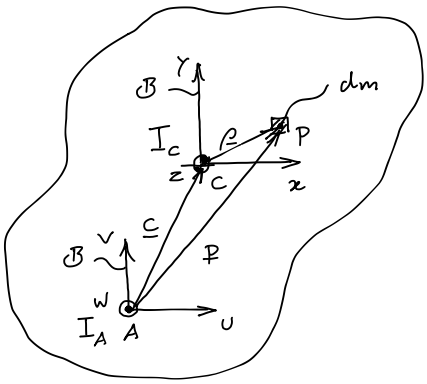


Figure 5.5: Parallel-axis theorem; the xyz coordinate system is body-fixed with its origin on the center of mass C of the body. The uvw coordinate system is shifted but has parallel axes directions, so shares triad $\mathcal{B}$.

The (absolute and relative) position vectors of a mass element dm and the body's center of mass are described by the shorthand notations:

$$\boldsymbol{c} := \boldsymbol{r}_{C/A} =: (u_C,\, v_C,\, w_C)^{\mathrm{T}} \tag{5.40}$$

$$\boldsymbol{p} := \boldsymbol{r}_{P/A} =: (u_P,\, v_P,\, w_P)^{\mathrm{T}} \tag{5.41}$$

$$\boldsymbol{\rho} := \boldsymbol{r}_{P/C} =: (x_P,\, y_P,\, z_P)^{\mathrm{T}} \tag{5.42}$$

We already know the body's mass m and its inertia tensor $\mathbf{I}_C$ with respect to its center of mass C:

$$\mathbf{I}_C = \int_m \tilde{\boldsymbol{\rho}}^{\mathrm{T}} \tilde{\boldsymbol{\rho}} \, \mathrm{d}m, \tag{5.43}$$

but we are looking for the inertia tensor $\mathbf{I}_A$ with respect to point A:

$$\mathbf{I}_A = \int_m \tilde{\mathbf{p}}^{\mathrm{T}} \tilde{\mathbf{p}} \, \mathrm{d}m, \tag{5.44}$$

We can use vector addition to link the two vectors:

$$\boldsymbol{p} = \boldsymbol{c} + \boldsymbol{\rho}, \tag{5.45}$$

which also means

$$\tilde{\mathbf{p}} = \tilde{\mathbf{c}} + \tilde{\boldsymbol{\rho}}. \tag{5.46}$$

Therefore, we can re-write the inertia tensor with respect to A:

$$\mathbf{I}_A = \int_m (\tilde{\mathbf{c}} + \tilde{\boldsymbol{\rho}})^{\mathrm{T}} (\tilde{\mathbf{c}} + \tilde{\boldsymbol{\rho}}) \, \mathrm{d}m, \tag{5.47}$$

$$= \int_m \left(\tilde{\mathbf{c}}^{\mathrm{T}} \tilde{\mathbf{c}} + \tilde{\mathbf{c}}^{\mathrm{T}} \tilde{\boldsymbol{\rho}} + \tilde{\boldsymbol{\rho}}^{\mathrm{T}} \tilde{\mathbf{c}} + \tilde{\boldsymbol{\rho}}^{\mathrm{T}} \tilde{\boldsymbol{\rho}} \right) \, \mathrm{d}m \tag{5.48}$$

By virtue of the definition of the center of mass, we know that

$$\int_m \boldsymbol{\rho} \, \mathrm{d}m = \mathbf{0} \quad \Rightarrow \int_m \tilde{\boldsymbol{\rho}} \, \mathrm{d}m = \mathbf{0}. \tag{5.49}$$

This results from the definition of $\boldsymbol{\rho}$ as the position vector of the mass element with respect to the center of mass C, expressed in the xyz coordinate system. Point C is at the origin, so its position vector is the zero vector. This is the same effect as what we observed in (5.8).

Therefore, we can re-write (5.48) as

$$\mathbf{I}_A = \tilde{\mathbf{c}}^{\mathrm{T}} \tilde{\mathbf{c}} \cdot m + \tilde{\mathbf{c}}^{\mathrm{T}} \left(\underbrace{\int_m \tilde{\boldsymbol{\rho}} \, \mathrm{d}m}_{=\mathbf{0}} \right) + \left(\underbrace{\int_m \tilde{\boldsymbol{\rho}}^{\mathrm{T}} \, \mathrm{d}m}_{=\mathbf{0}} \right) \tilde{\mathbf{c}} + \underbrace{\int_m \tilde{\boldsymbol{\rho}}^{\mathrm{T}} \tilde{\boldsymbol{\rho}} \, \mathrm{d}m}_{\mathbf{I}_C}. \tag{5.50}$$

and simplify to the parallel-axis theorem (writing out $\boldsymbol{c}$ again):

$$\mathbf{I}_A = \mathbf{I}_C + m \cdot \tilde{\mathbf{r}}_{C/A}^{\mathrm{T}} \tilde{\mathbf{r}}_{C/A}. \tag{5.51}$$

So, for example, the moment of inertia I_{ww} with respect to the w-axis is a function of the moment of inertia I_{zz} with respect to the z-axis, and the components of the vector $\boldsymbol{c} = (u_C, v_C, w_C)^{\mathrm{T}}$:

$$I_{ww} = I_{zz} + m(u_C^2 + v_C^2). \tag{5.52}$$

The reader may already recognize this equation from planar dynamics.

The part $(u_C^2 + v_C^2)$ is the projected distance between the origins A and C of the two coordinate systems in the uv-plane. So, the moment of inertia with respect to an axis that is shifted with respect to the body's center of mass C can be found by adding the mass times the squared distance between the respective axes to the moment of inertia with respect to C.

This implies that the moment of inertia with respect to a certain axis is always smallest if this axis passes through the center of mass of the rigid body.

The products of inertia can equally be extracted from (5.51) see Problem 5.17).

Problem 5.14 Show why a rigid body's moment of inertia with respect to a certain axis is always smallest if this axis passes through the center of mass of the body.

Problem 5.15 A homogeneous sphere of mass m and radius R has center of mass S and a sphere-fixed triad $\mathcal{K}$.

This sphere gets damaged: A piece of the sphere breaks off at a point that has the relative position vector ${}^{\mathcal{K}}\boldsymbol{r}_{Q/S} = (0, d, k)^{\mathrm{T}}$. This piece had mass m_Q and is modeled as a particle.

Calculate the inertia tensor ${}^{\mathcal{K}}\mathbf{I}_S$ of the damaged sphere with respect to S, with elements as functions of m, R, m_Q, d, k. ▷

Problem 5.16 This Problem is about the top of corks and skewers of Problem 1.8 on page 12. Calculate the entries of the top's inertia tensor ${}^{\mathcal{B}}\mathbf{I}_Q$ with respect to point Q, expressed in the body-fixed triad $\mathcal{B}$ that is associated with xyz, as functions of m_{k}, m_{s}, l, and Δ. ▷►

Problem 5.17 Use (5.51) to show that the products of inertia for a shifted point of reference, as depicted in Figure 5.5, are given by

$$\begin{aligned} I_{uv} &= I_{xy} - m u_C v_C \\ I_{uz} &= I_{xz} - m u_C w_C \\ I_{vw} &= I_{yz} - m v_C w_C. \end{aligned} \tag{5.53P}$$

5.2.2 Inertia in Rotated Triads

The entries of the inertia tensor will change depending on the orientation of the triad in which the tensor is calculated. We can find a relationship between the inertia tensor in one triad, ${}^{\mathcal{B}}\mathbf{I}_A$, and in a rotated one, ${}^{\mathcal{F}}\mathbf{I}_A$, both calculated with respect to the same point of reference A (which could be the center of mass or a fixed point). To do this, we will make use of the fact that kinetic energy does not change its value depending on which triad it is calculated in. In the following, we write only the part of kinetic energy that is due to rotation, T_{rot}, and we omit the subscript A, to shorten the equations.

Kinetic energy in both triads is identical, so

$$T_{\text{rot}} = \frac{1}{2}\,{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathrm{T}}\,{}^{\mathcal{F}}\mathbf{I}\,{}^{\mathcal{F}}\boldsymbol{\omega} = \frac{1}{2}\,{}^{\mathcal{B}}\boldsymbol{\omega}^{\mathrm{T}}\,{}^{\mathcal{B}}\mathbf{I}\,{}^{\mathcal{B}}\boldsymbol{\omega} \tag{5.54}$$

We know the general relationship of the expressions for vectors between the two triads, using rotation matrices. We can apply this to the angular velocity vector:

$${}^{\mathcal{B}}\boldsymbol{\omega} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}\,{}^{\mathcal{F}}\boldsymbol{\omega}. \tag{5.55}$$

When we substitute this for the expression for kinetic energy that is calculated in the $\mathcal{B}$-triad in (5.54), we obtain

$$T_{\text{rot}} = \frac{1}{2}({}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}\,{}^{\mathcal{F}}\boldsymbol{\omega})^{\mathrm{T}}\,{}^{\mathcal{B}}\mathbf{I}\,({}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}\,{}^{\mathcal{F}}\boldsymbol{\omega}) = \frac{1}{2}\,{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathrm{T}}\,\underbrace{({}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}^{\mathrm{T}}\,{}^{\mathcal{B}}\mathbf{I}\,{}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}})}_{{}^{\mathcal{F}}\mathbf{I}}\,{}^{\mathcal{F}}\boldsymbol{\omega} \tag{5.56}$$

By comparing with the expression for kinetic energy that is calculated in the $\mathcal{F}$-triad in (5.54), we find that ${}^{\mathcal{F}}\mathbf{I}$ must be:

$${}^{\mathcal{F}}\mathbf{I} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}^{\mathrm{T}}\,{}^{\mathcal{B}}\mathbf{I}\,{}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}} = {}^{\mathcal{F}}\mathbf{C}_{\mathcal{B}}\,{}^{\mathcal{B}}\mathbf{I}\,{}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}. \tag{5.57}$$

This is the relationship between the inertia tensor in one triad, ${}^{\mathcal{B}}\mathbf{I}$, and in a rotated one, ${}^{\mathcal{F}}\mathbf{I}$.

The relationship can also be inverted:

$${}^{\mathcal{B}}\mathbf{I} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}\,{}^{\mathcal{F}}\mathbf{I}\,{}^{\mathcal{F}}\mathbf{C}_{\mathcal{B}}. \tag{5.58}$$

Example 5.4

A rigid body is symmetric to two planes in such a way that it has a diagonal inertia tensor ${}^{\mathcal{B}}\mathbf{I}_C$ with respect to its body-fixed x, y, z-axes, which

also define triad $\mathcal{B}$:

$$ {}^{\mathcal{B}}\mathbf{I}_C = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}. $$

Due to accidental misalignment, the body is forced to rotate about an axis that is tilted with respect to its planes of symmetry. A triad $\mathcal{F}$ has one direction aligned with the rotation axis.

The rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}$ is given as

$$ {}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{pmatrix}. \tag{5.59E} $$

Calculate ${}^{\mathcal{F}}\mathbf{I}_C$, expressed in the $\mathcal{F}$-triad.

Exemplary solution

The inertia tensor can be calculated from:

$$ {}^{\mathcal{F}}\mathbf{I}_C = {}^{\mathcal{F}}\mathbf{C}_{\mathcal{B}}\ {}^{\mathcal{B}}\mathbf{I}_C\ {}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}} \tag{5.60E} $$

The rotation matrix ${}^{\mathcal{F}}\mathbf{C}_{\mathcal{B}}$ is the inverse of rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}$:

$$ {}^{\mathcal{F}}\mathbf{C}_{\mathcal{B}} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}^{-1} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix} \tag{5.61E} $$

Substituting all needed terms, the inertia tensor ${}^{\mathcal{F}}\mathbf{I}_C$ can be calculated from (5.60E) as:

$$ {}^{\mathcal{F}}\mathbf{I}_C = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy}\cos^2\gamma + I_{zz}\sin^2\gamma & (I_{yy} - I_{zz})\cos\gamma\sin\gamma \\ 0 & (I_{yy} - I_{zz})\cos\gamma\sin\gamma & I_{yy}\sin^2\gamma + I_{zz}\cos^2\gamma \end{pmatrix}, \tag{5.62E} $$

which can be written shorter as (using the addition theorems of the trigonometric functions):

$$ {}^{\mathcal{F}}\mathbf{I}_C = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_{yy}\cos^2\gamma + I_{zz}\sin^2\gamma & \frac{1}{2}(I_{yy} - I_{zz})\sin(2\gamma) \\ 0 & \frac{1}{2}(I_{yy} - I_{zz})\sin(2\gamma) & I_{yy}\sin^2\gamma + I_{zz}\cos^2\gamma \end{pmatrix}. \tag{5.63E} $$

5.2.3 Principal Moments of Inertia

Calculations and interpretations are greatly simplified if the inertia tensor is diagonal. Therefore, assuming we have an arbitrary inertia tensor ${}^{\mathcal{B}}\mathbf{I}$, we now aim to find out whether it is possible to find a triad $\mathcal{F}$ as-

sociated with coordinate system xyz, in which $\mathbf{I}$ is indeed diagonal:

$$^{\mathcal{F}}\mathbf{I} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}. \tag{5.64}$$

Three questions arise: 1. Does such a triad exist? 2. How do we find this triad $\mathcal{F}$, if it exists? And finally, 3. what are the values on the diagonal? We will find that indeed such a triad always exists, that its direction vectors are eigenvectors of $^{\mathcal{B}}\mathbf{I}$, and that the terms I_{xx}, I_{yy} and I_{zz} are the eigenvalues of $^{\mathcal{B}}\mathbf{I}$. This will be shown in the following.

We know the inertia tensor $^{\mathcal{B}}\mathbf{I}$ expressed in a triad $\mathcal{B}$ which is associated with coordinate system uvw. We just found in (5.57) a relationship to write an inertia tensor expressed in another triad. The problem is now inverse: We know the shape of the desired $^{\mathcal{F}}\mathbf{I}$, but we do not know the rotation matrix $^{\mathcal{F}}\mathbf{C}_{\mathcal{B}}$ yet.

By recalling Section 3.1, we start with constructing the rotation matrix from the (still unknown) unit vectors of the desired $\mathcal{F}$-triad, expressed in the $\mathcal{B}$-triad:

$$^{\mathcal{F}}\mathbf{C}_{\mathcal{B}} = \begin{pmatrix} ^{\mathcal{B}}\hat{\boldsymbol{f}}_1^{\mathrm{T}} \\ ^{\mathcal{B}}\hat{\boldsymbol{f}}_2^{\mathrm{T}} \\ ^{\mathcal{B}}\hat{\boldsymbol{f}}_3^{\mathrm{T}} \end{pmatrix}. \tag{5.65}$$

This allows us to write $^{\mathcal{B}}\mathbf{I}$ as

$$^{\mathcal{B}}\mathbf{I} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}\, {}^{\mathcal{F}}\mathbf{I}\, {}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}^{\mathrm{T}} = \begin{pmatrix} ^{\mathcal{B}}\hat{\boldsymbol{f}}_1 & ^{\mathcal{B}}\hat{\boldsymbol{f}}_2 & ^{\mathcal{B}}\hat{\boldsymbol{f}}_3 \end{pmatrix} \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \begin{pmatrix} ^{\mathcal{B}}\hat{\boldsymbol{f}}_1^{\mathrm{T}} \\ ^{\mathcal{B}}\hat{\boldsymbol{f}}_2^{\mathrm{T}} \\ ^{\mathcal{B}}\hat{\boldsymbol{f}}_3 \end{pmatrix}. \tag{5.66}$$

In order to find a triad $\mathcal{F}$ in which $^{\mathcal{F}}\mathbf{I}$ is indeed diagonal, we must look for an appropriate set of unit vectors.

We recognize (5.66) as a problem of eigendecomposition, as treated in Appendix A.7, with the associations:

$$^{\mathcal{B}}\mathbf{I} = \mathbf{A},\ ^{\mathcal{B}}\mathbf{C}_{\mathcal{F}} = \mathbf{Q},\ ^{\mathcal{F}}\mathbf{I} = \mathbf{\Lambda},\ ^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}^{\mathrm{T}} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}^{-1} = \mathbf{Q}^{-1}. \tag{5.67}$$

Every real symmetric matrix (and the inertia tensor is symmetric) possesses three real eigenvectors that are mutually orthogonal, so the inertia tensor is diagonalizable.

With this, we have found the answers to the three questions: 1. There is always a triad $\mathcal{F}$ in which the inertia tensor of a rigid body is diagonal. 2. Knowing the inertia tensor in an arbitrary triad $\mathcal{B}$, axis directions of $\mathcal{F}$ are given by the eigenvectors of the original inertia tensor ${}^{\mathcal{B}}\mathbf{I}$. These are called the *principal axes of inertia*. 3. The entries of the diagonal inertia tensor ${}^{\mathcal{F}}\mathbf{I}$ are the eigenvalues of the original inertia tensor ${}^{\mathcal{B}}\mathbf{I}$. These values are called the *principal moments of inertia*.

Note that it is possible that infinitely many triads exist where the inertia tensor is diagonal, for example for rotationally symmetric bodies.

To calculate the principal moments of inertia, we need to determine the roots of the characteristic polynomial of ${}^{\mathcal{B}}\mathbf{I}$, as outlined in Appendix A.6. To determine a possible triad $\mathcal{F}$, we have to solve the familiar problem:

$$\left({}^{\mathcal{B}}\mathbf{I} - \mathbf{E}\lambda_i\right) {}^{\mathcal{B}}\boldsymbol{s}_i = \mathbf{0}. \tag{5.68}$$

To achieve unit vectors for the triad directions, the eigenvectors ${}^{\mathcal{B}}\boldsymbol{s}_i$ need to be normalized to length 1. To do so, they are divided by their magnitude ($\hat{\boldsymbol{e}}_i = \boldsymbol{s}_i/|\boldsymbol{s}_i|$).

It is important to choose signs such that a right-handed coordinate system results. This final result is the set of unit vectors of the triad which is orientated in the principal directions, expressed in the $\mathcal{B}$-triad: ${}^{\mathcal{B}}\hat{\boldsymbol{b}}_1$, ${}^{\mathcal{B}}\hat{\boldsymbol{b}}_2$ and ${}^{\mathcal{B}}\hat{\boldsymbol{b}}_3$. These can be assembled to the rotation matrix ${}^{\mathcal{F}}\mathbf{C}_{\mathcal{B}}$.

Problem 5.18 We know the inertia tensor in one triad $\mathcal{F}$, and we are interested in the inertia tensor in another triad $\mathcal{G}$. What can you say about the eigenvalues of ${}^{\mathcal{G}}\mathbf{I}$ if you know the eigenvalues of ${}^{\mathcal{F}}\mathbf{I}$? What can you say about the eigenvectors of ${}^{\mathcal{G}}\mathbf{I}$ if you know those of ${}^{\mathcal{F}}\mathbf{I}$, with components expressed in $\mathcal{F}$?

Example 5.5 The inertia tensor ${}^{\mathcal{B}}\mathbf{I}_O$ of a rigid body around the origin of a body-fixed coordinate system with associated triad $\mathcal{B}$ equals

$${}^{\mathcal{B}}\mathbf{I}_O = J \begin{pmatrix} 20 & 4 & -4 \\ 4 & 34 & 0 \\ -4 & 0 & 34 \end{pmatrix}, \tag{5.69E}$$

where J is a constant.

Show algebraically that an axis in direction of the vector ${}^{\mathcal{B}}\boldsymbol{s} = (0,\, 1,\, 1)^{\mathrm{T}}$ is a principal axis of inertia of the combined rigid body.

Exemplary solution

The directions of the principal moments of inertia are provided by the eigenvectors of the inertia tensor. If ${}^{\mathcal{B}}\boldsymbol{s}$ points in such a direction, that

means that

$$ {}^{\mathcal{B}}\mathbf{I} \cdot {}^{\mathcal{B}}\boldsymbol{s} = \lambda \cdot {}^{\mathcal{B}}\boldsymbol{s}, \qquad \text{with scalar } \lambda. \tag{5.70E} $$

We check this for the given vector ${}^{\mathcal{B}}\boldsymbol{s}$ and inertia tensor ${}^{\mathcal{B}}\mathbf{I}$:

$$ J \begin{pmatrix} 20 & 4 & -4 \\ 4 & 34 & 0 \\ -4 & 0 & 34 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = J \begin{pmatrix} 0 \\ 34 \\ 34 \end{pmatrix} \Rightarrow \lambda = 34J. \tag{5.71E} $$

This means that (5.70E) is true. Therefore the vector ${}^{\mathcal{B}}\boldsymbol{s}$ points in direction of a principal axis of inertia of the rigid body.

Problem 5.19 Calculate the principal axes of inertia of the inertia tensor given in (5.63E) on page 161. You can use the MATLAB$^{\text{TM}}$ symbolic toolbox and the command `eig`, and choose a specific angle of $\gamma = \frac{\pi}{4}$. Before you start, formulate an expectation based on the problem, and check your result against that expectation.

Problem 5.20 Using the concept of eigenvalues, what can you say about the definiteness of the inertia tensor of an arbitrary rigid body, expressed in an arbitrary triad (see also Appendix A.5)? Compare your answer to the result of Problem 5.2 on page 149.

5.2.4 Equivalent Simpler Bodies

The principal moments of inertia can be interpreted in such a way that for any arbitrarily shaped rigid body, an *equivalent ellipsoid* exists that will have the same kinetic energy when rotating at the same angular velocity. An ellipsoid is symmetric to three planes, so it has a diagonal inertia tensor. The principal directions, the eigenvectors of the inertia tensor, define the symmetry planes of the equivalent ellipsoid.

Another, even simpler equivalent representation is that of three orthogonal *dumb-bells*, as illustrated in Figure 5.6. Each dumb-bell consists of two mass points symmetrically arranged to the origin of a coordinate system xyz, which is located in the body's center of mass.

A big advantage of parameterizing inertia tensors using a, b, c of an equivalent dumb-bells or ellipsoid versus using principal moments of inertia I_{xx}, I_{yy}, and I_{zz} is that the former intrinsically accounts for the *triangle inequality*, which states that the sum of any two principal moments of inertia must be larger than or equal to the third (e.g. $Ixx \leq Iyy + Izz$). This is helpful for *optimization* of body shapes and weight distributions.

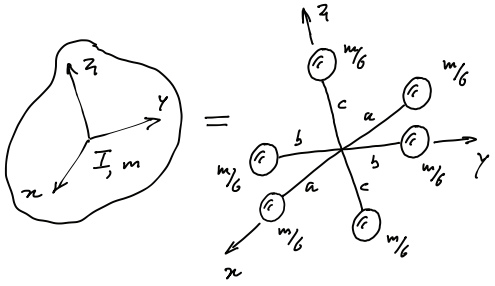


Figure 5.6: Equivalent mass moment of inertia representation for a rigid body by three orthogonal dumb-bells.

Problem 5.21 Show that the inertia tensor of a system of dumb-bells, as in Figure 5.6, with respect to their center of mass C can be found as

$$\mathbf{I}_C = \frac{m}{3}\begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & c^2+a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}. \tag{5.72P}$$

Problem 5.22 Use (5.72P) to show that dumb-bells parameters satisfy the triangle inequality.

Problem 5.23 Calculate a, b and c of the equivalent dumb-bells, as in Figure 5.6, for a rigid body of which mass m and the principal moments of inertia I_{xx}, I_{yy} and I_{zz} are given.

Problem 5.24 For the rigid body in Example 5.5 on page 163, use MATLAB$^{\text{TM}}$ to calculate the direction vectors of the xyz coordinate system, as well as locations a, b, and c of an equivalent dumb-bell representation.

5.2.5 Writing Kinetic Energy as a Product of Scalars

In (5.14) and (5.39), we found expressions for the kinetic energy T of a rigid body. Considering only the rotational part T_{rot}, the kinetic energy is in both cases (the reference point A would be C or a fixed point O):

$$T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega}^{\text{T}}\mathbf{I}_A\boldsymbol{\omega}. \tag{5.73}$$

This equation is greatly simplified if we choose a triad $\mathcal{F}$ for $\boldsymbol{\omega}$ and $\mathbf{I}_A$ where the $\hat{\boldsymbol{f}}_1$-direction is aligned with the body's instantaneous axis of rotation $\hat{\boldsymbol{a}}_i$, so it is also aligned with $\boldsymbol{\omega}$.

Kinetic energy can be calculated using expressions in this triad:

$$T_{\text{rot}} = \frac{1}{2} {}^{\mathcal{F}}\boldsymbol{\omega}^{\text{T}} {}^{\mathcal{F}}\mathbf{I} {}^{\mathcal{F}}\boldsymbol{\omega}. \tag{5.74}$$

As the $\hat{\boldsymbol{f}}_1$-direction is aligned with $\boldsymbol{\omega}$, the components of ${}^{\mathcal{F}}\boldsymbol{\omega}$ must be ${}^{\mathcal{F}}\boldsymbol{\omega} = \begin{pmatrix}\omega & 0 & 0\end{pmatrix}^{\text{T}}$, with scalar ω. So, kinetic energy due to rotation is:

$$T_{\text{rot}} = \frac{1}{2} \begin{pmatrix}\omega & 0 & 0\end{pmatrix} \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix} \begin{pmatrix}\omega \\ 0 \\ 0\end{pmatrix}. \tag{5.75}$$

Multiplication gives:

$$T_{\text{rot}} = \frac{1}{2} \begin{pmatrix}\omega & 0 & 0\end{pmatrix} \begin{pmatrix} I_{xx}\omega \\ I_{xy}\omega \\ I_{xz}\omega \end{pmatrix} = \frac{1}{2} \underbrace{I_{xx}}_{I_{\text{a}}} |\boldsymbol{\omega}|^2 \tag{5.76}$$

We note that the variable $I_{\text{a}} = I_{xx}$ is the moment of inertia about the instantaneous axis of rotation, as this is how we chose the axis directions. Also note that the magnitude $|\boldsymbol{\omega}|$ of the angular velocity vector does not change from one triad to another. So, we can calculate the magnitude in whichever triad the angular velocity vector is given.

Or in other words, if we write the vector $\boldsymbol{\omega}$ in the form of $\boldsymbol{\omega} = |\boldsymbol{\omega}|\hat{\boldsymbol{e}}_\omega$, then the vector $\hat{\boldsymbol{e}}_\omega$ is the unit vector of the instantaneous axis of rotation, and the kinetic energy due to rotation is

$$T_{\text{rot}} = \frac{1}{2} I_\omega |\boldsymbol{\omega}|^2, \tag{5.77}$$

with

$$I_\omega = \hat{\boldsymbol{e}}_\omega^{\text{T}} \mathbf{I} \hat{\boldsymbol{e}}_\omega. \tag{5.78}$$

This expression for kinetic energy is especially useful for problems where a body undergoes only *planar* movement (for example only in the xy-plane), because there the instantaneous axis of rotation is constant (in the example, it is the z-axis) and only a single moment of inertia of a rigid body needs to be considered throughout the problem. As soon as the instantaneous axis of rotation changes its location with respect to the body, it is often better to construct the entire inertia tensor and calculate kinetic energy using the previous relations.

Problem 5.25 For a problem in 2-D, how do the equations for the calculation of kinetic energy of a rigid body simplify? ▶

Problem 5.26 A rigid body is currently rotating with an angular velocity ${}^{\mathcal{B}}\boldsymbol{\omega} = \begin{pmatrix}\omega_x & \omega_y & \omega_z\end{pmatrix}^{\mathrm{T}}$. Body-fixed xyz-axes define triad $\mathcal{B}$. The body's products of inertia are zero with respect to xyz. Calculate the body's moment of inertia about the body's instantaneous axis of rotation, as a function of $\omega_x, \omega_y, \omega_z$ and the respective moments of inertia I_{xx}, I_{yy}, I_{zz}. ▶

Problem 5.27 Use (5.77) to prove the existence of an equivalent ellipsoid for any body.

5.3 The Principle of Work and Energy

5.3.1 Work of Internal and External Forces for Rigid Bodies

Section 2.2 showed how to calculate the power of a force $\boldsymbol{F}$ thas has a point of application with position vector $\boldsymbol{r}$ and velocity vector $\boldsymbol{v}$,

$$P := \boldsymbol{F}^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{F}^{\mathrm{T}}\dot{\boldsymbol{r}}, \tag{5.79}$$

and we also saw that the work done by the application of this force is:

$$W := \int_{t_0}^{t_1} P\mathrm{d}t = \int_{t_0}^{t_1} \boldsymbol{F}^{\mathrm{T}}\boldsymbol{v}\mathrm{d}t = \int_{A}^{B} \boldsymbol{F}^{\mathrm{T}}\mathrm{d}\boldsymbol{r}, \tag{5.80}$$

where the point of application travels from point A to B in the time interval from t_0 to t_1, with infinitesimal position vector increments $\mathrm{d}\boldsymbol{r}$.

For a system of particles, the principle of work and energy (2.47) states that both internal and external forces can do work and thereby change the kinetic energy of the system.

However, for a rigid body, all particles are rigidly connected. Therefore, the terms $\boldsymbol{f}_{ij}\cdot\mathrm{d}\boldsymbol{r}_i$ occur in equal and opposite pairs; the work done on particle i by the internal force $\boldsymbol{f}_{ij}$ is equal and opposite to the work done by $\boldsymbol{f}_{ji}$ on particle j, because there is no relative movement of the two particles in direction of the forces' line of action, so towards each other or away from each other (otherwise the body would not be rigid). That means that for a rigid body, only the work of external forces needs to be considered in the principle of work and energy:

$$\Delta T = W = \sum_{i=1}^{N} \int_{A_i}^{B_i} \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}_i. \tag{5.81}$$

In the case that all forces acting on the rigid body are conservative, conservation of energy, (2.50), applies.

◯ **Problem 5.28** Create a drawing to show that the work done by internal forces in a rigid body is zero.

◯ **Problem 5.29** A model plane is attached to the ceiling with a spring and can rotate and translate under the influence of gravity (like in Problem 3.23). At time t_0 the plane's center of mass has a velocity v_0 and the plane has no angular velocity. Later, at time t_1, the plane is back at its initial position and orientation. Does the plane have the same kinetic energy at t_1 and t_0? Is it possible it has nonzero angular velocity at t_1? ▷►

5.3.2 Work and Power of Moments

From Section 1.2.1, we recall that the moment of a force with respect to a point O is

$$\boldsymbol{M}_O = \boldsymbol{r} \times \boldsymbol{F}, \tag{5.82}$$

where the vector $\boldsymbol{r}$ is a vector from O to any point on the line of action of the force $\boldsymbol{F}$.

We choose O as a point *on the instantaneous axis of rotation* of the rigid body, and we assume that there is no translation along this axis. Then, with knowledge from Section 4.1.4, we can find the velocity of the point of application of the force by

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}. \tag{5.83}$$

So, the power can be re-written by substituting for velocity in (5.79):

$$P = \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{\omega} \times \boldsymbol{r}). \tag{5.84}$$

Using the tilde matrix notation, we can derive

$$P = \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{\omega} \times \boldsymbol{r}) = \boldsymbol{F}^{\mathrm{T}}(\tilde{\mathbf{r}}^{\mathrm{T}}\boldsymbol{\omega}) = (\boldsymbol{F}^{\mathrm{T}}\tilde{\mathbf{r}}^{\mathrm{T}})\boldsymbol{\omega} = (\tilde{\mathbf{r}}\boldsymbol{F})^{\mathrm{T}}\boldsymbol{\omega} = (\boldsymbol{r} \times \boldsymbol{F})^{\mathrm{T}}\boldsymbol{\omega}. \tag{5.85}$$

We recognize that the term on the right of this equation is the scalar product of the moment vector $\boldsymbol{M}_O$ with the angular velocity, so we can find power of a moment to be:

$$P = \boldsymbol{M}_O^{\mathrm{T}}\boldsymbol{\omega}, \tag{5.86}$$

So, we have found an alternative way of describing power of a force acting on a rigid body that performs pure rotation about an instantaneous axis. Note that in the case where there is also translation along the axis, further work is done by the force component in that direction.

As work is the integral of power with respect to time, we can find the work of a moment in a fashion analogous to a force, using the definitions of angular velocity $\boldsymbol{\omega}$ and infinitesimal angle increments $\mathrm{d}\boldsymbol{\theta}$ from Section 4.1:

$$W := \int_{t_0}^{t_1} P \,\mathrm{d}t = \int_{t_0}^{t_1} \boldsymbol{M}_O^{\mathrm{T}} \boldsymbol{\omega} \,\mathrm{d}t = \int_{\mathcal{A}}^{\mathcal{B}} \boldsymbol{M}_O^{\mathrm{T}} \,\mathrm{d}\boldsymbol{\theta}, \tag{5.87}$$

where the orientation of the rigid body changes from $\mathcal{A}$ to $\mathcal{B}$.

As stated in Section 1.2.2, also in the case where there is no net resultant force acting on the body, only a pure moment (for example applied by a rotary motor), one can always represent this moment vector by a couple of two forces that are equal and opposite, but do not share the same line of action. Therefore, the power and work of the moment is still calculated the same way, according to (5.86) and (5.87), respectively.

◯ **Problem 5.30** Calculate mechanical power generated by a free moment vector acting on a rigid body, if the moment vector points in direction of the body-fixed z-axis and the body is rotating about the body-fixed x-axis.

◯ **Problem 5.31** Calculate the mechanical power generated by a DC motor that is spinning at angular speed $|\Omega|$ and exerts a torque of magnitude $|\tau|$ on a load, for the case where the torque slows down the rotation.

5.4 Checking Plausibility and Frequent Mistakes

When calculating inertia tensors or kinetic energy, exemplary plausibility checks, in addition to Section 1.4, are to:

- check that the trace of any inertia tensor remains invariant to rotations, and equal to the sum of its principal moments of inertia (see Section A.6).
- check that the inertia tensor is symmetric.

Particularly frequent mistakes (and therefore potential reasons for the failure of a check) are

- using relative velocities (instead of absolute ones) for calculating kinetic energy.
- mixing (5.14) and (5.39). This can be recognized easily by always including the reference point as subscript to the inertia tensor.
- applying (5.39) although point O has nonzero velocity and is not the center of mass of the rigid body.

- attempting to calculate independent energy "components" for different coordinate directions using scalar equations, thereby losing contributions from off-diagonal elements in the inertia tensor.

5.5 Summary

Two alternative equations to calculate the kinetic energy of a rigid body were presented. A generic one where the reference is center of mass C:

$$T = \frac{1}{2} m |\dot{\boldsymbol{r}}_C|^2 + \frac{1}{2} \boldsymbol{\omega}^{\mathrm{T}} \mathbf{I}_C \boldsymbol{\omega}, \tag{5.88}$$

and one that can be applied if the body has point O without velocity:

$$T = \frac{1}{2} \boldsymbol{\omega}^{\mathrm{T}} \mathbf{I}_O \boldsymbol{\omega}. \tag{5.89}$$

The symmetric inertia tensor about a point P can be calculated via integration, and it contains the moments and products of inertia:

$$\mathbf{I}_P := \int_m \tilde{\boldsymbol{\rho}}^{\mathrm{T}} \tilde{\boldsymbol{\rho}} \, \mathrm{d}m = \begin{pmatrix} \int_m z^2 + y^2 \, \mathrm{d}m & -\int_m xy \, \mathrm{d}m & -\int_m xz \, \mathrm{d}m \\ -\int_m xy \, \mathrm{d}m & \int_m x^2 + z^2 \, \mathrm{d}m & -\int_m yz \, \mathrm{d}m \\ -\int_m xz \, \mathrm{d}m & -\int_m yz \, \mathrm{d}m & \int_m x^2 + y^2 \, \mathrm{d}m \end{pmatrix}. \tag{5.90}$$

The tensor has several important properties:

- The parallel-axis theorem allows re-writing moments of inertia with respect to a different point of reference.
- It is always possible to diagonalize the tensor by mapping it to a rotated triad, whereby the new axes point in direction of the eigenvectors (the principal directions) of the inertia tensor.
- It is also always possible to find a triad in which kinetic energy can be calculated by scalar multiplication, namely if one axis of the triad is aligned with the instantaneous axis of rotation.
- To each rigid body, there is an equivalent, simpler representation using an ellipsoid or a set of dumb-bells.

When applying the principle of work and energy to a rigid body, we found that the internal forces within the rigid body cannot do net work, because of the constant distances between all particles within the body.

5.6 Problems

5.6.1 Guided Problems

Problem 5.32 We will investigate kinetic energy of the model plane of Problem 3.23 on page 92. The plane's mass distribution is symmetric to the xz- and to the xy-plane, and the mass moments of inertia about the body-fixed axes are I_{xx}, I_{yy}, I_{zz}.

a. Establish the inertia tensor ${}^{\mathcal{B}}\mathbf{I}_C$ of the plane with respect to its center of mass C, expressed in the $\mathcal{B}$-triad, with elements as functions of I_{xx}, I_{yy}, I_{zz}.
b. Calculate the plane's kinetic energy T as a function of its angular velocity ${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$, the velocity ${}^{\mathcal{B}}\dot{\boldsymbol{r}}_C$ of the center of mass, mass m, and the inertia tensor ${}^{\mathcal{B}}\mathbf{I}_C$.
c. Calculate the position vector ${}^{\mathcal{N}}\mathbf{r}_{P/O}$ of attachment point P as a function of ${}^{\mathcal{B}}\mathbf{r}_{P/C}$, ${}^{\mathcal{N}}\mathbf{r}_{C/O}$, and the rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$.
d. Calculate the potential energy V_s stored in the spring and gravitational potential energy V_g of the plane as functions of the components of the position vector ${}^{\mathcal{N}}\boldsymbol{r}_{P/O} = (P_X, P_Y, P_Z)^\mathrm{T}$, the plane's center of mass coordinates C_X, C_Y, C_Z in XYZ, and the constant parameters k, l_0, m and g. Herein, define V_g to be zero when the center of mass $C = S$ of the plane is in the XY-plane. Be careful with signs.
Conduct at least three independent plausibility checks on each potential function. One of these checks must be whether the respective force vector acting on the plane is indeed the negative gradient of the potential. Be sure to calculate a predicted outcome *before* performing the check.
e. Calculate the power P generated by all external forces acting on the plane, as a function of ${}^{\mathcal{B}}\boldsymbol{r}_{C/O}$, ${}^{\mathcal{B}}\dot{\boldsymbol{r}}_{C/O}$, ${}^{\mathcal{B}}\boldsymbol{r}_{P/C}$, ${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$, the rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$, and the constants m, g, l_0, k [a].
f. Accidentally, a piece breaks off of the right wing.
Approximate the missing piece as a point mass of mass m_D that was located at point D, with position vector ${}^{\mathcal{B}}\boldsymbol{r}_{D/C} = (D_x, D_y, D_z)^\mathrm{T}$.
Calculate the new location of the center of mass Q with respect to point C, so the vector ${}^{\mathcal{B}}\boldsymbol{r}_{Q/C}$, as well as the new inertia tensor ${}^{\mathcal{B}}\mathbf{I}_{\mathrm{d},Q}$ with respect to point C. Provide all vector and matrix elements as functions of I_{xx}, I_{yy}, I_{zz}, D_x, D_y, D_z, m and m_D.
g. Independent of your results to the previous part, assume that in the damaged plane, the center of mass has moved to point ${}^{\mathcal{B}}\boldsymbol{r}_{Q/C} = (q_x, q_y, q_z)^\mathrm{T}$, and that the changed inertia tensor is given as ${}^{\mathcal{B}}\mathbf{I}_{\mathrm{d},S}$.
Calculate the damaged plane's inertia tensor ${}^{\mathcal{B}}\mathbf{I}_{\mathrm{d},Q}$ with respect to the new center of mass location Q, as a function of ${}^{\mathcal{B}}\mathbf{I}_{\mathrm{d},C}$, q_x, q_y, q_z, m and m_D.

h. State for each of the points O, P, C and Q whether they could be used as reference points to calculate kinetic energy of the damaged plane.
i. Explain in words how you would calculate the principal moments of inertia and the principal axes of inertia of the damaged plane. Do not actually calculate these.

[a] *Hint*: Note that forces act on different points.

Problem 5.33 A satellite (Figure 5.7) with body-fixed triad $\mathcal{F}$ is rotating in space with angular velocity ${}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$ relative to an inertial $\mathcal{N}$-triad. The reference coordinate system XYZ is associated with triad $\mathcal{N}$, while a local coordinate system xyz is associated with triad $\mathcal{F}$.

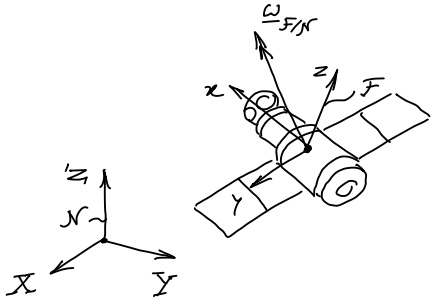


Figure 5.7: Satellite

The mass of the satellite is m, and its inertia tensor in the $\mathcal{F}$-triad about the center of mass S is given as a function of its mass as

$${}^{\mathcal{F}}\mathbf{I}_S = \frac{m}{256}\begin{pmatrix} 46 & 3\sqrt{3} & -15 \\ 3\sqrt{3} & 125/2 & 5\sqrt{3}/2 \\ -15 & 5\sqrt{3}/2 & 103/2 \end{pmatrix} \cdot \mathrm{m}^2$$

Angular velocity of the satellite at the considered instant is:

$${}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix}^{\mathrm{T}} \cdot \mathrm{s}^{-1},$$

and the absolute velocity vector $\boldsymbol{v}_S$ of center mass of the satellite is:

$${}^{\mathcal{F}}\boldsymbol{v}_S = \begin{pmatrix} 0 & 4\sqrt{3} & 4 \end{pmatrix}^{\mathrm{T}} \cdot \mathrm{m\,s}^{-1}.$$

a. Calculate the kinetic energy T of the satellite as a function of its mass m. Take care with units.
b. Calculate axis directions of a triad $\mathcal{G}$ such that ${}^{\mathcal{G}}\mathbf{I}_S$ is diagonal and the first axis is the one about which the satellite has the smallest moment

of inertia. Note that there is more than one such triad (why?) and choose it such that the scalar product of $\hat{\boldsymbol{f}}_3$ and $\hat{\boldsymbol{g}}_3$ is 3/4. [a]

c. Calculate ${}^{\mathcal{G}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$ and ${}^{\mathcal{G}}\boldsymbol{v}_S$.

d. Calculate the kinetic energy T again, this time using the $\mathcal{G}$-triad. Compare your result with the previously calculated T.

e. Repeat all calculations with a third triad $\mathcal{A}$ in which rotational kinetic energy can be calculated using only scalars.

[a] *Hint*: Try using the MATLAB™ function eig.

Problem 5.34 The prism provided in Figure 5.8 has a right-angled isosceles base and a uniform mass distribution with density ρ. An inertial triad $\mathcal{N}$, associated with axes XYZ, is defined with its Z-axis along the length of the prism, its Y-axis parallel to the bottom straight edge and its X-axis vertically up, parallel to its left straight edge. The origin of this coordinate system is the bottom left vertex O.

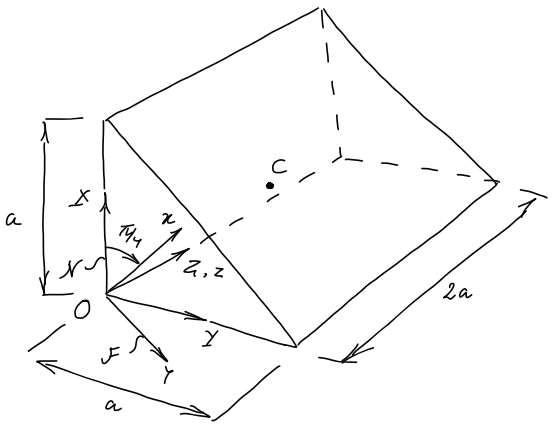


Figure 5.8: Schematic of an isosceles prism.

a. Derive the inertia tensor (mass moments of inertia) ${}^{\mathcal{N}}\mathbf{I}_O$ of the prism via integration.

b. A second coordinate system xyz with associated triad $\mathcal{F}$ is rotated about the Z-axis in positive direction by an angle of $\pi/4$ with respect to XYZ.
Using your answer to the previous part of this question, combined with suitable rotation matrices, calculate the inertia tensor ${}^{\mathcal{F}}\mathbf{I}_O$ of the prism.

c. A third coordinate system uvw has the same axes directions but is shifted with respect to xyz, such that it has its origin in the center of mass C of the prism.

Using your answer to the previous part of this question, calculate the inertia tensor ${}^{\mathcal{F}}\mathbf{I}_C$ of the prism.

d. Check your answers to the previous parts of this question for plausibility, using at least three different checks for each answer[a].
e. Calculate the unit vectors of a triad $\mathcal{G}$, in which the inertia tensor ${}^{\mathcal{G}}\mathbf{I}_O$ is a diagonal matrix. What are the values and meaning of the diagonal elements?

[a] *Hint*: For example, note that there is some symmetry in this body. What does this mean for which of the calculated inertia tensors?

Problem 5.35 We aim to apply the principle of work and energy to the tippe top from Problem 1.16 on page 32 and Problem 4.27 on page 134. The top spins on a flat surface under the influence of gravity, with field strength g.

a. Via integration, calculate the moment of inertia of a sphere that has a cap of height $(R - a)$ cut off about its symmetry axis. Make one or multiple drawings where you include and label *all* variables used in your calculations (including those that are infinitesimally small), so that they are clearly defined. Conduct at least three independent plausiblity checks.
b. Calculate the moment of inertia of the tippe top about its symmetry axis. Make again one or multiple drawings where you include and label *all* variables used in your calculations, so that they are clearly defined.
c. Implement a function in MATLAB$^{\text{TM}}$ or another software of your choice that calculates the kinetic and the potential energy of the tippe top as a function of the given constants and the current kinematic state, which is defined by Euler angles, angular velocity of the tippe top, velocity and position of point S. Be aware that S is not the center of mass. Conduct at least three independent plausibility checks using your implementation.
d. Implement a function in MATLAB$^{\text{TM}}$ or another software of your choice that calculates the power generated by all external forces acting on the tippe top, as a function of the given constants, the current kinematic state, and the friction coefficient μ. Assume only dry friction, and that the top is always slipping. Conduct at least three independent plausibility checks using your implementation.

Problem 5.36 Two cyclists are coasting (not pedaling, nor braking) on their bicycles close to each other. They want to increase their speed without using the pedals. After some thinking, they propose to pull against each other. They hypothesize that in this way, they put energy in the system, which will increase their average forward speed.

Assume there are no mechanical energy losses due to friction, air drag or slope resistance, then the system of the two bicycles plus the two cyclists is energy-conservative. Approximate each bicycle-rider combination as a point mass with a total mass of $m_A = 80\,\text{kg}$ and $m_B = 110\,\text{kg}$, respectively.

At the start of the maneuver, both bicycles have an initial speed of $v = 5\,\text{m/s}$ in forward direction. During the maneuver, rider A pulls rider B with a constant force of 100 N, and vice versa. The pulling maneuver stops when the bicycles are next to each other, that is when the relative distance of $s = 1.3\,\text{m}$ is reduced to zero.

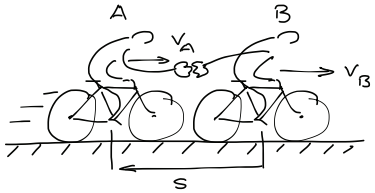


Figure 5.9: Two cyclist in motion pulling at each other to increase their average speed.

a. Determine the kinetic energy T_1 of the system before the pulling maneuver.
b. Determine the amount of work W that has been put into the system when the pulling operation is completed.
c. Determine the kinetic energy T_2 of the system when the pulling operation is completed.
d. Determine the speed v_{C1} of the combined center of mass of the system (CoM) before the pulling operation, and the CoM speed v_{C2} after the pulling operation[a].
e. Determine the speed of each individual cyclist v_{A2} and v_{B2} after the pulling operation.

[a] *Hint*: Draw a FBD of the combined system and apply Newton's second law.

Problem 5.37 A track cyclist is cycling in a velodrome on the straight with a constant forward speed of v_0 (Figure 5.10). At the end of the straight, he enters the curve with constant radius R, where he has to lean his bicycle to stay in the curve. He does not climb to a higher level in the curve, but stays at the same height as on the straight. Looking at his speedometer, which measures at the rear wheel, he notices that he has increased his forward speed although he did not add any more pedal energy into the system. At the end of the training, he discusses this with his coach, who, having been educated at TU Delft, after some thinking, solves the mystery.

Approximate the bicycle-rider system as one rigid body and neglect the

mass of the wheels. The mass of the system is m, and the mass moments of inertia about the body-fixed triad $\mathcal{B}$, with principal axes xyz, are given in the inertia tensor $\mathbf{I} = \text{diag}(A, B, C)$. The center of mass (CoM) of the system is at a height h from the ground when the cyclist is straight up. The rider is just coasting (not pedaling), and no energy is dissipated.

For parameter values use: $v_0 = 15$ m/s, $R = 25$ m, $m = 90$ kg, $\mathbf{I} = \text{diag}(9, 11, 3)$ kgm^2, $h = 0.9$ m, and $g = 9.81$ N/kg.

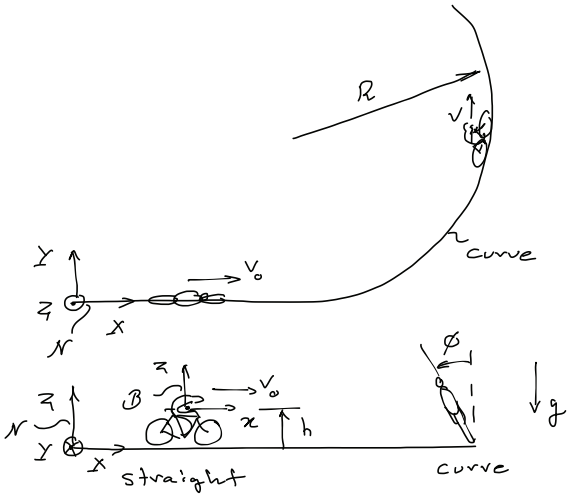


Figure 5.10: A cyclist in a velodrome, top and side view.

a. Determine the kinetic energy of the system T_0 when cycling on the straight.
b. Assume the rider is going through the curve with a constant forward speed v of his CoM. Determine the expression for the lean (roll) angle φ of the bicycle such that he will stay in the curve in steady motion ($\dot{\varphi} = 0$). Only for this part of the problem, approximate the rider as a particle on a massless stick (like an inverted pendulum), with mass concentrated in the CoM, and use the Newton-Euler approach.
c. Determine the angular velocity ${}^{\mathcal{B}}\boldsymbol{\omega}_{B/N}$ of the system with respect to the inertial triad $\mathcal{N}$, expressed in the body-fixed triad $\mathcal{B}$, when cycling in the curve, expressed in terms of the constant forward speed v in the curve, the constant roll angle φ, and the other parameters. Assuming the speed does not change much from straight to curve, calculate φ. Check the relative magnitudes of potential energy, the part of kinetic energy that is due to rotation about the CoM, and kinetic energy due to translation of the CoM. What is the dominant effect?
d. Determine the value of the forward speed v, using the constant roll

angle φ from the previous part. Explain the speed change in the curve as observed by the cyclist.

e. Continue to iterate: Update the roll angle φ using the forward speed v found in the previous part and compare with the initial roll angle. Redo the forward speed calculation in the curve with the new roll angle and compare your result with the previous one.

f. The sensor counts wheel revolutions. How is this related to v, and does this change in the curve?

5.6.2 Practice Problems

Problem 5.38 This Problem is about the ship installing a monopile of Problem 4.21 on page 130. Explain which of the points depicted in the figure (O, K, D, G, P) is/are allowed as reference point "?" for the monopile's inertia tensor $\mathbf{I}$ when calculating kinetic energy T of the monopile using

$$T = \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\mathbf{I}_{?}\boldsymbol{\omega}, \tag{5.91P}$$

where $\boldsymbol{\omega}$ is angular velocity of the monopile. ▷►

Problem 5.39 This Problem is about the T-shaped body of Problem 1.20 on page 36. Which statements is/are true about the body's kinetic energy T (Check all that apply)?

A. T is constant.
B. T is a vector.
C. T can be correctly calculated using the equation $T = \frac{1}{2}mv^2$, where v is the speed of the body's center of mass, and $m = m_1 + m_2 + m_3$.
D. None of the above

▷

Problem 5.40 This Problem is about the T-shaped body of Problem 1.20 on page 36. For the special case where $m_1 = m_2 = m_3$, which of the following statements is/are true? (Check all that apply):

A. $I_{yy} > I_{zz}$
B. $I_{xx} > I_{zz}$
C. $I_{xy} = I_{xz} = I_{yz}$
D. $I_{xy} = 0$
E. None of the above

▷

Problem 5.41 Is the instantaneous axis of rotation always equal to a

principal axis of inertia of a rigid body? If yes, why? If no, can you give counterexamples? ▷►

Problem 5.42 The rigid body depicted in Problem 1.20 on page 36 has three principal moments of inertia I_1, I_2, I_3 with respect to its center of mass, where it holds that $I_1 > I_2 > I_3$.

For a given angular momentum magnitude, in which case(s) is the kinetic energy of the body minimal (Check all that apply)?

A. in case the body is rotating purely about the axis associated with I_1.
B. in case the body is rotating purely about the axis associated with I_2.
C. in case the body is rotating purely about the axis associated with I_3.
D. in case the body has equal angular velocity components about all axes.
E. in case the magnitude of the angular velocity vector is minimal.
F. None of the above

▷

Problem 5.43 A rigid body has three principal moments of inertia I_1, I_2, I_3 with respect to its center of mass C, with $I_1 > I_2 > I_3$. Body-fixed axes xyz and its triad $\mathcal{B}$ are aligned with these principal directions in such a way that the body's inertia tensor with respect to the origin C of xyz is

$$^{\mathcal{B}}\mathbf{I}_C = \begin{pmatrix} I_3 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_1 \end{pmatrix} \tag{5.92P}$$

A student attempted to calculate the inertia tensor of the body in a different triad $\mathcal{F}$. The first direction of the $\mathcal{B}$-triad, expressed in $\mathcal{F}$, is $^{\mathcal{F}}\hat{\boldsymbol{b}}_1 = \frac{1}{5}(3,4,0)^{\mathrm{T}}$. The student found the following tensor:

$$^{\mathcal{F}}\mathbf{I}_C = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 + 2I_3 & I_2 - I_3 \\ 0 & I_2 - I_3 & 2I_2 + I_3 \end{pmatrix} \tag{5.93P}$$

Without knowing more about triad $\mathcal{F}$, check the plausibility of this result with three different, independent plausibility checks. The student has already checked units, so omit that. For each plausibility check, state a precise condition you expect to hold for $^{\mathcal{F}}\mathbf{I}_C$. Provide a complete reasoning why you expect this to hold. Only then, execute the check and conclude whether $^{\mathcal{F}}\mathbf{I}_C$ is plausible or not. ►

Problem 5.44 This Problem is about the accelerometer with a spherical mass of Problem 1.26 on page 41. We want to calculate the sphere's kinetic energy T using the following equation:

$$T = \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\mathbf{I}_P\boldsymbol{\omega} + \frac{1}{2}m|\boldsymbol{v}_P|^2, \tag{5.94P}$$

where $\boldsymbol{v}_P$ is the velocity of point P, $\mathbf{I}_P$ is the sphere's inertia tensor with respect to point P, and $\boldsymbol{\omega}$ is the sphere's angular velocity vector. For the calculation of T in (5.94P) to be correct, the following point(s) may be used for the placeholder P (Check all that apply):

A. S
B. A
C. O
D. C
E. D
F. None of the above

▷

Problem 5.45 For the accelerometer of Problem 1.26 and Problem 5.44 on page 41 and page 179, we want to calculate the angular velocity $\boldsymbol{\omega}$ of the sphere, as used in (5.94P). This vector can be correctly calculated as (Check all that apply):

A. $\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{K}/\mathcal{N}}$
B. $\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{C}/\mathcal{N}} \cdot \boldsymbol{\omega}_{\mathcal{K}/\mathcal{C}}$
C. $\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{C}/\mathcal{N}} - \boldsymbol{\omega}_{\mathcal{K}/\mathcal{C}}$
D. $\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{C}/\mathcal{N}} \times \boldsymbol{\omega}_{\mathcal{K}/\mathcal{C}}$
E. $\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{C}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{K}/\mathcal{C}}$
F. None of the above

▷

Problem 5.46 A homogeneous cylinder has radius r, height d and center of mass S. A cylinder-fixed coordinate system xyz has its x-axis aligned with the roational symmetry axis and its origin in S. It defines directions of a triad $\mathcal{B}$-triad. The inertia tensor of the cylinder about S is already given as ${}^{\mathcal{B}}\mathbf{I}_S$.

An additional particle m is attached to the cylinder in point P, which has coordinates $x_P = r$, $y_P = 0$, and $z_P = \frac{d}{2}$.

Calculate the inertia tensor ${}^{\mathcal{B}}\mathbf{I}^*_S$ of the composite body (cylinder and particle) in the $\mathcal{B}$-triad as a function of ${}^{\mathcal{B}}\mathbf{I}_S$, m, d, and r (Remark: It is possible that not all of these terms appear in the requested function).

Are x, y, z principal axes of inertia for this composite body (cylinder and particle)? Explain your answer, yes or no is not sufficient. ▶

Problem 5.47 This Problem is about the rotating chair with a person of Example 4.1 on page 106. The inertia tensor of the person about point O is defined by $\mathbf{I}_\mathrm{p}$ and the inertia tensor of the wheel about point W is defined by $\mathbf{I}_\mathrm{w}$. The wheel is rotationally symmetric and has homogeneous mass distribution. The velocity of point W is given by $\boldsymbol{v}_W$. For each option below, indicate whether or not it is a correct calculation of the kinetic energy of the system.

$$T = \frac{1}{2} m_\mathrm{w} \left| {}^{\mathcal{D}}\boldsymbol{v}_W \right|^2 \ldots$$

A. $\ldots + \frac{1}{2} \left({}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \right)^\mathrm{T} {}^{\mathcal{B}}\mathbf{I}_\mathrm{w} \, {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \frac{1}{2} \left({}^{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \right)^\mathrm{T} {}^{\mathcal{D}}\mathbf{I}_\mathrm{p} \, {}^{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}}$

B. $\ldots + \frac{1}{2} \left({}^{\mathcal{B}}\mathbf{I}_\mathrm{w} + {}^{\mathcal{B}}\mathbf{I}_\mathrm{p} \right) \cdot \left| \left({}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \right) \right|^2$

C. $\ldots + \frac{1}{2} \left({}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \right)^\mathrm{T} {}^{\mathcal{F}}\mathbf{I}_\mathrm{w} \, {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \frac{1}{2} \left({}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \right)^\mathrm{T} {}^{\mathcal{B}}\mathbf{I}_\mathrm{p} \, {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}}$

D. $\ldots + \frac{1}{2} \left({}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \right)^\mathrm{T} {}^{\mathcal{B}}\mathbf{I}_\mathrm{w} \, {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \frac{1}{2} \left({}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \right)^\mathrm{T} {}^{\mathcal{B}}\mathbf{I}_\mathrm{p} \, {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}}$

E. $\ldots + \frac{1}{2} {}^{\mathcal{B}}\mathbf{I}_\mathrm{w} \cdot \left| {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \right|^2 + \frac{1}{2} {}^{\mathcal{D}}\mathbf{I}_\mathrm{p} \left| {}^{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \right|^2$

F. $\ldots + \frac{1}{2} \left({}^{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \right)^\mathrm{T} {}^{\mathcal{B}}\mathbf{I}_\mathrm{w} \, {}^{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} + \frac{1}{2} \left({}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \right)^\mathrm{T} {}^{\mathcal{D}}\mathbf{I}_\mathrm{p} \, {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$

G. $\ldots + \frac{1}{2} \left({}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{F}} + {}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \right)^\mathrm{T} {}^{\mathcal{N}}\mathbf{I}_\mathrm{w} \left({}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{F}} + {}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \right) + \frac{1}{2} \left({}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \right)^\mathrm{T} {}^{\mathcal{N}}\mathbf{I}_\mathrm{p} \, {}^{\mathcal{N}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}}$ ▷

Problem 5.48 This Problem is about the hoverboard of Problem 3.26 on page 95. A student intends to calculate the kinetic energy of the complete system. Her idea is to add the kinetic energy contained in the wheels to the kinetic energy of the combined board-person body, using the following equation:

$$T = \frac{1}{2} m_w |\dot{\boldsymbol{r}}_A|^2 + \frac{1}{2} \boldsymbol{\omega}_1^\mathrm{T} \mathbf{I}_w \boldsymbol{\omega}_1 + \frac{1}{2} m_w |\dot{\boldsymbol{r}}_B|^2 + \frac{1}{2} \boldsymbol{\omega}_2^\mathrm{T} \mathbf{I}_w \boldsymbol{\omega}_2 + \frac{1}{2} m_p |\dot{\boldsymbol{r}}_P|^2 + \frac{1}{2} \boldsymbol{\omega}_3^\mathrm{T} \mathbf{I}_P \boldsymbol{\omega}_3, \tag{5.95P}$$

where point P is a placeholder.

The student simplifies the person's body as a point mass in S.

With this simplification, calculate the elements of the following tensors and vectors, in order for (5.95P) to be correct:

$$ {}^{?}\dot{\boldsymbol{r}}_A \, , \; {}^{?}\dot{\boldsymbol{r}}_B \, , \; {}^{?}\dot{\boldsymbol{r}}_P \, , \; {}^{?}\boldsymbol{\omega}_1 \, , \; {}^{?}\boldsymbol{\omega}_2 \, , \; {}^{?}\boldsymbol{\omega}_3 \, , \; {}^{?}\mathbf{I}_w \, , \; {}^{?}\mathbf{I}_P \, , \tag{5.96P}$$

Provide all answers as functions of the given constants R, d, h, m_p, m_w, the generalized coordinates $C_X, C_Y, \theta, \gamma, \alpha, \beta$, and the derivatives of the generalized coordinates with respect to time. The components of all vectors and the inertia tensors may be expressed in a triad of your choice, as long as (5.95P) still holds. Indicate in which triad your components are expressed, by a left superscript replacing the "?". ▷

Problem 5.49 This Problem is about the skateboard of Problem 1.24 on page 39. We now assume that the wheels of the skateboard *can* slip. Assume that the maximum friction force exerted by the ground is proportional to the normal force exerted by the ground, that the proportionality constant is the coefficient of dry friction μ, and that friction during slip opposes the direction of movement.

We consider only the left front wheel, and simplify the wheel to be thin, so that the ground contact is a point contact. The point K on the wheel that is currently in contact with the ground has the velocity $\boldsymbol{v}_K$.

Which statement(s) is/are always true about $\boldsymbol{v}_K$ and the friction force vector $\boldsymbol{F}_f$ acting on the wheel?

A. $\boldsymbol{F}_f \cdot \boldsymbol{v}_K = 0$
B. $|\boldsymbol{F}_f| = \frac{1}{4}\mu m_\mathrm{p} g$
C. If there is no slip, the friction force is zero.
D. The friction force does no work, regardless of whether there is slip or not.
E. If $\boldsymbol{v}_K = \boldsymbol{0}$, the wheel is currently not slipping.
F. None of the above

▷

Problem 5.50 Several students calculate the magnitude v of the velocity vector $\boldsymbol{v}$ of the center of mass of a mechanical system, as a function of given variables that include a length l, a position vector $\boldsymbol{r}$, mass m, an inertia tensor $\mathbf{I}$, an angular velocity vector $\boldsymbol{\omega}$ of magnitude ω, time t, a moment vector $\boldsymbol{M}$, a scalar force F, and acceleration of gravity g.

Without knowing more about this problem, which of these (intermediate) results is/are certainly incorrect[a]?

A. $v = 3\,\mathrm{N}$
B. $v = \boldsymbol{\omega} \times \boldsymbol{r}$
C. $v = 6.4\sqrt{2Fl/m + g}$
D. $v = 3.68$
E. $v = 6.87\sqrt{2Fl/m}\,[\mathrm{m/s}]$
F. $v = 45 \cdot [v]$
G. $v = lt\frac{\boldsymbol{M}}{\mathbf{I}}$
H. $v = 8\,\mathrm{m/s}$
I. $v = 2 \cdot t\,[\mathrm{s}]$
J. $v = \omega \cdot |\boldsymbol{r}|$
K. $v = 1\,\mathrm{m/s^3} \cdot t^2$
L. $v = 1\,\mathrm{m/s} \cdot \sin(\omega t)$

▷

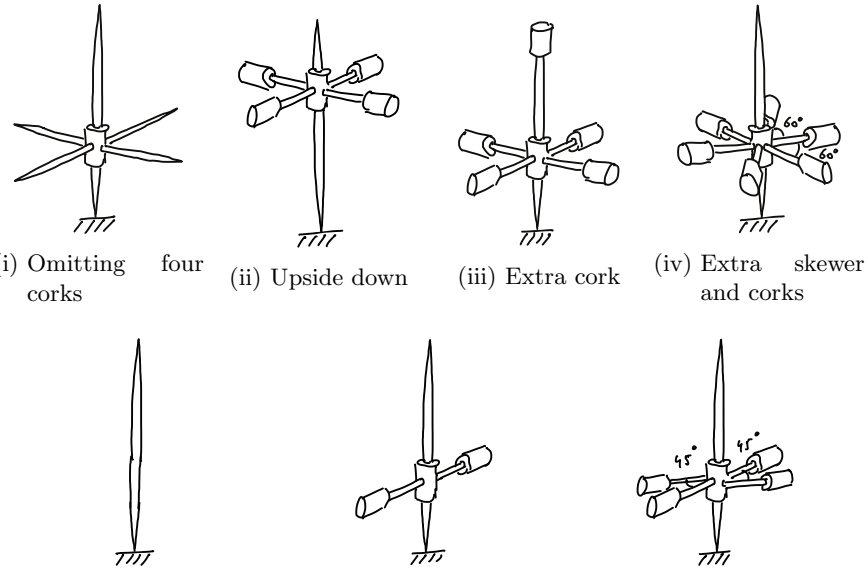
[a] *Hint*: Consult Section 1.1.4.

Problem 5.51 This Problem is about the top of corks and skewers of Problem 1.8 on page 12. The students want to modify the top to spin better. They know that [59, 36, 44] a top with principal moments of inertia $I_c > I_b \geq I_a$ with respect to point P and spinning about z will balance stably on point P if

- z is the axis of the top's largest principal moment of inertia $I_c > I_b$,
- $\omega^2 > \dfrac{4MgdI_a}{I_c^2}$ if $I_b = I_a$,
- $\omega^2 > \dfrac{Mgd}{I_c - I_b}$ if $I_b > I_a$,

where M is the top's mass, d is the distance between P and the top's center of mass, ω is the angular speed about z, and g is the gravitational constant.

Indicate which of the modifications shown in Figure 5.12 let the top balance better (so at lower ω), worse, or the same as the version in Figure 1.4.



(i) Omitting four corks (ii) Upside down (iii) Extra cork (iv) Extra skewer and corks

(v) One skewer (vi) Omit skewer 3 (vii) Skewer 2 and 3 closer

Figure 5.12: Modified top of skewers and corks.

▷

6 Newton-Euler

The Newton-Euler approach (as introduced in Chapter 2 for particles) is a fundamental method for deriving the equations of motion of rigid bodies. It combines Newton's second law of motion and Euler's second law of motion. The first step in each problem in this chapter will be drawing free-body diagrams. Also the inertia tensor from Chapter 5 will prove to be valuable again, here to calculate angular momenta of rigid bodies. Problem-specific choice of reference points and triads minimizes calculations and keeps equations compact. To this end, we will treat general and special rotating reference triads, and provide guidelines on advantageous choices for specific problems.

6.1 Newton's Second Law of Motion for Rigid Bodies

6.1.1 Newton's Second Law Using Inertial Triads

Section 2.1 stated Newton's second law of motion for a system of N particles. The same law applies to infinitely many particles that form a rigid body (So $N \to \infty$). So, the sum $\sum \boldsymbol{F}_{\text{ext}}$ of all external forces acting on a body governs its rate of change of linear momentum $\boldsymbol{p}$:

$$\sum \boldsymbol{F}_{\text{ext}} = \dot{\boldsymbol{p}}, \tag{6.1}$$

or, substituting the definition of linear momentum,

$$\sum \boldsymbol{F}_{\text{ext}} = m\ddot{\boldsymbol{r}}_C, \tag{6.2}$$

where m is the mass of the body and $\boldsymbol{r}_C$ is the position of the center of mass of the body.

◯ **Problem 6.1** A force acts on a rigid body. How does the acceleration of the body's center of mass change when the force vector acts at a different location of the body, but with unchanged magnitude and direction? ▷

In order to establish the three equations that Newton's second law provides for a rigid body in 3-D, the first step is to draw a free-body diagram (see Section 1.2.8), to inventory the external forces acting on the body.

Example 6.1 This Example is about the falling phone of Example 4.7 on page 123.

We model the phone as a rectangular block of homogeneous material and mass m. It is subjected to a gravity with field strength g in the negative Z direction.

By bringing his hand under the phone at instant t_i, the user manages to put a force $\boldsymbol{F}_C$ on point C.

Draw a free-body diagram of the phone at the instant t_i.

The components of the force $\boldsymbol{F}_C$ in the $\mathcal{B}$-triad at t_i are given as ${}^{\mathcal{B}}\boldsymbol{F}_C = (0,\, 0,\, F_C)^{\mathrm{T}}$, with magnitude F_C. Determine the acceleration vector ${}^{\mathcal{N}}\boldsymbol{a}_A$ of point A at t_i, with components expressed in the $\mathcal{N}$-triad, as functions of g, m, ω_{p}, b, d, h, H, F_C.

Exemplary solution

A FBD is drawn in Figure 6.1. All external forces are stated in the text: These are the weight, acting at the center of mass A, and the force exerted by the hand at vertex C. The FBD also includes all relevant dimensions, such that looking into the original figure is no longer needed for later moment calculations.

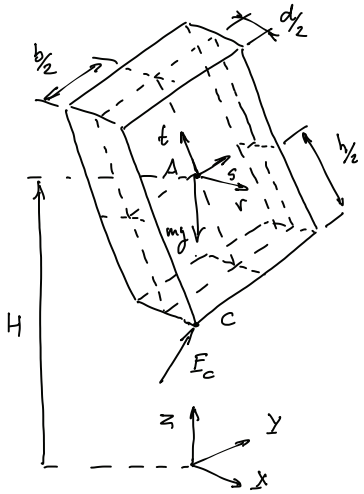


Figure 6.1: Free-body diagram of the falling smartphone.

The sum of external forces acting on the phone determines the acceleration

$\boldsymbol{a}_A$ of its center of mass A:

$$\sum {}^{\mathcal{N}}\boldsymbol{F} = m\,{}^{\mathcal{N}}\boldsymbol{a}_A. \tag{6.3E}$$

The only two external forces are drawn in the FBD, namely weight $\boldsymbol{F}_{\mathrm{g}}$ and the user's force $\boldsymbol{F}_C$. If we express all components in the inertial $\mathcal{N}$-triad, the sum of forces is

$${}^{\mathcal{N}}\boldsymbol{F}_{\mathrm{g}} + {}^{\mathcal{N}}\boldsymbol{F}_C = m\,{}^{\mathcal{N}}\boldsymbol{a}_A, \tag{6.4E}$$

with gravitational force $\boldsymbol{F}_{\mathrm{g}} = -mg\boldsymbol{n}_3$.

The force of the user's hand is given expressed in the $\mathcal{B}$-triad, so it needs to be mapped via the rotation matrix:

$${}^{\mathcal{N}}\boldsymbol{F}_C = {}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}}\,{}^{\mathcal{B}}\boldsymbol{F}_C = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}^{\mathrm{T}}\,{}^{\mathcal{B}}\boldsymbol{F}_C = \frac{F_C}{2}\begin{pmatrix}0\\-1\\\sqrt{3}\end{pmatrix}. \tag{6.5E}$$

Finally, both forces can be inserted in (6.4E), and the equation can be solved for the acceleration vector:

$$\Rightarrow \qquad {}^{\mathcal{N}}\boldsymbol{a}_A = \begin{pmatrix}0 & -\frac{F_C}{2m} & \frac{F_C}{2m}\sqrt{3} - g\end{pmatrix}^{\mathrm{T}}. \tag{6.6E}$$

Problem 6.2 This Problem is about the falling phone of Example 4.7 on page 123.

When the user's hand finally loses contact with the phone, point A is at a height H above the ground and it has a velocity vector, with components expressed in the $\mathcal{N}$-triad, of ${}^{\mathcal{N}}\boldsymbol{v}_A = (v_A,\, 0,\, 0)^{\mathrm{T}}$, with given magnitude v_A. The angular velocity is the same as at instant t_i. How much time Δt does the user have to catch the falling phone before it hits the ground (the XY-plane) flat face on its bh-side? Calculate Δt as a function of g, m, ω_{p}, v_A, b, d, h, H. ▶

6.1.2 Newton's Second Law for General Rotating Triads

Sometimes, it is convenient to calculate the acceleration $\ddot{\boldsymbol{r}}_C$ of a rigid body's center of mass using translating and rotating coordinate systems. We know how to find absolute acceleration of a point P, which in this case is the center of mass C of a rigid body, using (4.69):

$$\ddot{\boldsymbol{r}}_C = \ddot{\boldsymbol{Q}} + (\ddot{\boldsymbol{p}})_{\mathcal{F}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{p} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{p}) + 2\boldsymbol{\omega} \times (\dot{\boldsymbol{p}})_{\mathcal{F}}, \tag{6.7}$$

with the shorthand notation $\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$. Figure 4.9 on page 123 defines the meaning of the vectors used in this equation, with $\boldsymbol{r} = \boldsymbol{r}_C$.

Newton's Second Law (6.2) holds regardless of which triad is used to express vector components in, as long as all vectors (including the resulting force) have their components expressed in the same triad.

6.2 Euler's Second Law of Motion for Rigid Bodies

6.2.1 From Systems of Particles to Rigid Bodies

In Section 2.1, we also introduced Euler's second law of motion, which expands Newton's second law by a compact form to relate the sum of all external moments $\sum \boldsymbol{M}_{\text{ext}}$ acting on a system of particles. The same holds again if infinitely many particles form a rigid body, so we can also apply Euler's second law of motion to a rigid body:

$$\sum \boldsymbol{M}_{\text{ext},O} = \dot{\boldsymbol{H}}_O \text{ or } \sum \boldsymbol{M}_{\text{ext},C} = \dot{\boldsymbol{H}}_C. \tag{6.8}$$

Note that (6.8) should only be used if the reference point is a fixed point O or the center of mass C of the rigid body.

Problem 6.3 A satellite is orbiting Earth, with Earth's gravity acting on it. What can you say about the satellite's overall angular momentum with respect to its center of mass? Does it change over time? Also, what can you say about its overall angular momentum with respect to the Earth's center (considering that point fixed)? Does this change over time? Finally, what can you say about the satellite's overall angular momentum with respect to the Sun (considering the Sun fixed)? Does this change over time?

Problem 6.4 Under which condition(s) is the origin Q of a translating and rotating coordinate system an allowed choice as a point of reference for moment and angular momentum in Euler's second law?

In the following, we will show that we can find an expression to calculate angular momentum $\dot{\boldsymbol{H}}_O$ or $\dot{\boldsymbol{H}}_C$ of a rigid body without having to explicitly calculate the rate of change of angular momentum of each single particle in it.

6.2.2 Angular Momentum of a Rigid Body

To calculate angular momentum of a rigid body, we will start from the definition of angular momentum of a system of particles, as stated in Section 2.1.3. We consider the special case where the point of reference is chosen to be the center of mass C of the system of particles. Also, for a rigid body, $N \to \infty$.

We use again Figure 5.1 on page 146, and the same shorthand notations as in (5.2) for the absolute and relative position vectors of the center of mass C and a mass element $\mathrm{d}m$ at point P in the rigid body.

When using the center of mass as a reference point, we saw already in (2.18) and Problem 2.3 that we can choose either the absolute velocity of the element $\dot{\boldsymbol{r}}_P$ or the relative velocity $\dot{\boldsymbol{\rho}}$ in the calculation of angular momentum, they give the same result:

$$\boldsymbol{H}_C = \int_m \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}} \mathrm{d}m = \int_m \boldsymbol{\rho} \times \dot{\boldsymbol{r}}_P \mathrm{d}m. \tag{6.9}$$

The mass elements are part of a rigid body, so the relative movement of any element with respect to the center of mass C can be explained by a pure rotation about an axis through this point C (see (4.19)):

$$\dot{\boldsymbol{\rho}} = \boldsymbol{\omega}_\mathrm{b} \times \boldsymbol{\rho}. \tag{6.10}$$

Hereby, the angular velocity $\boldsymbol{\omega}_\mathrm{b}$ is the angular velocity of the rigid body. By substituting this in (6.9), we obtain:

$$\boldsymbol{H}_C = \int_m \boldsymbol{\rho} \times (\boldsymbol{\omega}_\mathrm{b} \times \boldsymbol{\rho}) \mathrm{d}m. \tag{6.11}$$

Using the tilde notation (see Section A.3), we can factor all variables that do not depend on the element outside of the integral:

$$\boldsymbol{H}_C = \int_m \tilde{\boldsymbol{\rho}} \cdot (-\tilde{\boldsymbol{\rho}} \boldsymbol{\omega}_\mathrm{b}) \mathrm{d}m = \underbrace{\left(\int_m \tilde{\boldsymbol{\rho}}^\mathrm{T} \tilde{\boldsymbol{\rho}} \mathrm{d}m \right)}_{\mathbf{I}_C} \cdot \boldsymbol{\omega}_\mathrm{b}. \tag{6.12}$$

We recognize the inertia tensor as found in Section 5.1.1. In this derivation, we also used very similar steps as in Section 5.1, when we derived kinetic energy of a rigid body.

This delivers the expression to calculate angular momentum of a rigid body, without having to explicitly calculate angular momentum of each single mass element in it. We only need to know the inertia tensor $\mathbf{I}_C$ and the body's (absolute) angular velocity $\boldsymbol{\omega}_\mathrm{b}$:

$$\boldsymbol{H}_C = \mathbf{I}_C \boldsymbol{\omega}_\mathrm{b}. \tag{6.13}$$

An analogous relationship applies to the case where the angular momentum is calculated about a fixed point O of the body's motion, in case such a point exists (See Problem 6.6 and Problem 6.7):

$$\boldsymbol{H}_O = \mathbf{I}_O \boldsymbol{\omega}_\mathrm{b}. \tag{6.14}$$

Example 6.2 A thin rod of mass m and length L is fixed at one of its ends in point P. We choose a local xyz-coordinate system with directions of triad $\mathcal{D}$, such that the rod's longitudinal axis is always aligned with the z-axis. The rod rotates with angular velocity ${}^{\mathcal{D}}\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^{\mathrm{T}}$, with components expressed in this local triad. Calculate the rod's angular momentum with respect to point P, with components also expressed in the local triad.

Exemplary solution

With the rod's inertia tensor,

$$ {}^{\mathcal{D}}\mathbf{I}_P = \frac{mL^2}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.15E} $$

the rod's angular momentum with respect to P can be calculated from (6.14), expressing all terms in the $\mathcal{D}$-triad:

$$ {}^{\mathcal{D}}\boldsymbol{H}_P = {}^{\mathcal{D}}\mathbf{I}_P\, {}^{\mathcal{D}}\boldsymbol{\omega} = \frac{mL^2}{3}\begin{pmatrix} \omega_1 & \omega_2 & 0 \end{pmatrix}^{\mathrm{T}}. \tag{6.16E} $$

Problem 6.5 Derive the units of angular momentum from (6.9) (see also Section 1.1.4). Does the result correspond with a derivation of units from (6.13)?

Problem 6.6 Show (6.14) from the definitions for particles.

Problem 6.7 If you know angular momentum $\boldsymbol{H}_C$ of some rigid body with respect to its center of mass C, how can you calculate this body's angular momentum $\boldsymbol{H}_O$ with respect to a fixed point O? Assume the vector from O to C is given as $\boldsymbol{r}_{C/O}$, and the absolute velocity of the center of mass is given as $\boldsymbol{v}_C$. Evaluate and check plausibility of your answer for two special cases: a) the body consists only of a single particle, b) C is itself this fixed point O.

6.2.3 Euler's Second Law for General Rotating Triads

We know from Euler's second law (6.8) that the net external moment $\sum \boldsymbol{M}_{\mathrm{ext},C}$ applied to a body is equal to the body's absolute rate of change of angular momentum $\dot{\boldsymbol{H}}_C = (\dot{\boldsymbol{H}}_C)_{\mathcal{N}}$. The derivative of (6.13), with respect to time is:

$$ \sum \boldsymbol{M}_{\mathrm{ext},C} = \dot{\boldsymbol{H}}_C = \dot{\mathbf{I}}_C\boldsymbol{\omega}_{\mathrm{b}} + \mathbf{I}_C\dot{\boldsymbol{\omega}}_{\mathrm{b}}. \tag{6.17} $$

This equation is complicated to apply immediately, because of the derivative $\dot{\mathbf{I}}_C$. Only in special cases, where $\dot{\mathbf{I}}_C$ is zero, it is convenient.

Example 6.3 A prominent special case where any relevant elements of $\mathbf{I}_C$ are constant is planar movement of a rigid body, where the dynamics can be analyzed in a 2-D projection. As the body will not change its moment of inertia in the particular plane, only this one (constant) moment of inertia is needed for calculations, and the problem is reduced to a scalar analysis.

Problem 6.8 For another special case, consider a perfect sphere of mass m and radius r that currently has an angular velocity $\boldsymbol{\omega}$ and an angular acceleration $\dot{\boldsymbol{\omega}}$. Calculate the sum of moments $\sum \boldsymbol{M}_{\text{ext},C}$ acting on the sphere with respect to its center of mass.[a]

[a] *Hint*: Note that whichever orientation the sphere assumes with respect to a triad, its inertia tensor in that triad remains identical. What does this mean for $\dot{\mathbf{I}}_C$?

In the general 3-D case, one generally avoids calculating $\dot{\mathbf{I}}_C$ directly. This is possible using the transport theorem. Often, it is convenient to calculate angular momentum in a rotating triad $\mathcal{D}$ in which the body's inertia tensor does not change. Obviously, this is always possible with a body-fixed triad.

However, in some special cases, there may also be other triads in which the inertia tensor is constant. For example, in the trivial case of a sphere, any arbitrary triad can be chosen. Or, in the case of axially symmetric bodies, any triad can be chosen where one axis direction is aligned with the body's axis of rotational symmetry (see Example 6.4 and Example 6.6).

We will now try to find a general expression for the rate of change of angular momentum using rotating triads.

To find the absolute rate of change of angular momentum, we can apply the transport theorem (with $\boldsymbol{A} \equiv \boldsymbol{H}_C$) from (4.28) on page 115 again, to obtain an alternative form of Euler's second law:

$$\sum \boldsymbol{M}_{\text{ext},C} = (\dot{\boldsymbol{H}}_C)_{\mathcal{N}} = (\dot{\boldsymbol{H}}_C)_{\mathcal{D}} + \boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \times \boldsymbol{H}_C. \tag{6.18}$$

Note that the angular velocity $\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}}$ is the angular velocity of the rotating triad, and *not* the angular velocity of the rigid body. The two can be, but do not need to be identical. Only in the case of a body-fixed triad, they are the same.

Note furthermore that $(\dot{\boldsymbol{H}}_C)_{\mathcal{D}}$ is the rate of change with respect to the rotating triad, so the relative change. Recall that the right subscript $\mathcal{D}$ has nothing to do with the left superscript $\mathcal{D}$ (which denotes in which triad directions vector components are projected).

The transport theorem can equally be applied to obtain $(\dot{\boldsymbol{H}}_O)_{\mathcal{N}}$:

$$\sum \boldsymbol{M}_{\text{ext},O} = (\dot{\boldsymbol{H}}_O)_{\mathcal{N}} = (\dot{\boldsymbol{H}}_O)_{\mathcal{D}} + \boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \times \boldsymbol{H}_O. \tag{6.19}$$

Problem 6.9 For which triad $\mathcal{D}$ is $\dot{\boldsymbol{H}} = (\dot{\boldsymbol{H}})_{\mathcal{D}}$ always true?

Problem 6.10 Explain the difference between the angular rotations $\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}}$ and $\boldsymbol{\omega}_{\text{b}}$ in this Section 6.2.3.

Example 6.4 A rotationally symmetric toy top is spinning on a horizontal surface. The top has mass m, its center of mass is at a distance l from the ground contact point O, the top's axial and transverse moments of inertia with respect to O are $I_{\text{a}O}$ and $I_{\text{t}O}$, respectively, and the gravitational acceleration constant is g. The top does not slip.

Determine the equations of motion of the top in terms of second derivatives of precession φ, nutation θ, and spin ψ.

Exemplary solution

We describe the top's motion by Euler angles type II: Precession with angle φ about the Z-axis (which points upwards), nutation with angle θ about the new x'-axis, and finally spin with angle ψ about the new $z = z''$-axis.

We name the intermediate triads $\mathcal{D}$ (performing only precession) and $\mathcal{G}$ (performing only precession and nutation), and the top-fixed triad is $\mathcal{B}$.

To solve the problem, we start with the FBD in Figure 6.2, including friction on the surface.

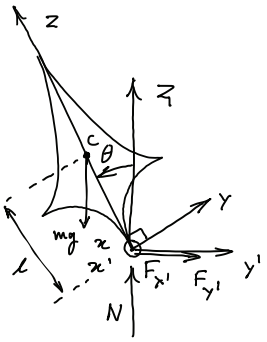


Figure 6.2: Free-body diagram of a top.

The next step would be to set up Newton's second law and Euler's second law. The reference point for Euler's second law is conveniently chosen such that there are as few unknowns in the equations as possible, which is the case if we choose the point O as reference for moments. Then, none of the interaction forces between the top and the ground create moments, the only force that creates a moment is the gravitational force. Newton's second law

could give us a relationship between external forces acting on the top and acceleration of its center of mass. The three resultant equations could be used to find the three components of the ground reaction force. However, we are not interested in the value of these forces. Therefore, we only apply Euler's second law, where the forces do not appear.

It is convenient to use a triad $\mathcal{G}$ that rotates with precession and nutation, but not with eigenrotation of the body. In this triad, the inertia tensor of the top is constant, because the body is axially symmetric. Therefore, there is no need to choose a fully body-fixed triad in this case, which would unnecessarily complicate calculations.

To populate (6.19), we need to calculate two things: The moment of all external forces that act on the top with respect to the point O, and the rate of change of angular momentum of the top, as a function of the angles, angular velocities, and angular accelerations.

We start with the left side, so the moment: Only gravity creates a moment. With the position vector $\boldsymbol{r}_C$ of the top's center of mass with respect to point O, top mass m, and the gravitational acceleration vector $\boldsymbol{g}$ acting in negative Z-direction, we obtain:

$$\sum {}^{\mathcal{G}}\boldsymbol{M}_{\text{ext},O} = {}^{\mathcal{G}}\boldsymbol{r}_C \times m\, {}^{\mathcal{G}}\boldsymbol{g} = \begin{pmatrix} 0 \\ 0 \\ l \end{pmatrix} \times mg \begin{pmatrix} 0 \\ -\sin\theta \\ -\cos\theta \end{pmatrix}. \tag{6.20E}$$

Now, we populate the right side of (6.19), which requires calculating the top's angular momentum with respect to point O. This in turn requires the inertia tensor and the angular velocity of the body.

As the top is symmetric with respect to the chosen axes of triad $\mathcal{G}$, its inertia tensor is diagonal. We call the moment of inertia about the z-axis, so the axial moment of inertia, $I_{\text{a}O}$ and the moment of inertia about any orthogonal axis $I_{\text{t}O}$, so we obtain the inertia tensor

$${}^{\mathcal{G}}\mathbf{I}_O = \begin{pmatrix} I_{\text{t}O} & 0 & 0 \\ 0 & I_{\text{t}O} & 0 \\ 0 & 0 & I_{\text{a}O} \end{pmatrix}. \tag{6.21E}$$

The top's angular velocity $\boldsymbol{\omega}_{\text{top}}$ can be assembled from three parts: Precession, nutation, and spin, about the respective axes. The vector's components in the $\mathcal{G}$-triad are:

$$\begin{aligned} {}^{\mathcal{G}}\boldsymbol{\omega}_{\text{top}} &= \dot{\varphi}\, {}^{\mathcal{G}}\hat{\boldsymbol{n}}_3 + \dot{\theta}\, {}^{\mathcal{G}}\hat{\boldsymbol{f}}_1 + \dot{\psi}\, {}^{\mathcal{G}}\hat{\boldsymbol{g}}_3 \\ &= \dot{\varphi} \begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix} + \dot{\theta} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \dot{\psi} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ \dot{\varphi}\sin\theta \\ \dot{\varphi}\cos\theta + \dot{\psi} \end{pmatrix}. \end{aligned} \tag{6.22E}$$

The triad $\mathcal{G}$ rotates less, it only performs precession and nutation movement, so its angular velocity with respect to the inertia triad is

$$ {}^{\mathcal{G}}\boldsymbol{\omega}_{\mathcal{G}/\mathcal{N}} = \begin{pmatrix} \dot{\theta} \\ \dot{\varphi}\sin\theta \\ \dot{\varphi}\cos\theta \end{pmatrix}. \tag{6.23E} $$

Now, we can calculate angular momentum as

$$ {}^{\mathcal{G}}\boldsymbol{H}_O = {}^{\mathcal{G}}\mathbf{I}_O \cdot {}^{\mathcal{G}}\boldsymbol{\omega}_{\text{top}} = \begin{pmatrix} I_{\text{t}O}\dot{\theta} \\ I_{\text{t}O}\dot{\varphi}\sin\theta \\ I_{\text{a}O}\dot{\varphi}\cos\theta + I_{\text{a}O}\dot{\psi} \end{pmatrix}. \tag{6.24E} $$

To calculate $(\dot{\boldsymbol{H}}_O)_{\mathcal{G}}$, we take the derivatives of the components of $\boldsymbol{H}_O$:

$$ {}^{\mathcal{G}}(\dot{\boldsymbol{H}}_O)_{\mathcal{G}} = \begin{pmatrix} \ddot{\theta}I_{\text{t}O} \\ \ddot{\varphi}I_{\text{t}O}\sin\theta + \dot{\varphi}\dot{\theta}I_{\text{t}O}\cos\theta \\ \ddot{\varphi}I_{\text{a}O}\cos\theta - \dot{\theta}\dot{\varphi}I_{\text{a}O}\sin\theta + I_{\text{a}O}\ddot{\psi} \end{pmatrix} \tag{6.25E} $$

Note how the transverse moment of inertia $I_{\text{t}O}$ remains constant in the rotating triad even though the triad is *not* body-fixed. It is constant due to the axial symmetry of the body. This illustrates why choosing a triad that does not spin with a body is particularly useful for axially symmetric bodies.

We insert all found terms into (6.19) to obtain three equations of motion:

$$ mgl\sin\theta = \ddot{\theta}I_{\text{t}O} + \frac{1}{2}\dot{\varphi}^2\sin(2\theta)(I_{\text{a}O} - I_{\text{t}O}) + I_{\text{a}O}\dot{\psi}\dot{\varphi}\sin\theta, \tag{6.26E} $$

$$ 0 = \ddot{\varphi}I_{\text{t}O}\sin\theta + \dot{\theta}\dot{\varphi}\cos\theta(2I_{\text{t}O} - I_{\text{a}O}) - I_{\text{a}O}\dot{\theta}\dot{\psi}, \tag{6.27E} $$

$$ 0 = \ddot{\psi} + \ddot{\varphi}\cos\theta - \dot{\theta}\dot{\varphi}\sin\theta. \tag{6.28E} $$

In order to simplify equations, we usually assume that the top spins much faster than it precesses, so $\dot{\psi} >> \dot{\varphi}$. Then, we notice that the terms occuring due to the cross product in (6.19) are much larger than those occuring to the relative derivative $(\dot{\boldsymbol{H}}_O)_{\mathcal{G}}$.

Example 6.5 This Example is about the spinning toy top of Example 6.4 on page 190. Investigate the special case where the top performs steady motion, meaning that θ, $\dot{\varphi}$, and $\dot{\psi}$ are constant. In which direction (clockwise or counter-clockwise looking from above) do you think the top will precess when you spin it with your right hand? Verify your theoretical prediction by spinning a real toy top.[a]

Exemplary solution

The term with $\dot{\varphi}^2$ is negligibly small compared to the other terms, and also $\ddot{\theta} = 0$, so only two terms remain in (6.26E). Spinning clockwise means

$\dot{\psi} < 0$. We see that for (6.26E) to hold with $\dot{\psi} < 0$, the precession also needs to be negative, because mgl is positive. So, $\dot{\varphi} < 0$, and the top will precess clockwise seen from above.

[a] *Hint*: Starting it with your right hand will make the top spin clockwise looking from above. Simplify the equation of motion for nutation of the top (6.26E) for steady motion and assuming it spins much faster than it precesses. Then, investigate signs of all variables in the remaining equation to determine the sign of precession speed.

Example 6.6 A pair of *control moment gyroscopes* is fitted within a wearable, backpack-like structure [43, 42] as shown in Figure 6.3. Its purpose is to support users with balance impairments by exchanging angular momenta during near-fall situations.

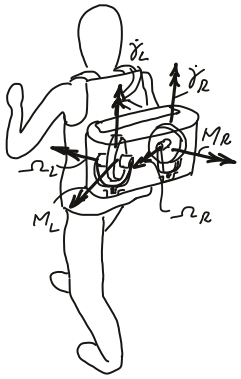


Figure 6.3: Gyroscopic backpack.

The device contains two flywheels that spin at very high angular speeds Ω_L and Ω_R. The speeds are maintained by integrated motors. The wheels are gimbaled such that additional motors can rotate them about axes that are aligned with the longitudinal (vertical) axis of the user's body and orthogonal to the spin axes. The gimbals allow changing the wheels' precession angles γ_L and γ_R with respect to the user. The flywheel spinning speeds are much higher than the gimbal motor speeds as well as angular velocity components of the user's upper body. Each flywheel has a moment of inertia J_s with respect to its spin axis.

We are interested in calculating the moment acting on the user when the motors control the angular velocities as depicted. In particular, the question is in which direction the user will be pushed.

To simplify calculations, the user's angular velocity and angular acceleration are assumed to be very small compared to the corresponding quantities of the wheels, and may therefore be neglected. We also assume that $\Omega_L = \Omega_R$

and $\dot{\gamma}_L = \dot{\gamma}_R$, and that all angular velocity components are momentarily constant.

Exemplary solution

Given the high spinning speeds, the respective angular momentum vectors $\boldsymbol{H}_L$ and $\boldsymbol{H}_{\boldsymbol{R}}$ of the wheels are always approximately aligned with the flywheel spin axes. While the gimbal motors re-orient the flywheels, they also force the associated angular momentum vectors to change their directions.

We start with the left flywheel (index L) and intend to calculate the moment needed to enforce the depicted rotation of the wheel about the gimbal axis. We make use of the fact that the flywheel is an axially symmetric body and choose a reference triad $\mathcal{L}$ that is fixed to the left gimbal and rotates with it. However, $\mathcal{L}$ does not rotate with the wheel about its spin axis. The triad's $\hat{\boldsymbol{l}}_1$ unit direction vector points in direction of the angular velocity vector of magnitude Ω_L, and the $\hat{\boldsymbol{l}}_3$ unit direction vector is aligned with the gimbal, such that it points upwards when the user's body is perfectly upright. In this triad, we can express the angular momentum vector of this wheel:

$$\boldsymbol{H}_L \approx J_s \Omega_L \hat{\boldsymbol{l}}_1. \tag{6.29E}$$

Here, we neglected all components of the wheel's angular velocity vector that are caused by $\dot{\gamma}_L$ or by angular velocity of the user, because these are all considered small in magnitude compared to the spin velocity.

Now, Euler's second law in the form (6.18) about the center of mass of the left gyroscope can be applied to calculate the required net moment $\boldsymbol{M}_L$ to enforce the movement:

$$\boldsymbol{M}_L = (\dot{\boldsymbol{H}}_L)_{\mathcal{L}} + \boldsymbol{\omega}_{\mathcal{L}/\mathcal{U}} \times \boldsymbol{H}_L = \boldsymbol{0} + \dot{\gamma}_L \hat{\boldsymbol{l}}_3 \times (J_s \Omega_L \hat{\boldsymbol{l}}_1). \tag{6.30E}$$

Here, the orientation of the user's upper body is described by the triad $\mathcal{U}$. We assume that the user moves slowly compared to the gyroscope, so we treat triad $\mathcal{U}$ as inertial. The reference point (the center of mass of the left wheel) is omitted here because the resulting moment is purely a free couple moment vector with no associated force. As both angular velocity components of the wheel are considered constant, only the gyroscopic portion, the second summand in Euler's second law, is nonzero. The vectors $\hat{\boldsymbol{l}}_3$ and $\hat{\boldsymbol{l}}_1$ are orthogonal to each other, and their cross product is the unit vector pointing in the $\hat{\boldsymbol{l}}_2$-direction. So, the gyroscopic moment results as:

$$\boldsymbol{M}_L = J_s \dot{\gamma}_L \Omega_L \hat{\boldsymbol{l}}_2. \tag{6.31E}$$

The moment vector is also depicted in the picture. We notice that this moment is not transmitted through either of the two motors, but through the bearings that support the wheel and the gimbal. The moment vector

can in fact be considerably higher than the peak moment of either of the two motors, given that it only depends on velocities.

The same procedure can be repeated for the right flywheel, where a different triad $\mathcal{R}$ is chosen that rotates with the right gimbal. The resulting two moment vectors both create a reaction moment vector on the user, which is equal to $-(\boldsymbol{M}_L + \boldsymbol{M}_R)$ and therefore points in the user's forward direction. This implies that the user, if passively reacting to this assistive moment, would be rotated towards the right.

6.2.4 Euler's Second Law for Body-Fixed Triads

Choosing a body-fixed triad $\mathcal{B} = \mathcal{D}$, we obtain a frequent special case of the equations in Section 6.2.3. In that case, the transport theorem (6.18) or (6.19) simplifies. This is because of two reasons:

- the inertia tensor $\mathbf{I}_C$ of the rigid body does not change with respect to triad $\mathcal{B}$, so $(\dot{\mathbf{I}}_C)_\mathcal{B} = \mathbf{0}$,
- angular velocity $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ of the rotating *triad* and angular velocity $\boldsymbol{\omega}_\mathrm{b}$ of the *body* are identical:

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{\omega}_\mathrm{b} = \boldsymbol{\omega}. \tag{6.32}$$

Applying the product rule of differentiation to the term $(\dot{\boldsymbol{H}}_C)_\mathcal{D}$ in (6.18) and using the above variables leads to

$$(\dot{\boldsymbol{H}}_C)_\mathcal{B} = \mathbf{I}_C \cdot (\dot{\boldsymbol{\omega}})_\mathcal{B} + (\dot{\mathbf{I}}_C)_\mathcal{B} \cdot \boldsymbol{\omega}. \tag{6.33}$$

To obtain $(\dot{\boldsymbol{\omega}})_\mathcal{B}$, we look at the transport theorem

$$(\dot{\boldsymbol{\omega}})_\mathcal{N} = (\dot{\boldsymbol{\omega}})_\mathcal{B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{\omega}. \tag{6.34}$$

As in the special case of a body-fixed triad, $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{\omega}$, we see that the second term is just the cross product of two identical vectors. This gives the zero vector. Therefore, in a body-fixed triad, we always have

$$(\dot{\boldsymbol{\omega}})_\mathcal{B} = (\dot{\boldsymbol{\omega}})_\mathcal{N}. \tag{6.35}$$

That means that the absolute angular velocity vector of a rigid body changes in the same way with respect to the body-fixed triad as with respect to the inertial triad.

With this, we see that in the case of a body-fixed triad, we can simplify (6.33) to

$$(\dot{\boldsymbol{H}}_C)_\mathcal{B} = \mathbf{I}_C \dot{\boldsymbol{\omega}}. \tag{6.36}$$

Substituting this in (6.18) we obtain:

$$\sum \boldsymbol{M}_{\text{ext},C} = \mathbf{I}_C \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_C \boldsymbol{\omega}). \tag{6.37}$$

In the same way, we can also derive the equation for a fixed reference point O:

$$\sum \boldsymbol{M}_{\text{ext},O} = \mathbf{I}_O \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_O \boldsymbol{\omega}). \tag{6.38}$$

Remember that both equations only hold in the special case where $\mathcal{B}$ is fixed to the body.

Example 6.7 This Example is about the washing machine with two masses of Example 5.3 on page 153. The motor is controlled such that the drum is spinning with *constant* angular velocity ω_x about the x-axis. Ignore the influence of gravity.

The axle of the drum is suspended by two spherical bearings at points P_1 and P_2, with respective position vectors $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ in the xyz coordinate system. Spherical bearings allow movement in all rotational directions. The bearings are arranged symmetrically to the origin O of the xyz coordinate system, such that each is located at the distance $2R$ from O. The bearing at P_2 is a floating bearing, which means it also allows motion in the x direction. Assume the reaction forces of both bearings have their points of application on the x-axis.

The inertia tensor ${}^{\mathcal{B}}\mathbf{I}_O$ of the combined body around the origin of the body-fixed triad is now given as ${}^{\mathcal{B}}\mathbf{I}_O = J \begin{pmatrix} 20 & 4 & -4 \\ 4 & 34 & 0 \\ -4 & 0 & 34 \end{pmatrix}$, where J is a constant. The vector from the origin O to the combined body's center of mass is given as ${}^{\mathcal{B}}\boldsymbol{r}_C = (r_x,\, r_y,\, r_z)^{\mathrm{T}}$ in the xyz coordinate system. The acceleration vector of the center of mass of such a body has already been calculated in Example 4.9 as ${}^{\mathcal{B}}\ddot{\boldsymbol{r}}_C = (0,\, -\omega_x^2 r_y,\, -\omega_x^2 r_z)^{\mathrm{T}}$.

Calculate the reaction force vector ${}^{\mathcal{B}}\boldsymbol{F}_1$ of the bearing at P_1, with components expressed in the body-fixed triad $\mathcal{B}$. Express its components as functions of the parameters M, R, J, the components r_x, r_y, r_z of ${}^{\mathcal{B}}\boldsymbol{r}_C$, and the constant angular velocity ω_x.

Start with a FBD of the combined body, from a perspective that shows the projection on the xz-plane (Remember there is no gravity). Indicate vector components pointing into the plane by $\otimes$ and those pointing out by $\odot$.

Exemplary solution

We use the FBD in Figure 6.4, the inertia tensor ${}^{\mathcal{B}}\mathbf{I}_O$ provided above, the acceleration $\ddot{\boldsymbol{r}}_C$ provided above, Newton's second law of motion, and Euler's second law of motion around the origin O of the xyz coordinate system to calculate the requested force vector. In drawing the FBD, we assumed that

the motor exerts a pure moment of magnitude M_m in x-direction.

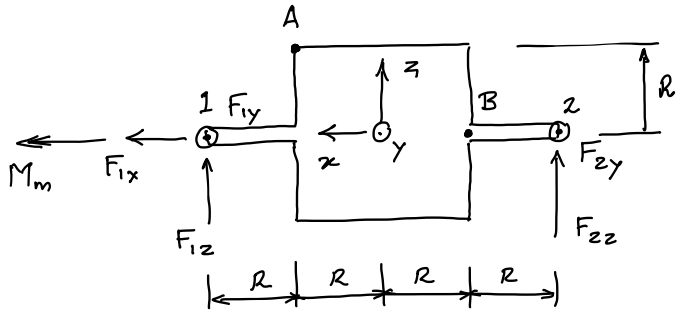


Figure 6.4: Free-body diagram for the washing machine.

Newton's second law provides three equations:

$$\sum \boldsymbol{F} = M\ddot{\boldsymbol{r}}_C = \boldsymbol{F}_1 + \boldsymbol{F}_2. \tag{6.39E}$$

Euler's second law provides the remaining three equations:

$$\sum \boldsymbol{M}_O = \dot{\boldsymbol{H}}_O = \boldsymbol{p}_1 \times \boldsymbol{F}_1 + \boldsymbol{p}_2 \times \boldsymbol{F}_2 + M_\mathrm{m}\hat{\boldsymbol{b}}_1, \tag{6.40E}$$

where angular momentum of the combined body is

$$\boldsymbol{H}_O = \mathbf{I}_O\boldsymbol{\omega}. \tag{6.41E}$$

Its rate of change is calculated from the transport theorem as:

$$\dot{\boldsymbol{H}}_O = (\dot{\boldsymbol{H}}_O)_\mathcal{B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{H}_O \tag{6.42E}$$

$$= \mathbf{I}_O\dot{\boldsymbol{\omega}} + \cancelto{0}{(\dot{\mathbf{I}}_O)_\mathcal{B}\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_O\boldsymbol{\omega}). \tag{6.43E}$$

Note that $(\dot{\mathbf{I}}_O)_\mathcal{B}$ is zero because the triad is body-fixed. As the angular velocity is constant, we obtain:

$$\dot{\boldsymbol{H}}_O = \boldsymbol{\omega} \times (\mathbf{I}_O\boldsymbol{\omega}) \tag{6.44E}$$

Substituting the given values in Newton's second law (6.39E), with all components expressed in the $\mathcal{B}$-triad, results in:

$$\begin{pmatrix} F_{1x} \\ F_{1y} \\ F_{1z} \end{pmatrix} + \begin{pmatrix} 0 \\ F_{2y} \\ F_{2z} \end{pmatrix} = M\omega_x^2 \begin{pmatrix} 0 \\ -r_y \\ -r_z \end{pmatrix}. \tag{6.45E}$$

Angular momentum is obtained from (6.41E):

$$ {}^{\mathcal{B}}\boldsymbol{H}_O = J \begin{pmatrix} 20 & 4 & -4 \\ 4 & 34 & 0 \\ -4 & 0 & 34 \end{pmatrix} \cdot \begin{pmatrix} \omega_x \\ 0 \\ 0 \end{pmatrix} = J\omega_x \begin{pmatrix} 20 \\ 4 \\ -4 \end{pmatrix}. \tag{6.46E} $$

Filling the values in for (6.40E) and (6.44E), Euler's second law can be completed, also in the $\mathcal{B}$-triad:

$$ \begin{pmatrix} M_{\mathrm{m}} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2R \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} F_{1x} \\ F_{1y} \\ F_{1z} \end{pmatrix} + \begin{pmatrix} -2R \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ F_{2y} \\ F_{2z} \end{pmatrix} = \begin{pmatrix} \omega_x \\ 0 \\ 0 \end{pmatrix} \times J\omega_x \begin{pmatrix} 20 \\ 4 \\ -4 \end{pmatrix}. \tag{6.47E} $$

Simplification of the six Newton-Euler equations leads to:

$$ F_{1x} = 0 \tag{6.48E} $$

$$ F_{1y} + F_{2y} = -M\omega_x^2 r_y \tag{6.49E} $$

$$ F_{1z} + F_{2z} = -M\omega_x^2 r_z \tag{6.50E} $$

$$ M_{\mathrm{m}} = 0 \tag{6.51E} $$

$$ -2RF_{1z} + 2RF_{2z} = 4J\omega_x^2 \tag{6.52E} $$

$$ 2RF_{1y} - 2RF_{2y} = 4J\omega_x^2 \tag{6.53E} $$

Finally solving this set of equations, we obtain the following expressions for the components of the force vector $\boldsymbol{F}_1$ of the left bearing, with directions as defined in the FBD:

$$ F_{1x} = 0, \tag{6.54E} $$

$$ F_{1y} = \omega_x^2(-\frac{Mr_y}{2} + \frac{J}{R}), \tag{6.55E} $$

$$ F_{1z} = \omega_x^2(-\frac{Mr_z}{2} - \frac{J}{R}). \tag{6.56E} $$

6.2.5 Euler Equations

In the previous section, the triad $\mathcal{B}$ was chosen to be body-fixed. If in addition, we choose it such that it always coincides with the principal axes of inertia of the body, we can further simplify the equations. (See Section 5.2 on how to make triad $\mathcal{B}$ coincide with the principal axes of inertia of the body).

With this choice, the inertia tensor is diagonal, so all off-diagonal elements are zero. Then, the vector equations (6.37) or (6.38) can be

written out to three short scalar equations:

$$\sum M_x = I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\omega_y\omega_z, \tag{6.57}$$

$$\sum M_y = I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_z\omega_x, \tag{6.58}$$

$$\sum M_z = I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\omega_x\omega_y. \tag{6.59}$$

Here, the vectors are split into their components in the rotating triad:

$$\sum {}^{\mathcal{B}}\boldsymbol{M}_{\text{ext},C} = \begin{pmatrix}\sum M_x & \sum M_y & \sum M_z\end{pmatrix}^{\mathrm{T}} \tag{6.60}$$

or

$$\sum {}^{\mathcal{B}}\boldsymbol{M}_{\text{ext},O} = \begin{pmatrix}\sum M_x & \sum M_y & \sum M_z\end{pmatrix}^{\mathrm{T}}, \tag{6.61}$$

and

$${}^{\mathcal{B}}\boldsymbol{\omega} = \begin{pmatrix}\omega_x & \omega_y & \omega_z\end{pmatrix}^{\mathrm{T}}. \tag{6.62}$$

Remember that the Euler equations are only valid if:

1. The reference triad is body-fixed
2. The inertia tensor is diagonal in this triad
3. The reference point for $\boldsymbol{H}$ and $\boldsymbol{M}$ is the center of mass C or a fixed point O.

Example 6.8 This Example is about the suspended model plane of Problem 3.23 on page 92. The plane's mass distribution is symmetric about the body-fixed axes, and its mass moments of inertia about these body-fixed axes are given as I_{xx}, I_{yy}, I_{zz}.

The current orientation of the plane is given by the rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$. The plane's current angular velocity vector, with shorthand notation $\boldsymbol{\omega} := \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$, has the components ${}^{\mathcal{B}}\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)^{\mathrm{T}}$.

The spring force acting on the plane is given by the vector ${}^{\mathcal{N}}\boldsymbol{F}_{\mathrm{s}}$ (with components expressed in the $\mathcal{N}$-triad).

Apply the Newton-Euler equations to calculate the angular acceleration vector ${}^{\mathcal{B}}\dot{\boldsymbol{\omega}} = (\dot{\omega}_x, \dot{\omega}_y, \dot{\omega}_z)^{\mathrm{T}}$, and the acceleration vector ${}^{\mathcal{N}}\ddot{\boldsymbol{r}}_C$ of the center of mass of the plane, as functions of the given variables and constants.

Exemplary solution

The FBD, as in Figure 6.5, shows that there are two external force vectors: The spring force $\boldsymbol{F}_{\mathrm{s}}$ and gravity $mg\hat{\boldsymbol{n}}_3$.

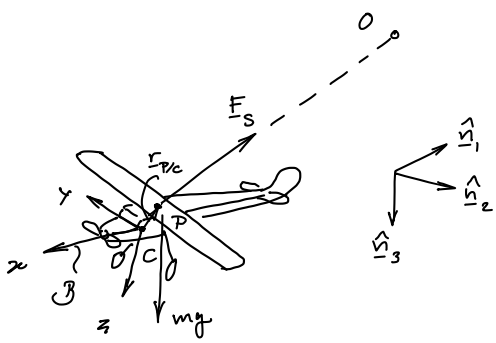


Figure 6.5: FBD of the model plane

Newton's second law of motion directly gives us the acceleration vector:

$$ {}^{\mathcal{N}}\ddot{\boldsymbol{r}}_C = \frac{1}{m}\left({}^{\mathcal{N}}\boldsymbol{F}_\mathrm{s} + mg\,{}^{\mathcal{N}}\hat{\boldsymbol{n}}_3\right) = \frac{1}{m}\,{}^{\mathcal{N}}\boldsymbol{F}_\mathrm{s} + \begin{pmatrix}0 & 0 & g\end{pmatrix}^\mathrm{T}. \tag{6.63E} $$

We can apply Euler's second law of motion in the simple form of the Euler equations in the $\mathcal{B}$-triad, because this triad is body-fixed and also aligned with the principal moments of inertia. As reference point, we can only use the center of mass here.

From the FBD, we see that gravity does not create a moment about C, only the spring force does. There are no further external moments. So, the sum of moments is

$$ \sum \boldsymbol{M}_C = \boldsymbol{r}_{P/C} \times \boldsymbol{F}_\mathrm{s}. \tag{6.64E} $$

The Euler equations (6.57)-(6.59) require the moment vector in $\mathcal{B}$. As the force vector is provided in $\mathcal{N}$ (in contrast to the local coordinates of point P), we have to apply a rotation matrix to calculate the sum of moments:

$$ \sum {}^{\mathcal{B}}\boldsymbol{M}_C = {}^{\mathcal{B}}\boldsymbol{r}_{P/C} \times ({}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}\,{}^{\mathcal{N}}\boldsymbol{F}_\mathrm{s}). \tag{6.65E} $$

With the shorthand definition of $\sum {}^{\mathcal{B}}\boldsymbol{M}_C =: (M_x, M_y, M_z)^\mathrm{T}$, we can finally solve the Euler equations for the plane's angular acceleration components:

$$ \dot{\omega}_x = \frac{1}{I_{xx}}(M_x - (I_{zz} - I_{yy})\omega_y\omega_z), \tag{6.66E} $$

$$ \dot{\omega}_y = \frac{1}{I_{yy}}(M_y - (I_{xx} - I_{zz})\omega_z\omega_x), \tag{6.67E} $$

$$ \dot{\omega}_z = \frac{1}{I_{zz}}(M_z - (I_{yy} - I_{xx})\omega_x\omega_y). \tag{6.68E} $$

6.3 Choosing Triads and Reference Points

When solving problems, it is important to choose a convenient coordinate system and reference points to simplify calculations. From the theory and examples in the previous sections, we can take some guidelines that often help choosing a good coordinate triad and reference point in diverse special cases:

- If one can easily determine the principal axes of inertia of a rigid body (for example because the body is symmetric), it is often convenient to choose a body-fixed triad where the axes coincide with these principal axes of inertia of the body. Then, one can directly apply the Euler equations (6.57), (6.58), and (6.59).
- If a body rotates about one, constant axis, it is sometimes convenient to choose a body-fixed triad where one axis is aligned with this rotation axis, and a reference point O on this axis. Note that the axis of rotation is not necessarily a principal axis of inertia. Consider for example a washing machine: If the rotor of the washing machine is *unbalanced* (the machine being loaded in some asymmetric way), the axis of rotation might not be the principal axis of inertia of the rotor. Therefore, we cannot apply the Euler equations, but we can apply Euler's second law in the form (6.38), where O is some arbitrary point on the axis.
- If a body is axially symmetric, it is often convenient to choose a triad $\mathcal{D}$ that might not be fully body-fixed, but where one axis is always aligned with the axis of rotational symmetry (the spin axis). The reason is that the inertia tensor in any such triad stays constant, even though the triad does not fully rotate with the body. A suitable choice to describe orientation of axially symmetric bodies is often precession-nutation-spin, with the body's local z-axis aligned with the symmetry axis. This choice is particularly useful if the body spins about its symmetry axis very fast, and there is no need to track the absolute orientation of the body, but only precession and nutation angle. See Example 6.4 or other examples like flywheels or *control moment gyroscopes*, as used to control attitude of spacecrafts (See [60] for more information on such systems). In those cases, we use the more general forms (6.18) or (6.19), but we make use of the fact that the inertia tensor does not change, so $(\dot{\boldsymbol{H}}_C)_{\mathcal{D}} = \mathbf{I}_C \dot{\boldsymbol{\omega}}_{\mathrm{b}}$.
- It is not necessary to formulate Newton's and Euler's second laws

in the same triads. Often, we calculate the acceleration of the CoM with components expressed in an inertial triad, while choosing rotating triads for Euler's second law.
- If a body is constrained to rotate about an axis that always passes through a fixed point, often this is a good point to choose as a reference O. Then, the constraint forces often contribute less to the moment about O, and it is more likely that Newton's second law and Euler's second law are decoupled.
- If there is no obvious fixed reference point (for example when considering an object in space), often the center of mass is a good choice as a reference point for moments and angular momenta.

In fact, there are often multiple good choices for triads and reference points. Therefore, one can try and do each problem with different choices, also providing the possibility to verify the obtained answers.

◯ **Problem 6.11** A student tries to derive the equations of motion of a rigid body. She knows that she can use the Newton-Euler approach, but she is unsure in which form to apply Euler's second law. She is considering two forms of the law, but looking for the underlying assumptions for each one:

Form 1:

$$\sum \boldsymbol{M}_{\mathrm{ext}A} = (\dot{\boldsymbol{H}}_B)_{\mathcal{G}_1} + \boldsymbol{\omega}_{\mathcal{G}_2/\mathcal{G}_3} \times \boldsymbol{H}_B, \text{ with } \boldsymbol{H}_B = \mathbf{I}_B \boldsymbol{\omega}_{\mathcal{G}_4/\mathcal{G}_5} \tag{6.69P}$$

Form 2:

$$\sum \boldsymbol{M}_{\mathrm{ext}A} = \mathbf{I}_D \dot{\boldsymbol{\omega}}_{\mathcal{G}_6/\mathcal{G}_7} + \boldsymbol{\omega}_{\mathcal{G}_8/\mathcal{G}_9} \times (\mathbf{I}_E \boldsymbol{\omega}_{\mathcal{G}_{10}/\mathcal{G}_{11}}) \tag{6.70P}$$

Here, $\sum \boldsymbol{M}_{\mathrm{ext}A}$ is the sum of external moments acting on the rigid body with respect to a point A, each $\boldsymbol{\omega}_{\mathcal{G}_\mathrm{i}/\mathcal{G}_\mathrm{j}}$ is the angular velocity vector of a triad $\mathcal{G}_\mathrm{i}$ with respect to a triad $\mathcal{G}_\mathrm{j}$, and each $\mathbf{I}_P$ is the inertia tensor of the body with respect to a point P.

State all relationships between the triads $\mathcal{G}_1$ to $\mathcal{G}_{11}$, and between the points A, B, D, E, and the conditions that they and/or the rigid body must meet to make sure each of the two stated forms of Euler's second law is always correct. ▶

6.4 Summary

Newton's and Euler's second laws of motion provide six equations that allow finding the relationships between linear and angular accelerations, linear and angular velocities, and position and orientation of a rigid body in response to external forces and moments.

To apply Euler's second law, we need the angular momentum of a rigid body, which is found using the body's angular velocity $\boldsymbol{\omega}_\mathrm{b}$ and the inertia tensor with respect to the center of mass C or a fixed point O on the body:

$$\boldsymbol{H}_C = \mathbf{I}_C \boldsymbol{\omega}_\mathrm{b} \text{ or } \boldsymbol{H}_O = \mathbf{I}_O \boldsymbol{\omega}_\mathrm{b}. \tag{6.71}$$

In many cases, it is easier to perform the calculations in rotating triads. Then, the transport theorem is used:

$$\sum \boldsymbol{M}_{\mathrm{ext},C} = (\dot{\boldsymbol{H}}_C)_\mathcal{D} + \boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \times \boldsymbol{H}_C \tag{6.72}$$

or

$$\sum \boldsymbol{M}_{\mathrm{ext},O} = (\dot{\boldsymbol{H}}_O)_\mathcal{D} + \boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \times \boldsymbol{H}_O. \tag{6.73}$$

It is important to remember that the angular velocity $\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}}$ here is the angular velocity of the rotating triad, and NOT the angular velocity of the rigid body.

However, often the equations are simplest by choosing a body-fixed triad. In that case, $\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} = \boldsymbol{\omega}_\mathrm{b} = \boldsymbol{\omega}$, so we obtain:

$$\sum \boldsymbol{M}_{\mathrm{ext},C} = \mathbf{I}_C \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_C \boldsymbol{\omega}) \tag{6.74}$$

or

$$\sum \boldsymbol{M}_{\mathrm{ext},O} = \mathbf{I}_O \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_O \boldsymbol{\omega}). \tag{6.75}$$

In the case this rotating triad is also aligned with the principal axes of inertia of the body, off-diagonal terms in the inertia tensor are zero, and we obtain the set of Euler equations.

6.5 Problems

6.5.1 Guided Problems

Problem 6.12 Our goal is to derive the equations of motion of the tippe top from Problem 1.16 on page 32 and Problem 4.27 on page 134, .

Type II Euler angles (precession φ, nutation θ, spin ψ) specify the orientation of the tippe top.

The spherical bottom has the center point S and radius R. The tippe top's center of mass C is located at a given distance d below S when all Euler angles are zero. The tippe top has given mass m and moments of inertia I_a about the axial symmetry axis and I_t about any transverse axis through C. The point on the tippe top that is currently on the ground is K.

We define again an inertial coordinate system XYZ with origin O, letting the Z-axis point up.

A local coordinate system xyz has the origin S and is fixed to the tippe top. It is associated with a triad $\mathcal{B}$.

Another triad $\mathcal{G}$ rotates with the first two Euler angles with respect to the inertial triad $\mathcal{N}$. So, the top spins with an angular speed $\dot{\psi}$ *relative* to triad $\mathcal{G}$, about the local z-axis.

a. Revise your free-body diagrams from Problem 1.16 on page 32, using any new insights you have gained since then. In each orthographic projection, include all of the above named axes and triad directions that are contained in the plane or orthogonal to it, and any further intermediate triads you may need to define. Make sure you include a projection onto the $\mathcal{G}$-triad directions. Also include all points S, K, C, O.
b. Revise your kinematic diagrams from Problem 4.27 on page 134, using any new insights you have gained since then. In each orthographic projection, include all axes and triad directions that are contained in the plane or orthogonal to it. Also include all points S, K, C, O.
c. Explain which of the points S, K, C, O could be used as possible reference points D to calculate angular momentum $\boldsymbol{H}_D$ of the tippe top using the equation $\boldsymbol{H}_D = \mathbf{I}_D \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$, where $\mathbf{I}_D$ is the respective inertia tensor.
d. Explain which of the points S, K, C, O could be used as possible reference points A to apply Euler's second law using the equation $\dot{\boldsymbol{H}}_A = \boldsymbol{M}_A$. Also state which moments need to be considered in $\boldsymbol{M}_A$, internal and/or external ones?
e. Calculate the angular momentum vector ${}^{\mathcal{G}}\boldsymbol{H}_C$ of the tippe top, expressed in the $\mathcal{G}$-triad, as a function of the given constants, of the angular velocity ${}^{\mathcal{G}}\boldsymbol{\omega}_{\text{top}} = {}^{\mathcal{G}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ of the top, and of the angular velocity ${}^{\mathcal{G}}\boldsymbol{\omega}_{\mathcal{G}/\mathcal{N}}$ of triad $\mathcal{G}$.
f. Apply the transport theorem to derive the rate of change of angular momentum ${}^{\mathcal{G}}\dot{\boldsymbol{H}}_C$ of the top, expressed in the $\mathcal{G}$-triad, as a function of ${}^{\mathcal{G}}\boldsymbol{H}_C$, ${}^{\mathcal{G}}(\dot{\boldsymbol{H}}_C)_{\mathcal{G}}$, and ${}^{\mathcal{G}}\boldsymbol{\omega}_{\mathcal{G}/\mathcal{N}}$.
g. We assume that the top is slipping. The components of a friction force vector should already be included in your FBDs. Assume that the friction force is proportional to the normal force exerted by the ground, that the proportionality constant is the coefficient of dry friction μ, and that friction opposes the direction of movement.
Calculate the friction force vector $\boldsymbol{F}_f$ as a function of μ, the velocity vector $\boldsymbol{v}_K$ of point K, and any additional variables or constants you defined in your FBDs and kinematic diagrams.
Also explain
   - why the assumption that the top is slipping is important for the correct calculation of friction force,
   - what the value of friction would be without slip,

- and how the assumption could be checked later, after simulation.

h. Revise and re-use your answers to homework of weeks 1 and 4, in order to calculate the needed variables on the left and right side of Euler's and Newton's second laws.

   - Start by revising your answer to homework problem 1c., such that you can calculate the components of the moment vector ${}^{\mathcal{G}}\boldsymbol{M}_C$, as needed to apply Euler's second law.
   - Then, revise your answer to homework 4 to calculate the angular velocity ${}^{\mathcal{G}}\boldsymbol{\omega}_{\text{top}}$ of the tippe top and the angular velocity ${}^{\mathcal{G}}\boldsymbol{\omega}_{\mathcal{G}/\mathcal{N}}$ of triad $\mathcal{G}$. Note that you need these vectors to calculate the rate of change of angular momentum.
   - Revise your answer to homework 4 to calculate the components of ${}^{\mathcal{G}}\boldsymbol{v}_K$, as needed for the friction force.
   - Finally, revise your answer to homework 4 to calculate the acceleration vector of the center of mass of the tippe top.

   Calculate all above *kinematic* variables as functions of the Euler angles, their first and second time derivatives, as well as a minimum number of further variables that you used to define the current kinematic state of the tippe top. Calculate all the above *kinetic* variables as functions of the Euler angles and further variables you defined in your FBDs.

i. Apply Newton's and Euler's second laws to derive the equations of motion of the top, which means: Derive equations that contain only the Euler angles, their first and second derivatives, the given constants, and a minimum number of further scalar variables you defined in your drawings.

j. Count the number of scalar equations that you obtained from Newton's and Euler's second laws. Then, count the number of scalar unknowns you would have to solve for during each step of numerical integration. List all these unknowns. [a]

k. Throughout all parts of this problem: Design at least five meaningful, independent plausibility checks of your choice to apply to final or intermediate results of your calculations. For example, check plausibility of your equations of motion for special cases.

[a] *Hint*: Remember that in numerical integration, the current kinematic state of the system is known, and we need to solve for accelerations and any unknown forces.

Problem 6.13 This is about the static and dynamic *balance* of the wheel of a car. The wheel, see Figure 6.6, rotates at the fixed center O about the inertial Z-axis, with a constant angular speed n. The wheel-fixed z-axis is aligned with the Z-direction.

The wheel is in static unbalance, such that the center of mass of the wheel has an offset $e = 0.2\,\text{mm}$ from the center of rotation O in the wheel-fixed y-direction.

The wheel has a mass $M = 11\,\text{kg}$, and its mass moments of inertia about the body-fixed xyz axes are, with respect to O: $I_{zz} = 0.6\,\text{kg}\,\text{m}^2$ and $I_{xx} = I_{yy} = 0.4\,\text{kg}\,\text{m}^2$. All products of inertia of the wheel are zero.

The tire on the wheel is of type 205/55R16. The "R" means that the tire has radial ply, and the numbers mean that the rim diameter is $d = 16\,\text{inches}$, the width of the tire is $w = 205\,\text{mm}$, and the sidewall height is $h = 205 \cdot 0.55\,\text{mm}$, which defines the tire outer radius $R = h + d/2$.

Now, in order to statically balance the wheel, a small mass m is placed on the rim at point P.

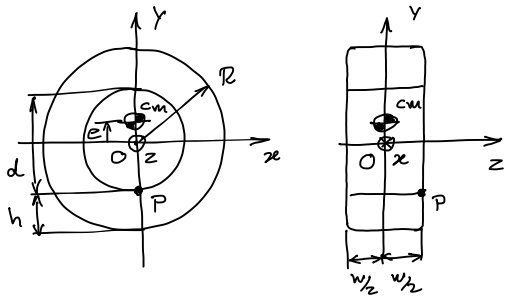


Figure 6.6: Car wheel in static unbalance, and point mass to statically balance the wheel.

Take care of units whenever you work with numerical values, and note that rpm is not a proper SI unit.

a. Determine the value for the mass m in P such that the wheel is statically balanced about the z-axis. After placing this mass, is the wheel also dynamically balanced?
b. Determine the angular momentum vector ${}^{\mathcal{B}}\boldsymbol{H}_O$ of the wheel with respect to the center of rotation O expressed in a body-fixed triad defined by xyz, where the static balance mass m is added in P, if the wheel rotates with a constant rotational frequency of $n = 1000\,\text{rpm}$.
c. Determine for the statically balanced wheel the torque components ${}^{\mathcal{N}}\boldsymbol{M}_x$ and ${}^{\mathcal{N}}\boldsymbol{M}_y$ which have to be supplied by the bearings in the axle support about the x- and y-axis, respectively, if the wheel rotates with a constant $n = 1000\,\text{rpm}$.

Problem 6.14 A horizontally oriented tube (Figure 6.7) rotates about the vertical Z-axis of an inertial coordinate system XYZ, with an angular velocity of magnitude ω. To this tube, a tube-fixed coordinate system uvw

is attached, associated with a triad $\mathcal{G}$. The Z- and the w-axes coincide and the u-axis lies in the longitudinal direction of the tube.

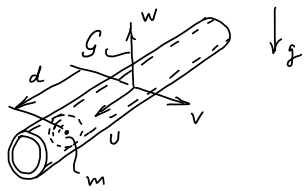


Figure 6.7: A rotating tube containing a particle.

A particle of mass m is present inside the tube. The position of this particle with respect to the tube's center of mass is described by the position vector ${}^{\mathcal{G}}\boldsymbol{\rho} = d \cdot {}^{\mathcal{T}}\hat{\boldsymbol{g}}_1$.

a. Define and draw an inertial coordinate system XYZ of which the Z-axis always coincides with w and the origin O is also in the center of mass of the tube. Assume this coordinate system is associated with a triad $\mathcal{N}$. Then, determine the rotation matrix ${}^{\mathcal{G}}\mathbf{C}_{\mathcal{N}}$.
b. Draw one FBD of the particle and also one FBD of the tube, as projections into the uw-plane. For the tube, assume it is supported by a suitable bearing structure and a motor enforces the rotation. Draw only force vectors with scalar labels, which means show all nonzero components along the axes directions.
c. Calculate the resultant force vector ${}^{\mathcal{G}}\boldsymbol{F}$ on the particle using Newton's Second Law and the body-fixed uvw system as a function of the parameters.
d. Compare your results of the two previous parts and show that these given values for the several parameters are a physically plausible combination: $m = 1\,\text{kg}$, $\omega = 3\,\text{s}^{-1}$, $\dot{\omega} = 0.5\,\text{s}^{-2}$, $d = -0.5\,\text{m}$, $\dot{d} = -1\,\text{m/s}$, $\ddot{d} = -0.2\,\text{m/s}^2$.[a]
e. Determine the position vector of the particle in your inertial coordinate system and apply Newton's Second Law again to determine the force components in the $\mathcal{N}$-triad. Use the rotation matrix you found in part a. to determine again the force components in the $\mathcal{B}$-triad.
f. In MATLAB$^{\text{TM}}$ or another program of your liking, create a function to calculate $\ddot{d}$ and the force components acting on the particle, expressed in the $\mathcal{G}$-triad. Assume there is only viscous friction with constant μ. The function should take three input arguments: A scalar d, a scalar $\dot{d}$, and a structure called "par", in which all system parameters are given. To create such a structure in MATLAB$^{\text{TM}}$, type:

```
1        % Here, parameters are defined in a structure called 'par':
2        par.m = 1; % tube mass. [m] = kg
```

```
3          par.omega = 3; % tube angular rate. [omega] = rad/s
4          par.domega = .5; % tube angular acceleration. [domega] = rad/s^2
```

Apply your function to first re-check the list of values given above. Then, change the angular rate and the angular acceleration and observe the change in outputs. Are the results congruent with your intuition?

g. Calculate angular momentum of the particle with respect to the origin O and apply Euler's Second Law to calculate the force components acting on the particle. Try this using both coordinate systems and compare your results to those found above.

[a] *Hint*: Show this by looking at the force components you defined in your particle FBD.

Problem 6.15 A person holds a *fidget spinner* of mass m loosely between two fingers (see Figure 6.8), in such a way that it can rotate freely about all axes, but the center of mass always remains stationary. We simplify the entire spinner as a massless structure connecting three particles of mass m at a distance r from the center.

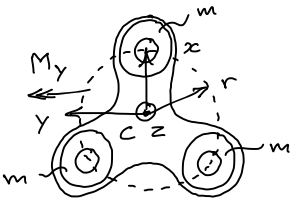


Figure 6.8: Fidget spinner.

We define a body-fixed coordinate system xyz with directions of triad $\mathcal{G}$ and origin in the center of mass C.

a. Calculate the spinner's inertia tensor in the $\mathcal{G}$-triad, with respect to C.
b. The spinner is not spinning about z. The person applies a moment M_y and has already enforced a rotation about the y-axis of magnitude ω_y. How will the instantaneous axis of rotation of the body change in the first instant? Apply Euler's second law in the body-fixed triad, to determine angular acceleration $\dot{\boldsymbol{\omega}}_{\mathrm{b}}$ of the body.
c. Now, the spinner is spinning about its z-axis with constant angular velocity of magnitude Ω. Use the inertia tensor to calculate the angular momentum vector of the spinner with respect to its center of mass, with components expressed in the $\mathcal{G}$-triad. Verify your calculation by calculating the sum of angular momenta of the individual particles.

d. If the spinner is spinning about its z-axis with constant angular rate Ω and the person enforces a rotation about the y-axis again of ω_y, how will the instantaneous axis of rotation of the body change in the first instant? Use a real fidget spinner to check signs of your answer.

e. Re-do the last part, but now use another triad $\mathcal{F}$. It rotates together with the spinner about the y- and x-axes, but it does not spin with the spinner about z. You should obtain the same result.[a]

f. Create a function in MATLAB$^{\text{TM}}$ or another program of your choice that outputs the angular acceleration vector in the $\mathcal{G}$-triad, while taking as inputs the current angular velocity vector in the $\mathcal{G}$-triad, the vector of external applied moments expressed in the same triad, and a parameter structure that contains the geometric and mass properties of the spinner.

g. Re-do the problem, but now model three thin rings of radius $r/3$ instead of particles, as depicted in the figure. Again use MATLAB$^{\text{TM}}$ or another program of your choice.

[a] *Hint*: Apply the transport theorem to relate $(\dot{\boldsymbol{\omega}}_{\text{b}})_{\mathcal{F}}$ with $(\dot{\boldsymbol{\omega}}_{\text{b}})_{\mathcal{N}}$. Be aware that $\boldsymbol{\omega}_{\text{b}} \neq \boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$.

Problem 6.16 A flywheel (simplified as a thin disk of homogeneous material, of mass m and radius r) is rotating about its symmetry axis $\hat{\boldsymbol{g}}_1$ with a constant speed Ω and about a gimbal axis (defined by the direction vector $\hat{\boldsymbol{g}}_3$) with a speed $\dot{\gamma}$ (Figure 6.9). Triad $\mathcal{G}$, rotates with the gimbal but not with the disk, such that $\boldsymbol{g}_3$ always stays parallel to the inertial direction $\hat{\boldsymbol{n}}_3$. The gimbal structure is idealized as massless, and we neglect any influence of gravity.

The assembly is used as a *control moment gyroscope* (*CMG*), which means that it generates gyroscopic moments on its support structure when a motor forces the flywheel to rotate about the gimbal axis $\hat{\boldsymbol{g}}_3$. In the following, we will investigate this effect.

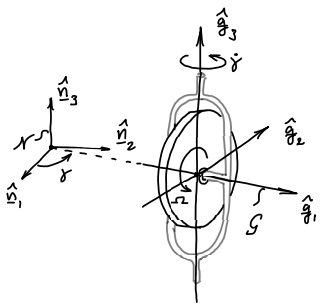


Figure 6.9: A control moment gyroscope (CMG) with its gimbal axis pointing in z-direction. The gimbal angle γ is measured with respect to the x-axis.

a. Calculate the angular momentum vector $\boldsymbol{H}_C$ of the disk with respect to its center of mass C in the gimbal-fixed $\mathcal{G}$-triad (denoted ${}^{\mathcal{G}}\boldsymbol{H}_C$) and in $\mathcal{N}$ (denoted ${}^{\mathcal{N}}\boldsymbol{H}_C$), as functions of m, r, Ω, γ, and $\dot{\gamma}$.
b. Use the transport theorem to calculate ${}^{\mathcal{G}}\dot{\boldsymbol{H}}_C$, the rate of change of angular momentum about C, with components expressed in $\mathcal{G}$.
c. Determine the external moment vector ${}^{\mathcal{N}}\boldsymbol{M}_C$ acting on the disk with respect to its center of mass, as a function of Ω, γ, $\dot{\gamma}$, and $\ddot{\gamma}$, with components expressed in the inertial triad.
d. Assuming the gimbal axis is supported by a pair of ball bearings that are arranged symmetrically (so one above and one below the flywheel) and at a distance a from each other, calculate the magnitude of the resultant radial force F_{r} on each bearing, as a function of m, r, Ω, γ, $\dot{\gamma}$, and $\ddot{\gamma}$.

6.5.2 Practice Problems

Problem 6.17 This Problem is about the skateboard of Problem 1.24 on page 39. The rider is simplified as a single rigid body of mass m_{p} and center of mass P, with inertia tensor $\mathbf{I}_P$. We neglect the mass and inertia of the skateboard including the wheels.

The following variables are given:

- ${}^{\mathcal{F}}\boldsymbol{v}_P$, which is the velocity of point P,
- ${}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$, the angular velocity of triad $\mathcal{F}$ with respect to triad $\mathcal{N}$,
- ${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{F}}$, the angular velocity of triad $\mathcal{B}$ with respect to triad $\mathcal{F}$,
- the rotation matrices ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}$, ${}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}}$ and ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$.

Calculate the *linear* momentum ${}^{\mathcal{N}}\boldsymbol{p}$ of the system as well as its *angular* momentum ${}^{\mathcal{B}}\boldsymbol{H}_P$ with respect to its center of mass P, as functions of ${}^{\mathcal{F}}\boldsymbol{v}_P$, ${}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$, ${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{F}}$, m_{p}, ${}^{\mathcal{B}}\mathbf{I}_P$, ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}$, ${}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}}$, and ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$. ▷

Problem 6.18 This Problem is about the hoverboard of Problem 3.26 on page 95. A student wants to apply Euler's second law to the person, using the equation

$$\dot{\boldsymbol{H}}_P = \boldsymbol{M}_P \tag{6.76P}$$

For (6.76P) to always be correct, the following point(s) may be used for the placeholder P (Check all that apply):

A. A
B. B
C. O
D. C
E. S
F. None of the above

▷

◯ **Problem 6.19** This Problem is about the hoverboard of Problem 3.26 on page 95. A student wants to calculate the angular momentum of the person with respect to a point P, using the equation

$$\boldsymbol{H}_P = \boldsymbol{I}_P \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{6.77P}$$

For (6.77P) to always be correct, the following point(s) may be used for the placeholder P (Check all that apply):

A. A
B. B
C. O
D. C
E. S
F. None of the above

▷

◯ **Problem 6.20** This Problem is about the hoverboard of Problem 3.26 on page 95. A student wants to calculate the rate of change of angular momentum of the person, using the equation

$$\dot{\boldsymbol{H}}_P = \boldsymbol{I}_P \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \tag{6.78P}$$

For (6.78P) to always be correct, the following point(s) may be used for the placeholder P (Check all that apply):

A. A
B. B
C. O
D. C
E. S
F. None of the above

▷

◯ **Problem 6.21** This Problem is about the T-shaped rigid body of Problem 1.20 on page 36. We are interested in the angle ϵ between the body's angular velocity vector $\boldsymbol{\omega}$ and the angular momentum vector $\boldsymbol{H}$. Which of these statements are true?

A. $\cos\epsilon = \boldsymbol{\omega}^{\mathrm{T}} \boldsymbol{H}$
B. $\sin\epsilon = |\boldsymbol{\omega} \times \boldsymbol{H}|$
C. None of the above

▷

◯ **Problem 6.22** This Problem is about the T-shaped rigid body of Problem 1.20 on page 36. The angular momentum of the rigid body with respect to its center of mass C is $\boldsymbol{H}_C$. Show that $|{}^{\mathcal{B}}\boldsymbol{H}_C|$ is always constant. ▶

Problem 6.23 This Problem is about the rotating chair with a person of Example 4.1 on page 106. The person can influence the rotation about the vertical axis by tilting the wheel. Gravity is not considered.

Draw a FBD of the wheel in a plane that is orthogonal to $\hat{\boldsymbol{d}}_2$. Represent all reactions by their resultant components acting in point W and leave out all reactions which you know to be zero from the type of constraints. ▶

Problem 6.24 This Problem is about the rotating chair with a person of Example 4.1 on page 106. Explain why in general $\sum \boldsymbol{M}_W \neq \sum \dot{\boldsymbol{H}}_W$, where $\boldsymbol{H}_W$ is the angular momentum about point W of the combined system, consisting of wheel and person, and $\sum \boldsymbol{M}_W$ is the sum of external moments acting on the system. ▶

Problem 6.25 This Problem is about the rotating chair with a person of Example 4.1 on page 106. Gravity is not considered. The rotation matrix ${}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}}$ is given as

$$ {}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}} = \begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{pmatrix}. \tag{6.79P} $$

Re-use the FBD from Problem 6.23. The rate of change of angular momentum of the wheel about point W can be calculated as

$$ {}^{\mathcal{F}}\dot{\boldsymbol{H}}_W = {}^{\mathcal{F}}\mathbf{I}_{\mathrm{w}}\, {}^{\mathcal{F}}(\dot{\boldsymbol{\omega}}_1)_{\mathcal{F}} + {}^{\mathcal{F}}\boldsymbol{\omega}_2 \times \left({}^{\mathcal{F}}\mathbf{I}_{\mathrm{w}}\, {}^{\mathcal{F}}\boldsymbol{\omega}_1 \right), \tag{6.80P} $$

where $\mathbf{I}_{\mathrm{w}}$ is the inertia tensor of the wheel about point W. Calculate the correct vectors ${}^{\mathcal{F}}\boldsymbol{\omega}_1$, ${}^{\mathcal{F}}(\dot{\boldsymbol{\omega}}_1)_{\mathcal{F}}$ and ${}^{\mathcal{F}}\boldsymbol{\omega}_2$ that need to be used in this equation. Give the components as functions of ψ, $\dot{\psi}$, $\ddot{\psi}$, θ, $\dot{\theta}$, $\ddot{\theta}$, Ω, $\dot{\Omega}$, and the given constants h and b (Remark: None of these terms has to necessarily appear in the requested functions). ▶

Problem 6.26 This Problem is about the rotating chair with a person of Example 4.1 on page 106.

Calculate the reaction moment component that the wheel exerts on the person about the vertical Z-axis, as a function of Ω, ψ, θ and their derivatives. Based on this, determine in which direction the person will start rotating on the chair if starting from rest with $\theta = 0$, and then enforcing a positive $\dot{\theta}$ on the spinning wheel.

The wheel's moment of inertia about its rotational symmetry axis is I_{a}, and about any transverse axis is I_{t}.

You may re-use results from Problem 6.25. Also, to simplify, assume that $\dot{\Omega} = 0$ and $\Omega >> \dot{\psi}$ and $\Omega >> \dot{\theta}$.

Problem 6.27 This Problem is about the quadcopter on a pyramid of Problem 3.35 on page 100.

Explain why it is advantageous that adjacent rotors are rotating in opposing directions in most quadcopters. ▶

Problem 6.28 If a rotor is unbalanced in such a way that its center of mass is not on its axis of rotation, we call this phenomenon *static unbalance*. If the rotor has off-diagonal terms in its inertia tensor, it is called *dynamically unbalanced*. What do you think is a suitable choice of reference triad and reference point for unbalanced rotors? Which forces and/or moments need to act on the rotor in order to keep an unbalanced rotor spinning? What happens to the axis of rotation if the rotor is spinning and all forces and/or moments are suddenly released?

Problem 6.29 A very simple rotor consists only of two identical particles, each of mass m. The masses are each located at a distance of l from the axis of rotation, opposite each other. So, both masses are located in the same plane that also contains the axle. However, the masses are also separated by a distance d in direction of the axle. You can see that this system is in static balance regardless of the rotor angle. What are the moments of the bearings that support the axle when the rotor is rotating at a constant angular velocity of magnitude ω about its axle?

Problem 6.30 This Problem is about the falling wireless phone of Example 4.7 on page 123. Consider the same forces as in Example 6.1. Determine the phone's angular acceleration vector ${}^{\mathcal{B}}\dot{\boldsymbol{\omega}}_{\mathrm{p}}$ at t_i, with components expressed in the $\mathcal{B}$-triad, as functions of g, m, ω_{p}, b, d, h, H, F_C (Remark: None of these terms has to necessarily appear in the requested functions). ▶

Problem 6.31 The cone depicted in Figure 6.10 is rolling on a flat surface. It has mass m, uniform density ρ, base radius R, height H and center of mass C.

The figure indicates an inertial XYZ coordinate system (associated with a triad $\mathcal{N}$), and two rotating triads $\mathcal{D}$ and $\mathcal{F}$. For triad $\mathcal{D}$, $\hat{\boldsymbol{d}}_2$ is always aligned with the cone's contact line with the ground, and $\hat{\boldsymbol{d}}_3$ is always vertical. For triad $\mathcal{F}$, $\hat{\boldsymbol{f}}_2$ is always normal to the cone's base, and $\hat{\boldsymbol{f}}_1$ always points in the direction of $\hat{\boldsymbol{d}}_1$.

The cone rolls (possibly with slipping) on the XY-plane. It is rotating about the Z-axis with angular velocity of magnitude $\dot{\psi}$, and about its symmetry axis with angular velocity of magnitude $\dot{\gamma}$. The cone is subjected to gravity with field strength g in the negative Z direction.

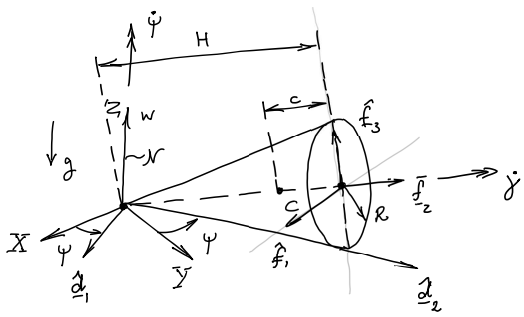


Figure 6.10: Cone

Apply Newton's second law to calculate the resultant force vector (sum of forces) acting on the cone, with components expressed in the $\mathcal{D}$-triad and as functions of $\psi, \dot{\psi}, \dot{\gamma}, \ddot{\psi}, \ddot{\gamma}$, and the given constants (Remark: None of these terms has to necessarily appear in the requested functions). ▶

Problem 6.32 This Problem is about the rolling cone on a flat surface of Problem 6.31 on page 213. At a specific instant in time t_1, the rotation matrix that transforms vectors with components expressed in the $\mathcal{N}$-triad to their representation with components expressed in the $\mathcal{F}$-triad is given as

$${}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}} = \frac{1}{4}\begin{pmatrix} 2\sqrt{2} & 2\sqrt{2} & 0 \\ -\sqrt{6} & \sqrt{6} & 2 \\ \sqrt{2} & -\sqrt{2} & 2\sqrt{3} \end{pmatrix}.$$

Also the moment of inertia I_a of the cone about its symmetry axis, and the moment of inerta I_{tC} about any transverse axis through its center of mass C are given. To simplify calculations, take a cone with special dimensions, $H = \sqrt{3}R$, such that the angle between $\hat{\boldsymbol{d}}_2$ and $\hat{\boldsymbol{f}}_2$ is $\pi/6$.

Apply Euler's second law about point O to calculate the sum of moments acting on the cone about point O, with components expressed in the $\mathcal{F}$-triad, as functions of $\psi, \dot{\psi}, \ddot{\psi}, \dot{\gamma}, \ddot{\gamma}$ and the given constants (Remark: None of these terms has to necessarily appear in the requested functions). ▶

Problem 6.33 This Problem is about the unbalanced cylindrical rotor of Problem 1.22 on page 37. The rotor's inertia tensor about the origin of the rotor-fixed coordinate system xyz, with components expressed in the associated triad, is given as:

$$\mathbf{I}_{\mathrm{D}} = \frac{mR^2}{1000}\begin{pmatrix} 1000 & 5 & 5\sqrt{3} \\ 5 & 2000 & 1 \\ 5\sqrt{3} & 1 & 2000 \end{pmatrix}.$$

As the off-diagonal elements of its inertia tensor are nonzero, the drum is

in moment unbalance, which can cause undesired vibrations when the rotor is in motion.

Again, a weight of mass m_a is to be added to the rotor at a strategic place A, described by the angle φ and the distance d. This time, this weight should compensate for moment unbalance, such that all off-diagonal terms in the inertia tensor $\mathbf{I}_C$ of the combined body (rotor plus mass m_a) are zero. This is done so that the bearings P_1 and P_2 do not have to support any net moment due to rotation of the drum.

Determine m_a and provide a possible location A given by φ, and d as functions of R and m (Remark: None of these terms has to necessarily appear in the requested functions).[a] ►

[a] *Hint*: In general, the results to compensate for static and moment unbalance (which means dynamically balancing) of such a rotor with a single added mass are not identical.

Problem 6.34 This Problem is about the Earth orbiting the Sun of Problem 3.36 on page 101. A comet hits Delft. Before its destruction, the measurement equipment of the TU Delft achieves estimating the impact force ${}^{\mathcal{T}}\boldsymbol{F}_{\mathrm{c}}$ acting on D. Also the force ${}^{\mathcal{T}}\boldsymbol{F}_{\mathrm{g}}$ exerted on Earth due the Sun's gravity is given.

Do not assume anymore that Earth is a sphere, but a rigid body having an inertia tensor ${}^{\mathcal{T}}\mathbf{I}_E$ with respect to Earth's center of mass E. Points E and D are still in the same location as before with respect to the Sun, and the relative position of point D with respect to point E is ${}^{\mathcal{T}}\boldsymbol{r}_{D/E}$. Neglect the difference between center of mass and center of gravity, such that the resultant force from the Sun's gravity acts in E. The current angular velocity of Earth with respect to the inertial triad $\mathcal{N}$ is ${}^{\mathcal{T}}\boldsymbol{\omega}_{\mathcal{T}/\mathcal{N}}$. The components of all given vectors are expressed in the $\mathcal{T}$-triad.

Calculate the absolute acceleration vector ${}^{\mathcal{T}}\boldsymbol{a}_E$ of point E, as well as the angular acceleration ${}^{\mathcal{T}}(\dot{\boldsymbol{\omega}})_{\mathcal{T}/\mathcal{N}}$ of Earth at the considered instant, with components expressed in the $\mathcal{T}$-triad. State your answer as vector functions of ${}^{\mathcal{T}}\mathbf{I}_E$, ${}^{\mathcal{T}}\boldsymbol{F}_{\mathrm{c}}$, ${}^{\mathcal{T}}\boldsymbol{F}_{\mathrm{g}}$, ${}^{\mathcal{T}}\boldsymbol{r}_{D/E}$, ${}^{\mathcal{T}}\boldsymbol{\omega}_{\mathcal{T}/\mathcal{N}}$, and m. ▷►

Problem 6.35 This Problem is about the disk in steady roll of Problem 1.9 on page 19. Given acceleration vector ${}^{\mathcal{N}}\ddot{\boldsymbol{r}}_C$ of point C, calculate the ground reaction force ${}^{\mathcal{N}}\boldsymbol{F}_P$ as a vector function of ${}^{\mathcal{N}}\ddot{\boldsymbol{r}}_C$ and constants m, g. ▷►

Problem 6.36 This Problem is about the disk in steady roll of Problem 1.9 on page 19. Given the relative position vector ${}^{\mathcal{B}}\boldsymbol{r}_{P/C}$ of point P with respect to point C, the ground reaction force vector ${}^{\mathcal{N}}\boldsymbol{F}_P$ acting on the wheel, the wheel's angular velocity ${}^{\mathcal{B}}\boldsymbol{\omega}$, the rotation matrix ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}$, and the wheel's inertia tensor ${}^{\mathcal{B}}\mathbf{I}_C$, determine the wheel's angular acceleration ${}^{\mathcal{B}}\dot{\boldsymbol{\omega}}$ as a vector function of these terms and the constants m, g. ▷►

Problem 6.37 This Problem is about the disk in steady roll of Problem 1.9 on page 19. A student calculated that for the wheel to indeed perform the described motion, the following relationship must hold:

$$8g\cos\theta + (5r\sin(2\theta) - 12R\sin\theta)\,\frac{v_C^2}{(R-\cos\theta)^2} = 0. \tag{6.81P}$$

Perform two different checks of plausibility of this equation. Explain your reasoning. ▶

7 Generalized Coordinates and Virtual Work

Generalized coordinates offer a condensed form of describing the possible kinematic configuration of a system. We will see the relation between generalized coordinates and degrees of freedom of a system, and we will see that it depends on the type of constraints whether it is possible to choose a minimal set of generalized coordinates for a particular system. Then, we will apply the principle of virtual work, in combination with d'Alembert's principle, to derive the equations of motion. We will see that these principles are particularly advantageous for systems with many ideal constraints, because ideal constraint forces do not appear in the equations. This contrasts the Newton-Euler approach, where one would, in general, need to draw a free-body diagram of each rigid body or independent particle in the system, complicating its use for systems that contain many constraints, and thereby many constraint forces and moments.

7.1 Configuration Space

7.1.1 Degrees of Freedom

The number of *degrees of freedom* (DoF) of a rigid body in configuration space is the number of independent possibilities for a system to move. So it is the minimum number of parameters required to specify the (instantaneous) kinematic configuration.

An unconstrained rigid body in 3-D has three translational and three rotational DoF, so in total six DoF. A node (or point) only has three translational ones. In 2-D, a rigid body has only two translational and one rotational DoF, so in total three DoF. A node has only two translational ones in 2-D (see Table 7.1).

A *constraint* acting on the body, such as imposed by a joint or a massless link, restricts absolute or relative movements and thereby reduces the number of DoF. There are many types of constraints. An indica-

Table 7.1: Degrees of freedom of rigid bodies and nodes.

	3-D	2-D
Rigid body	#DoF = 6	#DoF = 3
Node	#DoF = 3	#DoF = 2

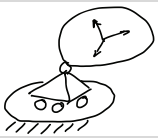
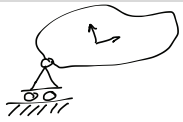
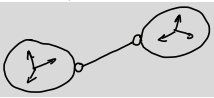
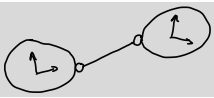
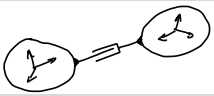
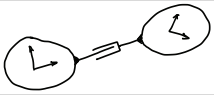
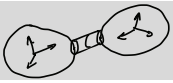
tion for the number of DoF of a system with any kind of constraint as provided by:

$$\#\text{DoF} \geq F = 6b + 3k - \sum_{i=1}^{N} \alpha_i c_i \quad \text{(in 3-D)}, \tag{7.1}$$

$$\#\text{DoF} \geq F = 3b + 2k - \sum_{i=1}^{N} \alpha_i c_i \quad \text{(in 2-D)}. \tag{7.2}$$

Here b is the number of bodies, k the number of nodes, α_i is the number of degrees of freedom that constraint i prevents and c_i is the number of constraints of type i. Some typical constraints are listed in Table 7.2.

Table 7.2: An overview of types of constraints with the number α_i of DoF they prevent. This overview is non-exhaustive; Look carefully how many DoF a constraint prevents in each specific situation.

Constraint	3-D		2-D	
Spherical joint		$\alpha_i = 3$		$\alpha_i = 2$
Roller spherical joint		$\alpha_i = 1$		$\alpha_i = 1$
Massless link		$\alpha_i = 1$		$\alpha_i = 1$
Cylindrical joint		$\alpha_i = 4$		$\alpha_i = 2$
Hinge		$\alpha_i = 5$		$\alpha_i = 2$
etc.	...	...	...	...

Calculating the number F from (7.1) or (7.2) does not directly yield the number of DoF, F is only a lower bound. The numbers are only

equal if each constraint in the system has a unique function. If any constraints are redundant, fewer degrees of freedom are constrained.

Example 7.1 Three rigid bodies in 3-D are connected to each other via spherical joints, as illustrated in Figure 7.1. Applying the formula (7.1) with number of rigid bodies $b = 3$, number of nodes $k = 0$ and number of spherical ($\alpha = 3$) joints $c = 3$, we calculate $F = 9$. Indeed, we can recognize nine independent ways of the system to move, by considering the dashed triangle between the bodies. This triangle can assume any position and orientation in space, so these are already six DoF. Then, each of the rigid bodies can rotate with respect to this triangle about a line through its connecting joints. These are three additional DoF. In this case, the formula yielded the correct number of DoF, and none of the constraints was redundant.

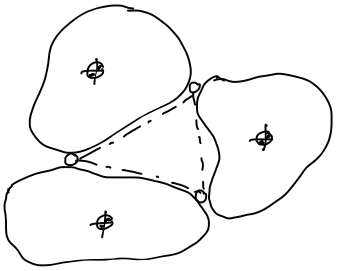


Figure 7.1: Example of a 3-D system with three rigid bodies and three joints that has nine degrees of freedom

Example 7.2 Figure 7.2 shows two rigid bodies connected by spherical joints. Applying the formula in a similar way as in Example 7.1 yields $F = 0$. However, we can intuitively see that this is not the number of DoF.

The leftmost joint takes away three DoF of that system, constraining translation of the connected points on the two bodies with respect to each other. The second joint takes away only two additional degrees of freedom, because the two bodies can still rotate with respect to each other, about the line between the two joints. So, the two first joints act like a hinge. However, there is a statically indeterminate situation, because the distance between the two joints is imposed by each of the two rigid bodies. As a result, there could be compressive or tensile forces in the rigid bodies due to over-constraining them. The third and fourth joints have no unique function, so they do not take away any further DoF. They only add to more possible stresses inside the bodies.

These considerations result in seven degrees of freedom for the system. For example, these could be described by the first body's free rotations and

translations (as it is not attached to the fixed world), and the relative rotation of the second body about the line between the joints.

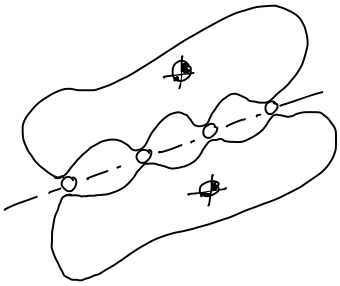


Figure 7.2: Example of a 3-D system with two rigid bodies and four joints that has seven degrees of freedom

Note: When interpreting whether something is a node or a joint, there is sometimes freedom of choice. However, a body/node is always connected to another body/node via some sort of constraint that imposes a limitation to their relative movement. So, one needs to make sure to alternate between body/node and constraint.

Example 7.3 Three rigid bodies and a node are connected via spherical joints and massless links to each other and to the ground as shown in Figure 7.3.

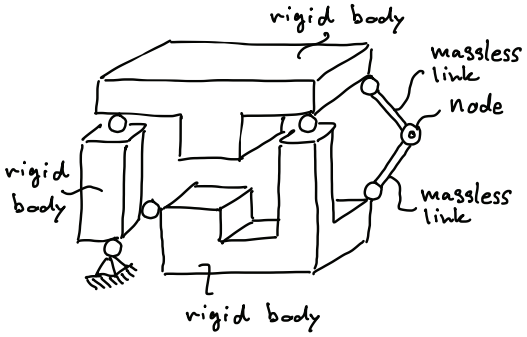


Figure 7.3: Three rigid bodies and a node connected via spherical joints and massless links to each other and to the ground.

Here, one can see the alternation between body/node and constraint for the chain on the right. The massless links, including their joints, form constraints.

◯ **Problem 7.1** Determine the number of DoF of the system shown in Figure 7.3[a].

[a] *Hint*: Note that this is a very similar system to the one in Example 7.1. The three rigid bodies are just drawn more elaborately.

◯ **Problem 7.2** A double pendulum is mounted on a massless cart. The system is restricted to move in 2-D. How many DoF does the system have? Verify using (7.2).

In general, if a system has at least one DoF, this means the system can move, it is a *mechanism*. If it has one DoF, it exhibits constrained motion. An example for constrained motion is a nut on a bolt, because rotating the nut also translates it along the axis. Another example is the gearwheels in a gearbox, because rotating one gearwheel rotates the other gearwheels in a given ratio. If a system has no DoF, it is completely constrained, which means it cannot move at all. Such a system is a *structure* and not a mechanism.

◯ **Problem 7.3** Draw an example system that has zero DoF and that is statically indeterminate.

7.1.2 Generalized Coordinates

Generalized coordinates are parameters that describe the kinematic configuration of a system. They can, but do not need to correspond to commonly employed coordinates, such as $(x,\, y,\, z)$ in Cartesian or (r, θ, h) in cylindrical coordinates (see Section 4.3 for more examples), in order to represent the relative or absolute position of particles or rigid bodies. The n-dimensional *configuration space* of a system is the space in which the n generalized coordinates $(q_1,\, q_2 \dots q_j \dots q_n)$ are defined.

We often attempt to find a set of generalized coordinates such that they are *independent*. In that case, the number n of generalized coordinates equals the number of degrees of freedom. Sometimes, however, we need more generalized coordinates to describe the kinematics of a system. In that case, we can be sure there will be m relationships between generalized coordinates in velocity space, so they cannot be chosen independently of each other anymore. Such a relationship between general coordinates is called a *constraint equation*, and there will be just as many constraint equations as we have chosen excess generalized coordinates. So, in general, the number of DoF is

$$\#\text{DoF} = n - m. \tag{7.3}$$

◯ **Example 7.4** In Figure 7.4, we see an example of a system of two point masses in 2-D, which have four DoF, so the minimum number of generalized coordinates q_j is four.

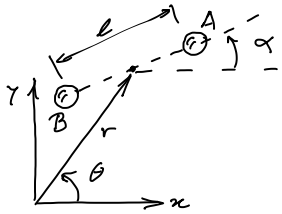


Figure 7.4: System in 2-D with four degrees of freedom.

One possible choice for the generalized coordinates would be the x and y Cartesian coordinates of the two particles. An alternative set could be the dimensions θ, r, α, and l, as indicated in the figure. However, these are just two possible sets of generalized coordinates. Many other sets could equally describe the kinematics of this system, as long as they indeed completely define the kinematic configuration. For example, this would also be a valid choice for the q_j:

$$q_1 := l, \tag{7.4E}$$
$$q_2 := r, \tag{7.5E}$$
$$q_3 := \theta + \alpha, \tag{7.6E}$$
$$q_4 := \sin\alpha. \tag{7.7E}$$

This demonstrates that the generalized coordinates may also be a combination of other parameters (like q_3) or nonlinear functions (like q_4). Also, they need not be valid globally, as the Cartesian positions of both particles would. Instead, they can be a local set that is only valid in a certain region (for example, in order to avoid ambiguity in q_4, one could restrict the range of angles to $-\frac{\pi}{2} \leq \alpha < \frac{\pi}{2}$).

We already saw several frequently used generalized coordinates in this book, besides Cartesian positions. For example, Euler angles can be used as a minimal set of (local) generalized coordinates to describe the orientation of a rigid body. Also here, we saw that restriction of the intervals of a particular set of Euler angles is necessary to avoid singularities and ambiguity. To describe orientation globally, one would choose different sets of Euler angles, each valid in a different region.

◎ **Example 7.5** Another example for generalized coordinates that describe orientation of a rigid body are the principal axis $\hat{\boldsymbol{a}}$ and the principal angle φ. This is a set that is valid globally. However, it appears that there

are already four parameters in this description (three components of the direction vector plus the angle). A rigid body has only three rotational degrees of freedom. Therefore, it must be possible to formulate exactly one constraint equation that links the generalized coordinates. And in fact, such a constraint equation is given by choosing a direction vector with unit length ($\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = 1$).

As the generalized coordinates must completely define the kinematic configuration of a system, the Cartesian coordinates of all relevant points in the system can always be given as functions of the generalized coordinates, and possibly also of time t (in case there are constraints that depend on time). So, for a system of N particles, the position vectors of all particles can be found as

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1,\, q_2,\, q_3 \ldots q_n,\, t), \qquad i = (1 \ldots N), \tag{7.8}$$

Each particle has three Cartesian coordinates in 3-D. Therefore, (7.8) delivers $3N$ scalar equations. These equations, which map from configuration space to Cartesian space[1], are called the *transformation equations*.

Example 7.6 This Example is about the two point masses in 2-D with 4 DoF of Example 7.4 on page 222. For this example system in Figure 7.4, $N = 2$ and (7.8) reduces to finding the x and y coordinates of the two particles. So, in total there are four transformation equations. For example, the Cartesian y coordinate of particle A is given by

$$y_A = r \sin\theta + \frac{1}{2} l \sin\alpha. \tag{7.9E}$$

Substituting the generalized coordinates we defined in (7.4E) to (7.7E) results in

$$y_A = \underbrace{r}_{q_2} \sin \underbrace{(\theta)}_{q_3 - \arcsin(q_4)} + \frac{1}{2} \underbrace{l}_{q_1} \underbrace{\sin\alpha}_{q_4}. \tag{7.10E}$$

All four transformation equations:

$$x_A = q_2 \cos(q_3 - \arcsin(q_4)) + \frac{1}{2} q_1 \sqrt{1 - q_4^2}, \tag{7.11E}$$

$$y_A = q_2 \sin(q_3 - \arcsin(q_4)) + \frac{1}{2} q_1 q_4, \tag{7.12E}$$

$$x_B = \ldots, \tag{7.13E}$$

$$y_B = \ldots. \tag{7.14E}$$

[1] Euclidian space described by Cartesian coordinates

◯ **Problem 7.4** Find x_B and y_B to complete (7.13E) and (7.14E).

Sometimes, for brevity of notation we write the generalized coordinates as one vector, so $\boldsymbol{q} = (q_1, q_2 \dots q_n)^{\mathrm{T}}$. However, note that this is not a true vector, because all elements in it can have different units, and because the general rules of vector algebra do not apply. For example, the three Euler angles are valid choices for generalized coordinates. However, when writing them in vector form, the vectors may not be added.

The dynamic *state* of a system is defined by the coordinates and their velocities. The *state equations* are determined by the equations of motion, as for example derived from Newton-Euler. These are second-order differential equations in the coordinates, or first-order differential equations in the velocities.

7.1.3 Classification of Constraints

There are different classifications of constraints. In this book, we will particularly distinguish between *holonomic* and *non-holonomic* and between *scleronomic* and *rheonomic* constraints.

A *holonomic* (*hólos* is *whole* in Greek) constraint Φ between generalized coordinates q_j can always be written in the form:

$$\Phi = \Phi(q_1, q_2 \dots q_j \dots q_n, t) = 0. \tag{7.15}$$

If all constraints imposed on a system are holonomic, the system itself is also defined as holonomic. In those cases, it is always possible to find a minimal set of generalized coordinates, such that the number n of generalized coordinates is equal to the number of degrees of freedom.

◎ **Example 7.7** Consider a particle that is constrained to move on a spherical constraint surface (like a person walking around on the surface of the Earth). We can describe the position of the person with three Cartesian coordinates x, y, z using a coordinate system that has its origin in the center of the sphere. However, we know that we have chosen one coordinate too many, because there are only two degrees of freedom. Indeed, we can find one constraint equation, using the radius R of the sphere:

$$\Phi = x^2 + y^2 + z^2 - R^2 = 0. \tag{7.16E}$$

This is a holonomic constraint, because it can be written in terms of the generalized coordinates, as required in (7.15). Therefore, we know we can find a lower-dimensional set of generalized coordinates. One possibility is to

choose x and y and to solve (7.16E) for z. This is not so convenient because z is ambiguous due to the square. Therefore, such a set can only locally be applicable, for example for the Northern Hemisphere. An alternative, globally valid set of generalized coordinates for this example would be latitude and longitude.

Problem 7.5 This Problem is about the ship installing a monopile of Problem 4.21 on page 130.

Redundant generalized coordinates to analyze the monopile's motion are chosen as $\boldsymbol{q} = (X_D,\, Y_D,\, Z_D,\, X_G,\, Y_G,\, Z_G,\, \gamma,\, \theta,\, \psi)^{\mathrm{T}}$, which are the X, Y, and Z-coordinates of points D and G with respect to origin O, and the Precession-Nutation-Spin (type II) Euler angles γ, θ, ψ of the monopile with respect to triad $\mathcal{N}$.

Write the constraint equation that describes the fact that points D and G are linked by the cable of length l, in the form of a function

$$\Phi(X_D,\, Y_D,\, Z_D,\, X_G,\, Y_G,\, Z_G,\, \gamma,\, \theta,\, \psi,\, l) = 0. \tag{7.17P}$$

▷▶

A *non-holonomic* constraint f cannot be used to reduce the number of generalized coordinates, but it does reduce the velocity space. So, a different number of DoF exists in configuration space and in velocity space. Note that by default, in this book DoF without modifier refers to DoF in the velocity space. The velocity space is reduced by as many dimensions as there are independent non-holonomic constraints. Remember that the dynamic state of a system is defined by the coordinates *and* their velocities.

In mechanics, a non-holonomic constraint can usually be defined as a linear constraint in velocity space:

$$f = \left(\sum_{j=1}^{n} a_j(q_1,\, q_2 \ldots\, q_n,\, t)\dot{q}_j \right) + a_o(q_1, q_2 \ldots q_n, t) = 0, \tag{7.18}$$

where the coefficients a_j and the offset a_o can be functions of the generalized coordinates q_j and time t.

Non-holonomic constraints only exist in velocity space, not in configuration space; they cannot be integrated. So, the system has more degrees of freedom in configuration space than in velocity space. It must be stressed that also holonomic constraints can be written in velocity space. The difference is that a holonomic constraint *can* be integrated to obtain a formula as in (7.15).

Example 7.8 Consider a mass at position x, moving with a constant speed $\dot{x} = c_1$. This constraint can be written in the form of (7.18), with $n = 1$, $\dot{q}_1 = \dot{x}$, $a_1 = 1$ and $a_o = -c_1$:

$$f = \dot{x} - c_1 = 0. \tag{7.19E}$$

From this equation, one could expect the constraint to be non-holonomic. However, the equation can be integrated, such that it also fits (7.15):

$$\int f \, \mathrm{d}t = \int \dot{x} \, \mathrm{d}t - \int c_1 \, \mathrm{d}t = 0 \tag{7.20E}$$

$$\Phi = x + c_2 - c_1 \, t = 0, \tag{7.21E}$$

such that this constraint is in fact a *holonomic constraint*.

Example 7.9 Consider a two-wheeled trailer being pulled by a car on a horizontal surface (Figure 7.5). The wheels can rotate independently of each other, but they do not slip. We want to establish the constraint equation describing the absence of lateral slip, in terms of a suitable set of generalized coordinates and their derivatives.

We define an inertial coordinate system XYZ, a trailer-fixed local coordinate system xyz, and two associated triads: $\mathcal{N}$ and $\mathcal{B}$, respectively.

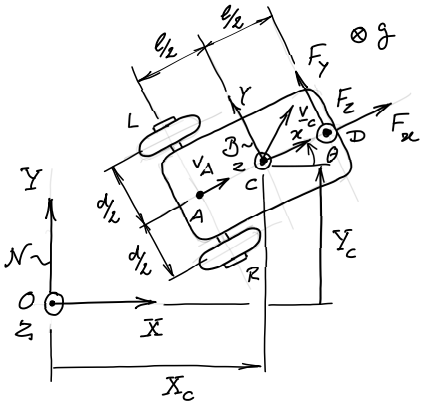


Figure 7.5: A trailer getting pulled by a car.

We start by defining generalized coordinates. We have to choose minimally three generalized coordinates to describe any possible configuration of the trailer's chassis on level ground, for example the Cartesian coordinates $q_1 = X_C$ and $q_2 = Y_C$ of its center of mass and the current heading $q_3 = \theta$. We would need additional three generalized coordinates to also describe the rotations of the wheels, but for the remainder of this problem we only look at the chassis.

We already intuitively see that in velocity space, the trailer has one degree of freedom less: The trailer cannot choose independent derivatives of these generalized coordinates, for example it cannot move sideways without maneuvering back and forth as well (as in parallel parking).

This constraint is imposed by the non-slip condition of the wheels in lateral direction, and it can be described in mathematical form: We first note that the rolling contact of each wheel constrains the center of this wheel to move in direction of the plane in which the wheel is contained. That means the velocity vector of this point is always orthogonal to the axle of the wheel. Second, we note that both wheels' centers are on the rear axle, so the center A of this axle can only move orthogonal to the axle.

This can be expressed by saying that the scalar product of the axle direction vector $\hat{\boldsymbol{b}}_2$ and the velocity vector of point A is always zero:

$$\boldsymbol{v}_A \cdot \hat{\boldsymbol{b}}_2 = 0. \tag{7.22E}$$

We now need to transform this equation to configuration space.

The velocity $\boldsymbol{v}_A$ is given as a function of the generalized coordinates using the transport theorem:

$${}^{\mathcal{B}}\boldsymbol{r}_A = {}^{\mathcal{B}}\boldsymbol{r}_C + {}^{\mathcal{B}}\boldsymbol{r}_{A/C} \Rightarrow {}^{\mathcal{B}}\boldsymbol{v}_A = {}^{\mathcal{B}}\boldsymbol{v}_C + \cancelto{0}{{}^{\mathcal{B}}(\dot{\boldsymbol{r}}_{A/C})_{\mathcal{B}}} + {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathrm{t}} \times {}^{\mathcal{B}}\boldsymbol{r}_{A/C}, \tag{7.23E}$$

with the trailer's angular velocity vector ${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathrm{t}} = (0, -\frac{1}{2}l\dot{\theta}, 0)^{\mathrm{T}}$, the velocity vector ${}^{\mathcal{B}}\boldsymbol{v}_C = (\dot{X}_C, \dot{Y}_C, 0)^{\mathrm{T}}$ of point C, and the relative position vector ${}^{\mathcal{B}}\boldsymbol{r}_{A/C} = (-\frac{1}{2}l, 0, Z_A - Z_C)^{\mathrm{T}}$. Substituting this and the unit vector ${}^{\mathcal{B}}\hat{\boldsymbol{b}}_2 = (0, 1, 0)^{\mathrm{T}}$ in (7.22E), we obtain the constraint equation in the requested form:

$$f(X_C, Y_C, \theta, \dot{X}_C, \dot{Y}_C, \dot{\theta}) = \dot{X}_C \sin\theta - \dot{Y}_C \cos\theta + \tfrac{1}{2}l\dot{\theta} = 0 \tag{7.24E}$$

It is not possible to integrate this constraint to obtain an equation of the form in (7.15). If the constraint could be integrated to configuration space, this would imply that we could not independently choose the position of the trailer's center of mass and the trailer's orientation. We know that this is not true, because the trailer can in principle get to any point and head in any direction, just some maneuvering might be necessary to get there. The fact that we cannot integrate it means the constraint is non-holonomic.

So, one degree of freedom in velocity space is lost by the non-holonomic constraints of non-slip in lateral direction for the wheels, which translates into one non-holonomic constraint equation for the trailer.

◯ **Problem 7.6** Extract the coefficients from (7.24E) to write it in the form of (7.18). ▷

Problem 7.7 A wheel is rolling on the ground (the XY-plane), constrained to be always upright (so the axle is parallel to the XY-plane). So, it resembles the one in Figure 3.6 on page 82, but with constant $\theta = \pi/2$. The configuration of the wheel is described by the Cartesian coordinates X_C and Y_C of the center of mass C, the precession angle φ, and the angle ψ describing the rotations performed by the wheel. The angles are defined in Figure 3.6. Make a top-view and a side-view drawing of this special case. The wheel has zero lateral slip, which means the wheel cannot move sideways instantaneously. Determine the coefficients a_j of a function f_{l} that describes this constraint, in the form of (7.18).

Problem 7.8 For the wheel in Problem 7.7, we are interested in knowing how many rotations the wheel has made, as described by the fourth generalized coordinate ψ. Formulate a second constraint function f_{r} that forces the wheel to roll without slipping in the longitudinal direction. Determine again the coefficients a_j as in (7.18).

In the general case, both holonomic and non-holonomic constraints can explicitly depend on time, so the functions (7.15) and (7.18) contain the variable t. In case this variable is indeed explicitly needed, and the time-dependency is not only given by the fact that the generalized coordinates q_j are functions of time, then the constraints are *rheonomic* (*rhéo* is *to flow* in Greek). In contrast, a *scleronomic* (*sclerés* is *hard* in Greek) constraint does not explicitly depend on time. Any combinations of the two classifications, holonomic/non-holonomic and rheonomic/scleronomic, are possible.

Example 7.10 An example of a rheonomic constraint is a linear motor that imposes a time-dependent distance $l(t)$ between two particles. This constraint is also holonomic.

An example of a scleronomic constraint is a joint or a link. Another example is the rolling wheel in Figure 3.6 on page 82, which has two scleronomic non-holonomic constraints, imposed by the non-slip conditions with the ground in circumferential and lateral direction.

Problem 7.9 This Problem is about the free rolling non-slipping disk of Example 3.6 on page 82. What is the minimal number n of generalized coordinates for this physical system? How many scalar constraint equations m will exist between this minimal set of generalized coordinates? ▷

Problem 7.10 This Problem is about the scooter with springboard of Problem 1.31 on page 43. We consider the mechanical system consisting of scooter and person.

Classify the following constraints in the system as to whether they are holonomic and/or scleronomic: Constraint imposed because of...

- hinge connecting rear wheel and footboard
- wheel rolling on the ground without slip in circumferential direction
- wheel not slipping laterally on the ground
- ground surface not deforming in normal direction

▶

7.2 Virtual Work

7.2.1 Virtual Work in Cartesian Space

From Section 2.2, we can calculate the work increment $\mathrm{d}W_i$ that a force $\boldsymbol{F}_i$ performs on a system, if the force's point of application travels with infinitesimal position vector increment $\mathrm{d}\boldsymbol{r}_i$:

$$\mathrm{d}W_i = \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}_i. \tag{7.25}$$

As a thought experiment, we can also consider an infinitesimally small, *virtual* displacement of the point of application, and the work that would be done by force $\boldsymbol{F}_i$ in that case, so the *virtual* work δW_i:

$$\delta W_i = \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i. \tag{7.26}$$

This is a thought experiment, so the $\delta \boldsymbol{r}_i$ do not need to be congruent with the actual displacements $\mathrm{d}\boldsymbol{r}_i$. However, as we will see, this *thought experiment* of virtual work will allow us to find equations of motion of a dynamic system.

If there are multiple forces acting on the system, the virtual work δW is the sum of the work done by each of these N forces, so

$$\delta W = \sum_{i=1}^{N} \delta W_i = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i. \tag{7.27}$$

If a system is currently in *static* equilibrium, then the net virtual work of all N forces acting on it must be zero, and vice versa. We obtain the *principle of virtual work*:

$$\text{Static equilibrium } \Leftrightarrow \delta W = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i = 0. \tag{7.28}$$

In case the N virtual displacements are independent, one could for example choose all of the virtual displacements $\delta \boldsymbol{r}_i$ in (7.28), except for a single one, orthogonal to their respective forces $\boldsymbol{F}_i$, such that no

virtual work is done by these forces. Then, the sum in (7.28) reduces to the scalar product of the one remaining virtual displacement and its force. For this scalar product to be zero regardless of the choice of virtual displacement, it follows that the force must be zero. This can be repeated for all other $N-1$ displacements and forces, so it becomes evident that all forces must be zero. Note that this procedure is only possible because all displacements were independent.

◯ **Example 7.11** We want to apply (7.28) to a system of two unconstrained particles, as in Figure 7.4. Forces act on each of the particles (not drawn), namely force $\boldsymbol{F}_A$ acting on particle A, and force $\boldsymbol{F}_B$ acting on particle B. Following (7.28), the virtual work of these forces must be zero in order for the system to be in static equilibrium:

$$\text{Static equilibrium } \Leftrightarrow \delta W = \boldsymbol{F}_A \cdot \delta \boldsymbol{r}_A + \boldsymbol{F}_B \cdot \delta \boldsymbol{r}_B = 0. \tag{7.29E}$$

In this case, the particles can move fully independently from each other. So, there is no constraint on the choice of neither $\delta \boldsymbol{r}_A$ nor $\delta \boldsymbol{r}_B$, they are independent. If, for example, we choose $\delta \boldsymbol{r}_A = \boldsymbol{0}$ and $\delta \boldsymbol{r}_B$ to be aligned with the applied force in point B, so $\delta \boldsymbol{r}_B = k\boldsymbol{F}_B$ with infinitesimally small, but nonzero scalar k, we obtain

$$k|\boldsymbol{F}_B|^2 = 0. \tag{7.30E}$$

Therefore, we see that force $\boldsymbol{F}_B$ must be the zero vector for the system to be in static equilibrium. We can make analogous choices and see that $\boldsymbol{F}_A$ must be zero as well for static equilibrium.

The same can be done for systems that contain not only two, but N particles.

This means that in this special case of independent virtual displacements, (7.28) reduces to requesting that all $\boldsymbol{F}_i$ must be zero (Remember that each $\boldsymbol{F}_i$ here is the shorthand notation for the *sum* of forces acting in particle i).

Once the movements are not independent anymore, this reduction does not hold anymore.

◯ **Example 7.12** Consider again the system in Figure 7.4, but now we specify l to be constant, removing one degree of freedom of the system. This can be visualized as a massless link between the particles.

Now, there will be a relationship between $\delta \boldsymbol{r}_A$ and $\delta \boldsymbol{r}_B$ that needs to be taken into account. We cannot freely choose the virtual displacements anymore, which in turn means that not all forces applied to the individual particles need to be zero for static equilibrium. Indeed we know that if the forces are solely applied in direction of the link, in an equal and opposite

fashion (so the special case where $\boldsymbol{F}_B = -\boldsymbol{F}_A$), then the system is still in a state of equilibrium.

A solution to this issue is to incorporate the constraint equations, in order to consider only displacements that are congruent with these constraints in (7.27). This will allow us to find the conditions on the forces for static equilibrium.

Problem 7.11 For the system of Example 7.4 shown in Figure 7.4, consider l to be constant, as enforced by a massless rod between the particles. Choose allowed displacements for (7.29E), in order to find the conditions on $\boldsymbol{F}_B$ and $\boldsymbol{F}_A$ such that the system is in static equilibrium[a].

[a] *Hint*: Choose a new coordinate system where one axis is parallel to the rod, and consider virtual displacements orthogonal and parallel to the rod that are congruent with the constraint.

It is inconvenient to apply the principle in the form of (7.28) for three reasons: First, the virtual displacements $\delta\boldsymbol{r}_i$ are in most cases not independent of each other, because also the virtual displacements have to obey the constraints of the system. In that case, we need to specify the relationships between the individual virtual displacements $\delta\boldsymbol{r}_i$.

Second, some forces do not do any net virtual work and do not need to be considered.

Third, (7.28) is only applicable in the case of static equilibrium, so it does not hold anymore once the system moves. Therefore, we aim to extend the use of the principle of virtual work to dynamic systems.

The following three subsections will deal with each of the limitations.

7.2.2 Virtual Work in Configuration Space

A more convenient solution to deal with constraints and the dependency they cause between coordinates in Cartesian space is to transform the principle of virtual work to a minimal configuration space, where generalized coordinates are *independent*. Then, we can again consider fully independent virtual displacements, because a minimal set of generalized coordinates will automatically respect the given constraints. Note that finding such a minimal set of generalized coordinates is not always possible. In that case, we still need to be careful to apply only allowed (coupled) virtual displacements.

A key concept of virtual displacements is that they are infinitesimally small, such that they can be superimposed even if the system itself is nonlinear. A mechanical system is described by n generalized coordinates q_j, with $j = 1 \dots n$. There are N forces acting on it, with cor-

responding points of application $\boldsymbol{r}_i$. We know we can formulate these points as functions of the generalized coordinates using (7.8). Now, all these q_j undergo virtual displacements δq_i, simultaneously. We are interested in the virtual work of the forces.

As the virtual displacements of the n generalized coordinates in configuration space are infinitesimally small, we can find the corresponding Cartesian virtual displacement of the point of application $\boldsymbol{r}_i$ of the i-th force using superposition of individual virtual displacements of the generalized coordinates. Remember this is a *virtual* displacement, so the points of application of the forces do not actually need to move in this way.

Example 7.13 On the left of Figure 7.6, we see a roller joint and a massless rod balancing on it, connecting two particles A and B. We are interested in the virtual displacements of the two particles, as functions of virtual displacements in a set of generalized coordinates.

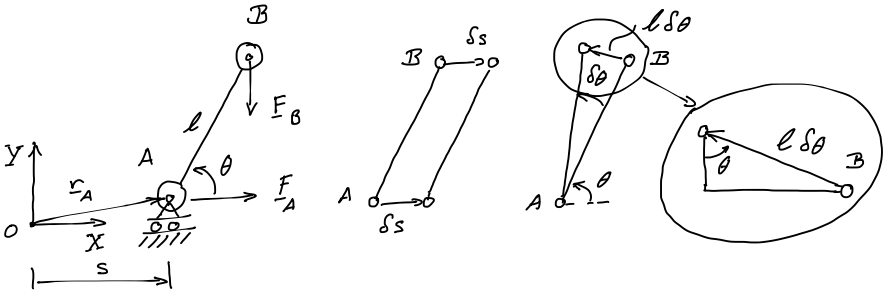


Figure 7.6: Left: Two particles A and B, each with an external force acting on it, and constrained by a massless link and a roller joint. Right: Virtual displacements of points A and B in directions s and θ.

This system has only holonomic constraints, so we know that we can choose a minimal set of generalized coordinates. A possible minimal set consists of distance s and angle θ. On the right of Figure 7.6, we see the allowed displacements of the two points, in direction of the generalized coordinates. The virtual displacements $\delta\boldsymbol{r}_A$ and $\delta\boldsymbol{r}_B$ of point A and B can be found by superposition of the two individual virtual displacements:

$$\delta\boldsymbol{r}_A = \delta\boldsymbol{r}_{A\theta} + \delta\boldsymbol{r}_{As}, \tag{7.31E}$$

$$\delta\boldsymbol{r}_B = \delta\boldsymbol{r}_{B\theta} + \delta\boldsymbol{r}_{Bs}, \tag{7.32E}$$

where the individual terms are found, using geometric relationships of the

two independent virtual displacements, as

$$\delta \boldsymbol{r}_{A\theta} = \boldsymbol{0}, \tag{7.33E}$$

$$\delta \boldsymbol{r}_{As} = \begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathrm{T}} \delta s, \tag{7.34E}$$

$$\delta \boldsymbol{r}_{B\theta} = \begin{pmatrix} -l\sin\theta & l\cos\theta \end{pmatrix}^{\mathrm{T}} \delta\theta, \tag{7.35E}$$

$$\delta \boldsymbol{r}_{Bs} = \begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathrm{T}} \delta s. \tag{7.36E}$$

A more formal alternative to the geometric approach is to use partial derivatives. We know that the virtual displacements are infinitesimally small, so we can find the virtual displacement of a point with position vector $\boldsymbol{r}_i(q_1,\, q_2 \ldots q_i)$ in a scleronomic system as

$$\delta \boldsymbol{r}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j. \tag{7.37}$$

Problem 7.12 Using (7.37), validate the relationships found in (7.33E) to (7.36E) for the system of Example 7.13.

As for the coordinates or the virtual displacements, also each three-dimensional force vector $\boldsymbol{F}_i$ in Cartesian space can be linked to a corresponding n-dimensional *generalized force* vector in configuration space $\boldsymbol{Q}_i = (Q_{i1},\, Q_{i2} \ldots Q_{ij} \ldots Q_{in})^{\mathrm{T}}$. Note again that the $\boldsymbol{Q}_i$ are not true vectors, like $\boldsymbol{q}$, and that they are usually not actual forces (with the unit of Newtons) because the generalized coordinates are usually not actual coordinates (with the unit of meters). We only define $\boldsymbol{Q}_i$ for brevity of notation, but we cannot perform any vector operations on it.

In analogy to virtual displacements, there are again two ways to find the corresponding generalized force $\boldsymbol{Q}_i$ for a force $\boldsymbol{F}_i$: A geometric, intuitive way, and one using partial derivatives. In both cases, the link between the two spaces is that the virtual work done must be identical:

$$\delta W_i = \boldsymbol{Q}_i \cdot \delta \boldsymbol{q} = \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i. \tag{7.38}$$

or alternatively, written in components:

$$\delta W_i = \sum_{j=1}^{n} Q_{ij} \delta q_j = \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i. \tag{7.39}$$

Remember that $\delta \boldsymbol{r}_i$ must be the virtual displacement of the point of application of the force $\boldsymbol{F}_i$.

The first, intuitive way to find the Q_{ji} associated with $\boldsymbol{F}_i$ is by calculating the work done by the forces for each virtual displacement δq_j, and then comparing coefficients in (7.39).

Example 7.14 This Example is about the massles rod on a roller joint of Example 7.13 on page 232. We aim to find the total generalized force vector $\boldsymbol{Q} = (Q_s, Q_\theta)^{\mathrm{T}}$ that is equivalent to the two forces.

We can write virtual work using the (yet unknown) generalized forces Q_s and Q_θ:

$$\delta W = Q_s \delta s + Q_\theta \delta\theta. \tag{7.40E}$$

This must be equal to virtual work done by the two forces in Cartesian space:

$$\delta W = \boldsymbol{F}_A \cdot \delta \boldsymbol{r}_{As} + \boldsymbol{F}_A \cdot \delta \boldsymbol{r}_{A\theta} + \boldsymbol{F}_B \cdot \delta \boldsymbol{r}_{Bs} + \boldsymbol{F}_B \cdot \delta \boldsymbol{r}_{B\theta}. \tag{7.41E}$$

By looking at the figure and considering the direction of the virtual displacements, we notice that the second summand must be zero, because there is no displacement of the point of application of the force in point A when only θ varies and s is constant. Also the third summand must be zero, because the virtual displacement is orthogonal to the force vector. So, the scalar product is zero and no virtual work can be done. This reduces the equation to

$$\delta W = \boldsymbol{F}_A \cdot \delta \boldsymbol{r}_{As} + \boldsymbol{F}_B \cdot \delta \boldsymbol{r}_{B\theta}. \tag{7.42E}$$

We see the same when we substitute equations (7.33E)-(7.36E) in (7.41E), and we obtain the overall equations:

$$\delta W = \boldsymbol{F}_A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \delta s + \boldsymbol{F}_B \cdot \begin{pmatrix} -l \sin\theta \\ l \cos\theta \end{pmatrix} \delta\theta \tag{7.43E}$$

$$= \begin{pmatrix} F_A \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \delta s + \begin{pmatrix} 0 \\ -F_B \end{pmatrix} \cdot \begin{pmatrix} -l \sin\theta \\ l \cos\theta \end{pmatrix} \delta\theta \tag{7.44E}$$

$$= \underbrace{F_A}_{Q_s} \delta s + (\underbrace{-F_B l \cos\theta}_{Q_\theta}) \delta\theta. \tag{7.45E}$$

By comparing coefficients in (7.45E) and (7.40E), we find the desired terms for Q_s and Q_θ.

The alternative approach follows exactly the same reasoning, but makes use of (7.37). We are interested in the generalized force Q_j acting in direction of a certain generalized coordinate q_j, knowing all forces $\boldsymbol{F}_i$ acting in Cartesian space. We know that the virtual work of both must be the same, so

$$\delta W = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i \tag{7.46}$$

and equally

$$\delta W = \sum_{j=1}^{n} Q_j \delta q_j. \tag{7.47}$$

Using both equations and substituting (7.37), we obtain

$$\sum_{i=1}^{N} \boldsymbol{F}_i \cdot \left(\sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j \right) = \sum_{j=1}^{n} Q_j \delta q_j. \tag{7.48}$$

Re-grouping the terms in this equation leads to

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{N} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \right) \delta q_j = \sum_{j=1}^{n} Q_j \delta q_j. \tag{7.49}$$

So, when we compare coefficients, we see that

$$Q_j = \sum_{i=1}^{N} \left(\boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \right). \tag{7.50}$$

This defines the generalized force Q_j.

If we are interested only in one particular Cartesian force vector $\boldsymbol{F}_i$, we can find its equivalent generalized force vector $\boldsymbol{Q}_i$ in configuration space using

$$Q_{ij} = \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}. \tag{7.51}$$

Problem 7.13 Apply (7.50) to Example 7.13, in order to verify the relationships in (7.45E) found with the intuitive approach.

Problem 7.14 This Problem is about the top of corks and skewers of Problem 1.8 on page 12. The top's configuration is described by its precession angle φ, the nutation angle θ, the spin angle ψ, and the X and Y positions X_P and Y_P of point P, such that the vector of generalized coordinates is $\boldsymbol{q} = (\varphi,\, \theta,\, \psi,\, X_P,\, Y_P)^{\mathrm{T}}$.

The rotation matrix ${}^{\mathcal{N}}\mathbf{C}_{\mathcal{F}}$ is given as:

$${}^{\mathcal{N}}\mathbf{C}_{\mathcal{F}} = \begin{pmatrix} \cos\varphi & -\sin\varphi\cos\theta & \sin\varphi\sin\theta \\ \sin\varphi & \cos\varphi\cos\theta & -\cos\varphi\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \tag{7.52P}$$

where triad $\mathcal{F}$ does not spin along with the top, but does perform the same precession and nutation. Using Virtual Work, calculate the components of the generalized force vector $\boldsymbol{Q}_g$ associated with the gravitational force acting on the top, as functions of m_{s}, m_{k}, d, g, φ, and θ. ▷▶

7.2.3 Virtual Work of Ideal Constraint Forces

Whenever constraints act on a system, there are potentially also constraint forces or moments. In fact most of the forces introduced when drawing free-body diagrams are constraint forces, because one cuts parts loose from each other at the constraints, such as contact points or joints. In effect, drawing free-body diagrams exposes the constraint forces. All constraints where the constraint forces do no (virtual) work are defined as *ideal constraints*.

Example 7.15 We consider a roller joint, as depicted in an example system in Figure 7.7.

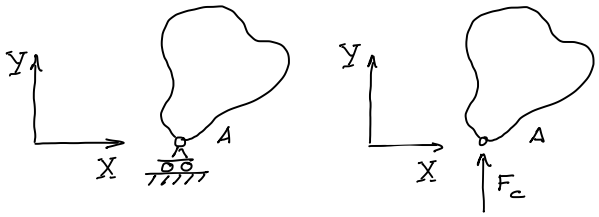


Figure 7.7: Left: A rigid body on an ideal roller joint. Right: FBD of the body.

The constraint force has the form $\boldsymbol{F}_\mathrm{c} = (0,\, F_\mathrm{c})^\mathrm{T}$, and it acts in point A. The virtual work that is done by this force can be found by

$$\delta W_\mathrm{c} = \boldsymbol{F}_\mathrm{c} \cdot \delta \boldsymbol{r}_A. \tag{7.53E}$$

Given that the virtual displacement is always orthogonal to the constraint force, the scalar product is zero, so no virtual work is done.

Example 7.16 We can also find the virtual work done by internal forces in a massless link, as depicted in an example system in Figure 7.8.

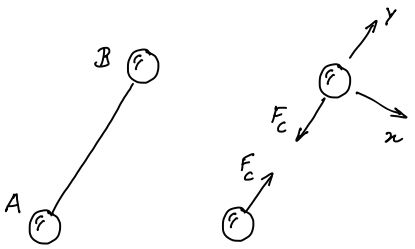


Figure 7.8: Left: A link connecting two particles. Right: FBD of the two particles.

In this case, we need to consider two forces: The one exerted on particle A

and the one on particle B:

$$\delta W_{\mathrm{c}} = \boldsymbol{F}_{\mathrm{c}A} \cdot \delta \boldsymbol{r}_A + \boldsymbol{F}_{\mathrm{c}B} \cdot \delta \boldsymbol{r}_B. \tag{7.54E}$$

We know that the force the link exerts on particle A must be equal and opposite to the one exerted on particle B, so $\boldsymbol{F}_{\mathrm{c}B} = -\boldsymbol{F}_{\mathrm{c}A}$. This allows us to reduce (7.54E) to

$$\delta W_c = \boldsymbol{F}_{\mathrm{c}A} \cdot (\delta \boldsymbol{r}_A - \delta \boldsymbol{r}_B). \tag{7.55E}$$

We may do calculations in any triad of reference, so we choose a triad where one axis y is aligned with the massless link. We know that this is also the line of action of the forces in the link. Simultaneously, we know that there cannot be any relative movement of the particles in y-direction, because then the length of the link would have to change. So, while $(\delta \boldsymbol{r}_A - \delta \boldsymbol{r}_B)$ can only have components in directions orthogonal to y, $\boldsymbol{F}_{\mathrm{c}A} = (0,\, F_{\mathrm{c}})^{\mathrm{T}}$ has only a component in direction of y. Therefore, the scalar product is again zero, and no virtual work can be done.

Other examples of ideal constraints are rolling contact without slip, or spherical joints. A non-ideal constraint could for example be a roller joint with friction. In that case, the friction force can do (virtual) work.

Problem 7.15 This Problem is about the free rolling non-slipping disk of Example 3.6 on page 82. Assume one of the generalized coordinates is the angle θ. Determine the contributions of the interaction forces between the disk and the ground to the generalized force Q_θ associated with this generalized coordinate. ▶

As they have so different properties in terms of virtual work, we will distinguish from now on between general *applied* forces $\boldsymbol{F}_{\mathrm{a}i}$ and ideal constraint reaction forces $\boldsymbol{R}_i$. Note that applied forces in a system of particles or rigid bodies can be internal or external for the system.

7.2.4 Virtual Work and d'Alembert's Principle

In the principle of virtual work, we have only considered static equilibrium. In order to extend the principle to the case of dynamically moving systems, we can combine it with *d'Alembert's principle*. Jean le Rond d'Alembert (1717-1783) re-interpreted Newton's second law by moving all terms to the left side of the equation [41]. For one particle i, with a resultant force $\boldsymbol{F}_i$ acting on it, the law reads:

$$\boldsymbol{F}_i - m\ddot{\boldsymbol{r}}_i = \boldsymbol{0}. \tag{7.56}$$

If we now re-interpret the term $-m\ddot{\boldsymbol{r}}_i$ as an *inertial force*, we can consider (7.56) as *dynamic equilibrium* of forces, just as we did for static

equilibrium. Note however, that this is just an interpretation, the term $-m\ddot{\boldsymbol{r}}_i$ is only a pseudo- or fictitious force, needed for interpretation. We *never* draw these "forces" into a free-body diagram.

Problem 7.16 Consider a point mass of mass m that is connected to a fixed point O via a massless rigid rod of length l, and which moves in the xy-plane under the influence of gravity g, as shown in Figure 7.9. The location of the point mass is described by the angle φ of the rod with respect to the vertical.

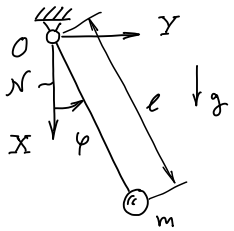


Figure 7.9: A pendulum consisting of a point mass and a massless rigid rod.

Draw a free-body diagram of the point mass. Do not include D'Alembert forces in your diagram. ▶

If we consider again the acceleration of a point in a rotating coordinate system, we can interpret diverse acceleration terms in terms of fictitious *d'Alembert forces*:

$$-m\ddot{\boldsymbol{r}}_P = -m\left(\ddot{\boldsymbol{R}} + (\ddot{\boldsymbol{\rho}})_{\mathcal{B}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + 2\boldsymbol{\omega} \times (\dot{\boldsymbol{\rho}})_{\mathcal{B}}\right). \tag{7.57}$$

Note to *not confuse* the vector $\ddot{\boldsymbol{R}}$ and the ideal constraint force $\boldsymbol{R}_i$ that acts on point i, they are unrelated[2].

The term $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$ can be interpreted as a *centrifugal force*, which is in equilibrium with the forces needed to keep the point on a circular trajectory. The centrifugal force is only a pseudo-force needed for the thought experiment of dynamic equilibrium.

Similarly, the term $-2m\boldsymbol{\omega} \times (\dot{\boldsymbol{\rho}})_{\mathcal{B}}$ can be interpreted as a *Coriolis force*. Also this force is purely a pseudo force, needed to explain movement in a rotating coordinate system. For example, this force explains why the plane in which Foucault's pendulum [18] oscillates appears to change its direction. In reality, there is no such force, and the observed

[2] As the alphabet has only 26 letters, we sometimes use definitions that are only locally valid. Indices could be a solution, but also make reading more cumbersome.

change in direction is only due to the fact that the observer is standing inside a rotating coordinate system (Earth).

D'Alembert's principle and the principle of virtual work can be combined, in order to yield for one single particle the condition for dynamic equilibrium:

$$\delta W_i = (\boldsymbol{F}_i - m_i \ddot{\boldsymbol{r}}_i) \cdot \delta \boldsymbol{r}_i = 0. \tag{7.58}$$

In the context of virtual work, as explained in Section 7.2.3, it is advisable to distinguish between applied forces and ideal constraint forces. Therefore, we re-write (7.58) to

$$\delta W_i = (\boldsymbol{F}_{\mathrm{a}i} + \boldsymbol{R}_i - m_i \ddot{\boldsymbol{r}}_i) \cdot \delta \boldsymbol{r}_i = 0. \tag{7.59}$$

This allows us to eliminate the ideal constraint forces, because they do no virtual work, so

$$\delta W_i = (\boldsymbol{F}_{\mathrm{a}i} - m_i \ddot{\boldsymbol{r}}_i) \cdot \delta \boldsymbol{r}_i = 0. \tag{7.60}$$

If we sum over all N particles, we obtain

$$\delta W = \sum_{i=1}^{N} (\boldsymbol{F}_{\mathrm{a}i} - m_i \ddot{\boldsymbol{r}}_i) \cdot \delta \boldsymbol{r}_i = 0. \tag{7.61}$$

Another name of this combination is *Lagrange's form of d'Alembert's principle*.

Note that static equilibrium is a special case of (7.61), where all accelerations $\ddot{\boldsymbol{r}}_i$ are zero.

With the principle of virtual work and d'Alembert's principle combined as in (7.61), again only allowed displacements can be applied, as done earlier for static equilibrium. However, this time, the procedure will provide the instantaneous relationships between forces and accelerations (given current configuration and velocities of the system), i.e. the *equations of motion* of the system.

◯ **Problem 7.17** Would (7.61) still hold if one accidentally included the ideal constraint forces as well within the $\boldsymbol{F}_{\mathrm{a}i}$?

To apply (7.61) to a practical problem, it is again convenient to map it to configuration space. Therefore, we need to find also the generalized forces associated with the inertial or d'Alembert forces of the system, the same way as was done for applied forces. This will be outlined in the following.

7.2.5 Generalized d'Alembert Forces

The condition for dynamic equilibrium (7.61) can also be written as a function of generalized forces Q_j and generalized d'Alembert's (pseudo) forces A_j.

The virtual displacements $\delta \boldsymbol{r}_i$ in Cartesian space and the virtual displacements δq_j in configuration space are related by (7.37), while generalized forces Q_j in configuration space are related to Cartesian forces $\boldsymbol{F}_i$ by (7.50).

Therefore, we can re-write (7.61) in configuration space as:

$$\delta W = \sum_{j=1}^{n} \left(Q_j + A_j(q_j,\, \dot{q}_j,\, \ddot{q}_j) \right) \delta q_j = 0, \tag{7.62}$$

where a symbol is introduced for the generalized d'Alembert forces

$$A_j = -\sum_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}. \tag{7.63}$$

In the special case where the generalized coordinates q_j are *independent*, the variations δq_j can be chosen arbitrarily, and therefore all $(Q_j + A_j)$ in (7.62) must be zero. This directly leads to n equations of motion:

$$Q_j + A_j = 0, \qquad j = 1 \ldots n. \tag{7.64}$$

It is important to remember this only holds for the case of independent q_j, otherwise only allowed variations of q_j may be considered in (7.62).

In general, we will find as many independent equations of motion as we have DoF. So, in case we have chosen a minimal set of generalized coordinates, we will find exactly n independent equations of motion. The equations, in combination with any constraint equations, allow (numerical) integration to find the movement as a function of time. Note that the equations of motion are often coupled at acceleration level. To allow for integration, one would normally try to decouple them and solve for the accelerations as functions of the current configuration, velocities, and applied forces. In the case where constraint equations exist, it is convenient to re-write them at acceleration level as well, to enable combination.

Example 7.17 This Example is about the massles rod on a roller joint of Example 7.13 on page 232. For the example system, we can find the

virtual work of all applied forces and all d'Alembert forces as

$$\delta W = Q_s \delta s + Q_\theta \delta\theta - m\ddot{\boldsymbol{r}}_A \cdot \delta \boldsymbol{r}_A - m\ddot{\boldsymbol{r}}_B \cdot \delta \boldsymbol{r}_B = 0. \tag{7.65E}$$

In order to obtain the accelerations, we take the derivatives of the position vectors in Cartesian space, which are

$$\boldsymbol{r}_A = \begin{pmatrix} s & 0 \end{pmatrix}^{\mathrm{T}}, \qquad \boldsymbol{r}_B = \begin{pmatrix} s + l\cos\theta & l\sin\theta \end{pmatrix}^{\mathrm{T}}. \tag{7.66E}$$

Starting with particle A, we find its acceleration as $\ddot{\boldsymbol{r}}_A = (\ddot{s},\, 0)^{\mathrm{T}}$. We can do the same for particle B (See Problem 7.18).

Applying the principle of virtual work and d'Alembert's principle, we know

$$\delta W = (Q_s + A_s)\delta s + (Q_\theta + A_\theta)\delta\theta = 0. \tag{7.67E}$$

As virtual displacements for the generalized coordinates s and θ can be chosen independently, both terms in brackets must equal zero.

Substituting all forces (applied forces and d'Alembert forces), we find that the net generalized force component $Q_s + A_s$ in s-direction is

$$Q_s + A_s = F_A - 2m\ddot{s} + ml\ddot{\theta}\sin\theta + ml\dot{\theta}^2\cos\theta. \tag{7.68E}$$

Therefore, we find

$$F_A - 2m\ddot{s} + ml\ddot{\theta}\sin\theta + ml\dot{\theta}^2\cos\theta = 0. \tag{7.69E}$$

This is one of the two equations of motion of the system. The same way, we can find the generalized force component in θ-direction, to obtain the second equation of motion (see Problem 7.19).

The equations of motion here are indeed coupled at acceleration level, so to enable integration one would try to decouple them and solve for the accelerations $\ddot{s}$ and $\ddot{\theta}$ as separate functions of the current configuration (s, θ), velocities $(\dot{s}, \dot{\theta})$, and the applied forces F_A and F_B.

Problem 7.18 Find the acceleration of particle B in Figure 7.6, as a function of the generalized coordinates s, θ and their derivatives $\dot{s}, \dot{\theta}, \ddot{s}, \ddot{\theta}$, and the constant parameter l. You can make use of (7.66E).

Problem 7.19 Apply the principle of virtual work and d'Alembert's principle to find the second equation of motion for the example system in Figure 7.6, complementing (7.69E).

Problem 7.20 This Problem is about the governor of Example 2.2 on page 48. The spring is not present anymore, as in $k = 0$, and also $\dot{\psi}$ is constant and equal to ω. For this movement, there exist constant "equilibrium" values for θ such that this angle remains constant. Use the principle

of virtual work, together with d'Alembert inertial forces, to calculate these equilibrium values as a function of ω and the given constant parameters m, L, g.

Include a drawing of virtual displacements in your reasoning. (Hint: when $\dot{\psi}$ is constant then $\ddot{\psi} = 0$). ▶

7.3 Summary

The number of degrees of freedom of a mechanical system is equal to the number of independent possibilities for a system to move. This number can be different in configuration space and in velocity space, although we normally refer to DoF in velocity space.

Generalized coordinates are parameters that locally describe the kinematic configuration of a system. So, they span the configuration space. Often, it is possible and advisable to choose a minimal set of generalized coordinates, which are then independent of each other. If this is not possible, m constraint equations between the n generalized coordinates remain, such that the number of DoF equals $n - m$.

In the case of holonomic constraints, the constraint equation can be written as a function only of the generalized coordinates, not of their derivatives. Only if all constraints are holonomic, a system is holonomic. Non-holonomic constraints only exist in velocity space, they cannot be integrated to equivalent functions in configuration space. Therefore, for non-holonomic systems, it is not possible to eliminate some seemingly redundant generalized coordinates by formulating them as functions of others, so one cannot choose a minimal set of generalized coordinates.

Virtual displacements $\delta \boldsymbol{r}_i$ in Cartesian space are related to virtual displacements δq_j in configuration space by:

$$\delta \boldsymbol{r}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j. \tag{7.70}$$

The principle of virtual work enables the mapping of forces and moments in Cartesian space to generalized forces in configuration space. The virtual work of both has to be identical, which means that

$$\delta W_i = \boldsymbol{Q}_i \cdot \delta \boldsymbol{q} = \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i. \tag{7.71}$$

Then, the components of $\boldsymbol{Q}_i$ can be found by comparing coefficients. Alternatively, one can directly use partial derivatives to find the equivalent generalized force vector $\boldsymbol{Q}_i$ to a Cartesian force vector $\boldsymbol{F}_i$ from

$$Q_{ij} = \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}. \tag{7.72}$$

In combination with d'Alembert's principle, the principle of virtual work delivers the equations of motion for a system of particles, using generalized d'Alembert forces A_j:

$$\delta W = \sum_{j=1}^{n} (Q_j + A_j)\, \delta q_j = 0. \tag{7.73}$$

We will always obtain n equations of motion, which need to be combined with any remaining constraint equations to solve.

7.4 Problems

7.4.1 Guided Problems

Problem 7.21 We consider again a centrifugal governor as in Example 2.2, which usually controls the driving power in a steam engine. The original type, introduced by James Watt, is shown in Figure 7.10.

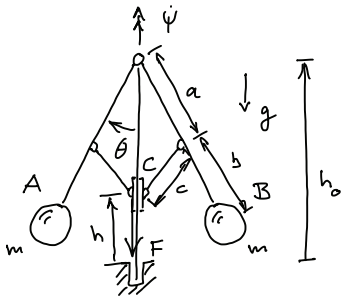


Figure 7.10: Centrifugal governor.

The central spindle, to which the arms carrying the two balls are hinged, rotates at a rate $\dot{\psi}$ that is proportional to the speed of the engine.

If this rotation is uniform, the balls, under the action of gravity and centrifugal "force", take up a definite "equilibrium" position, which depends on the speed $\dot{\psi}$. If the speed increases, the balls diverge outwards, raising the collar C. To this collar, some system of levers can be connected which can turn a valve so as to reduce the supply of steam. Conversely, when the speed diminishes, the collar descends, and the supply of steam is reinforced.

A force F is applied at the collar C in the downward direction, representing the reaction of the system of levers and valve.

The angle θ denotes the inclination of the arms with respect to the spindle, the angle ψ denotes the rotation of the spindle, and h is the height of the collar C with respect to the support.

In a number of steps, we will derive the equations of motion of this system.

To simplify equations, you may assume $a = b = c$, that all parts of the system are massless except the two balls, and that these balls are particles, each having a mass m.

a. State how many degrees of freedom this system has. Also state two different possible minimal sets of generalized coordinates.
b. Choose a fixed point O and an inertial triad $\mathcal{N}$. Include orthographic drawings where you clearly define the kinematic configuration of the governor with respect to your chosen fixed point and your triad. Make sure your drawings are generic, which may require defining and including further triads.
c. Consider the generalized coordinates $\boldsymbol{q} = (\psi, \theta)^{\mathrm{T}}$ for this system. Calculate the components of the position vectors ${}^{\mathcal{N}}\boldsymbol{r}_{C/O}$, ${}^{\mathcal{N}}\boldsymbol{r}_{A/O}$, and ${}^{\mathcal{N}}\boldsymbol{r}_{B/O}$ of the collar and of the two balls, respectively, with respect to your point O, expressed in your triad $\mathcal{N}$. Express these components as functions of the generalized coordinates (and of the given constants).
d. Use virtual work, your answer to part c, and partial derivatives to calculate the contribution of the force F to the "vector" of generalized applied forces $\boldsymbol{Q}$.
e. Visualize the virtual displacements of the collar height h resulting from virtual displacements of the generalized coordinates in two separate drawings, one for each generalized coordinate. You may use 3-D perspective, and you may also include additional enlarged drawings of details where needed.
f. Use the drawings created in part e to determine the virtual work δW done by the force F for each of the separate virtual displacements. Then, derive the contribution of F to $\boldsymbol{Q}$ again from your answer. Compare to the result you found in part d.
g. Again use two different methods (as above) to derive the contribution of gravity acting on the particles A and B to the generalized applied forces $\boldsymbol{Q}$.
h. Use the transport theorem and suitable triads to calculate the acceleration vectors of particles A and B expressed in a clearly defined triad of your choice, with components as functions of the generalized coordinates ψ and θ and their first and second derivatives with respect to time (and of the given constants).
i. Use your answer to part h to calculate the d'Alembert "force" vectors acting on the particles A and B, with components expressed in a clearly defined triad of your choice. Then, use again two different

methods (as above) to determine the generalized applied d'Alembert "forces" $\boldsymbol{A}$. Be aware that these are all just pseudo-forces.

j. Use your responses to the previous parts, the principle of virtual work, and d'Alembert's principle to derive the equations of motion for the system in terms of the generalized coordinates $\boldsymbol{q} = (\theta, \psi)^{\mathrm{T}}$, under the influence of force F and gravity. You do not need to decouple the equations, which means you may leave them as second-order *implicit* differential equations.

k. Apply at least five independent plausibility checks to verify your equations of motion. Include at least one check where you make a clear prediction involving *static* equilibrium.

l. Apply Newton's second law, as an alternative to derive the equations of motion of the system, and compare to your above result. Do not forget to draw clear free-body diagrams first, and do *not* include any d'Alembert "forces" in these drawings.

Problem 7.22 Consider the 2-D backhoe system with attached roller of radius r from Figure 7.11. The roller rotates about D and rolls perfectly without slip with respect to the ground in the contact point P. A hydraulic actuator controls the length l.

For this system a set of generalized coordinates $\boldsymbol{q}$ has been selected such that $\boldsymbol{q} = (\alpha, \beta, \gamma, s, l, \varphi)^{\mathrm{T}}$.

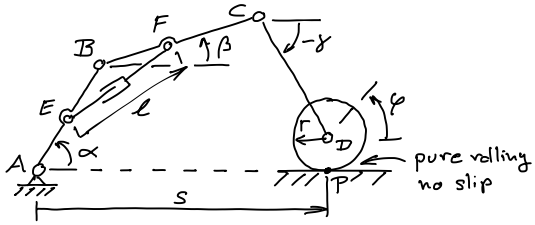


Figure 7.11: Backhoe with roller.

a. How many degrees of freedom does this system have? Distinguish the cases where the actuator imposes a certain l and where l is free.

b. Derive the constraint equation(s) that express(es) a relation between the given generalized coordinates and classify this/these constraint(s). Classify in terms of holonomic/non-holonomic and rheonomic/scleronomic. To simplify your calculations, you may assume that the position vectors from any one point to another fulfill $|\boldsymbol{r}_{E/A}| = |\boldsymbol{r}_{B/E}| = |\boldsymbol{r}_{F/B}| = |\boldsymbol{r}_{C/F}| = \frac{1}{2}|\boldsymbol{r}_{D/C}| = L$, where L is a constant.

c. Give an example for a minimal set of generalized coordinates for this system.

Problem 7.23 A rigid 2-D pendulum swing is supported by a cable (Figure 7.12). The support point P of the rigid pendulum, with coordinates (X_P, Y_P), can move freely along the cable.

The cable is supported in two fixed points A and B with known coordinates (X_A, Y_A) and (X_B, Y_B), respectively. We assume that the cable will always be in tension, and that it is inextensible, with a total length of l. The distance between P and A is denoted by s.

The angle of the rigid pendulum with the down vertical is φ, with pendulum length r, and the center of mass is in C with coordinates X_C, Y_C. The swing has mass m and negligible mass moment of inertia with respect to its center of mass.

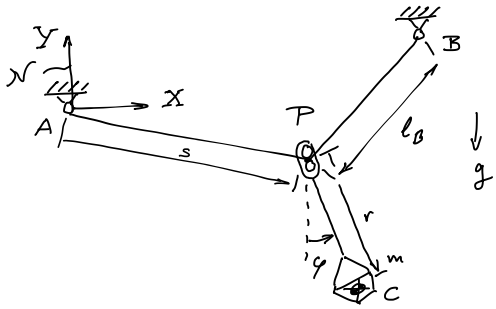


Figure 7.12: A cable supported swing.

Gravity with acceleration constant g acts in the negative Y-direction. For this system, a set of generalized coordinates $\boldsymbol{q}$ has been selected such that $\boldsymbol{q} = (X_C, Y_C, X_P, Y_P, s, \varphi)^{\mathrm{T}}$.

a. How many degrees of freedom does this system have?
b. Derive the constraint equation(s) that express(es) a relation between the given generalized coordinates and classify this/these constraint(s). Classify in terms of holonomic/non-holonomic.
c. Give an example for a minimal set of generalized coordinates for this problem.
d. Use the principle of virtual work and d'Alembert's principle to derive the equations of motion for the system in terms of the generalized coordinates $\boldsymbol{q} = (s, \varphi)^{\mathrm{T}}$. To make your expressions less lengthy, assume that the cable is perfectly tight (although this is not realistic), so the distance between A and B is equal to l.

Problem 7.24 Consider a single-speed bicycle, with the right pedal being in the bottom position, see Figure 7.13. One person is holding the bicycle upright and with both wheels aligned. A second person pushes backwards on the right pedal.

Given is the wheel radius r_w, the pedal crank length r_p, the front chain wheel radius r_f and the rear chain wheel radius r_r. Consider the wheels massless.

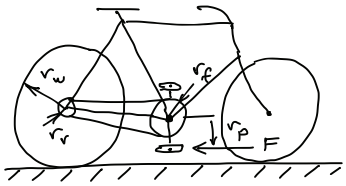


Figure 7.13: Bicycle.

a. In which direction will the bicycle start to move (potentially depending on the given parameters)? Use the principles of d'Alembert and of virtual work in your argumentation. Also define an xyz-coordinate system to express your displacements and rotations in.
b. In which direction will a standard Dutch city bicycle start to move: forward or backward? First just guess using your mechanical intuition. Next go outside and measure for a typical Dutch standard bike values for r_w, r_p, r_f and r_r and check your intuitive guess with the results as found in part a. Finally, take a bicycle and confirm your result experimentally.
c. What will happen if the right crank is positioned horizontally and towards the front (so the right pedal is at the most forward position), and the force on the pedal points downwards? In what direction will the bicycle start to move?

7.4.2 Practice Problems

Problem 7.25 This Problem is about the scooter with springboard of Problem 1.31 on page 43. We consider the mechanical system consisting of scooter and person. Classify each one of the following forces as to whether it is conservative, an ideal constraint force, an external force, and/or a d'Alembert force:

- force in leaf spring,
- weight of the person's left arm,
- muscle force of the person's biceps,
- shear force in point B in y-direction,
- normal force exerted by ground on front wheel.

▶

Problem 7.26 This Problem is about the hoverboard of Problem 3.26 on page 95. A student found the following constraint equation for the hoverboard:

$$\dot{C}_X \cos\theta - \dot{C}_Y \sin\theta = 0 \tag{7.74P}$$

Check the plausibility of this equation with three different, independent plausibility checks. For each check, first state a precise condition you expect to hold for the equation, and a reasoning why you expect this to hold. Only then, perform the check and draw a conclusion. ▶

Problem 7.27 This Problem is about the hoverboard of Problem 3.26 on page 95. is the constraint of (7.74P) on page 248 a holonomic constraint? ▷

Problem 7.28 This Problem is about the hoverboard of Problem 3.26 on page 95. Calculate the expression for yaw rate $\dot{\theta}$ of the hoverboard, as a function of $\dot{\alpha}, \dot{\beta}, R$, and d. ▷

Problem 7.29 This Problem is about the skateboard of Problem 1.24 on page 39. The yaw rate of the skateboard $\dot{\psi}$ is kinematically determined by the forward speed v of the skateboard and the truck steer angle δ. For small steer angles, this can be approximated by the linear kinematic relation $\dot{\psi} = (-2v/l)\delta$.

However, this is only a linear approximation for small angles. Determine for large angles the yaw rate $\dot{\psi}$ of the skateboard as a function of v, l, δ. ▷

Problem 7.30 This Problem is about the quadcopter on a pyramid of Problem 3.35 on page 100. The orientation of the quadcopter is now described by Euler Type II angles:

$$
{}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}} = \begin{pmatrix} \cos\psi\cos\alpha - \sin\psi\cos\theta\sin\alpha & -\cos\psi\sin\alpha - \sin\psi\cos\theta\cos\alpha & \sin\psi\sin\theta \\ \sin\psi\cos\alpha + \cos\psi\cos\theta\sin\alpha & -\sin\psi\sin\alpha + \cos\psi\cos\theta\cos\alpha & -\cos\psi\sin\theta \\ \sin\theta\sin\alpha & \sin\theta\cos\alpha & \cos\theta \end{pmatrix}. \tag{7.75P}
$$

Here, ψ is the precession angle, θ is the nutation angle, and α is the spin angle.

In point A, which is located in the center of the first rotor (at a distance $\frac{L}{2}$ from C, see figure), a force of magnitude F_A is applied which is always directed in the inertial Z- direction. Calculate the generalized forces Q_θ and Q_ψ associated with this force, as functions of F_A, ψ, θ, α, and the given constant L (Remark: None of these terms has to necessarily appear in the requested functions). ▶

8 Lagrange's Equations

Lagrange's method extends the concept of virtual work and enables the derivation of equations of motion for arbitrary dynamic systems in a very systematic way. Its key idea is to link generalized forces to kinetic energy via variational operations. In the case of independent generalized coordinates, ideal constraint forces do not need to be considered. Calculations can be further simplified by re-writing conservative forces in terms of their potential. This makes Lagrange's method particularly suited to derive the equations of motion for conservative systems with many ideal constraints. However, also in the case of dependent generalized coordinates, the Lagrangian framework enables a solution, using Lagrange multipliers.

8.1 Lagrange for Independent Coordinates

8.1.1 D'Alembert Forces and Kinetic Energy

An important finding so far was that d'Alembert's principle and the principle of virtual work can be combined, in order to yield the condition for dynamic equilibrium (7.61) or (7.62). Remember that as $\boldsymbol{F}_{\mathrm{a}i}$ we need to include only the applied forces and no ideal constraint forces, because those cannot do virtual work for any virtual displacements $\delta\boldsymbol{r}_i$ that obey the constraints (see Problem 7.17).

In this notation, finding the equations of motion requires explicitly finding all the d'Alembert forces and their generalized equivalents. This can be cumbersome.

A very elegant and easy-to-apply solution to this issue was presented by Joseph-Louis Lagrange (1736 - 1813) in [38]. Lagrange wrote the generalized d'Alembert forces A_j as functions of kinetic energy T of the mechanical system:

$$A_j = \frac{\partial T}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right). \tag{8.1}$$

A proof of this relationship is provided below. The advantage of re-writing the d'Alembert terms this way is that we can derive the equations of motion of systems of particles or rigid bodies in a very formalized way, using the system's overall kinetic energy. If we look back at the notation of virtual work in (7.62), with (8.1) we can re-write it as

$$\delta W = \sum_{j=1}^{N} \left(Q_j + \frac{\partial T}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \right) \delta q_j = 0,\ j = 1 \dots n. \tag{8.2}$$

For the case of *independent* generalized coordinates, we find the fundamental relationship:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j. \tag{8.3}$$

Remember that independent generalized coordinates are only possible if the number of generalized coordinates is equal to the number of DoF of a system in velocity space. That means that a system with non-holonomic constraints cannot be treated in this way.

The proof for (8.1) will be given in the following. First, we prove that for each single particle

$$A_{ij} = -m_i \ddot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} = \frac{\partial T_i}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T_i}{\partial \dot{q}_j} \right). \tag{8.4}$$

Here, the position vector $\boldsymbol{r}_i$ of the i-th particle is expressed as a function of generalized coordinates, so $\boldsymbol{r}_i = \boldsymbol{r}_i(q_1,\, q_2 \dots q_j \dots q_n,\, t)$. From Section 2.2, we remember that kinetic energy of a particle i is

$$T_i = \frac{1}{2} m_i |\dot{\boldsymbol{r}}_i|^2 \tag{8.5}$$

$$= \frac{1}{2} m_i \left(\dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i \right). \tag{8.6}$$

Now, we will calculate all derivatives in (8.4). Note that the partial derivative, as noted by the operator ∂, is different from the total derivative by d. This is explained in more detail in Appendix B.

The first term of Lagrange's notation (8.4) is the partial derivative of T_i with respect to q_j, so $\frac{\partial T_i}{\partial q_j}$. Applying the product rule of differentiation to (8.6) gives

$$\frac{\partial T_i}{\partial q_j} = \frac{1}{2} m_i \left(\frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j} \cdot \dot{\boldsymbol{r}}_i + \dot{\boldsymbol{r}}_i \cdot \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j} \right) \tag{8.7}$$

$$= m_i \dot{\boldsymbol{r}}_i \cdot \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j}. \tag{8.8}$$

The second term in Lagrange's notation (8.4) contains the partial derivative of T_i with respect to $\dot{q}_j$, so $\frac{\partial T_i}{\partial \dot{q}_j}$. The procedure is very similar:

$$\frac{\partial T_i}{\partial \dot{q}_j} = \frac{1}{2} m_i \left(\frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_j} \cdot \dot{\boldsymbol{r}}_i + \dot{\boldsymbol{r}}_i \cdot \frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_j} \right) \tag{8.9}$$

$$= m_i \dot{\boldsymbol{r}}_i \cdot \frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_j}. \tag{8.10}$$

The position vector $\boldsymbol{r}_i$ of the i-th particle is expressed as a function of generalized coordinates and time t, so $\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2 \dots q_j \dots q_n, t)$. If we take the derivative with respect to time, the chain rule gives:

$$\dot{\boldsymbol{r}}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \boldsymbol{r}_i}{\partial t}. \tag{8.11}$$

The partial derivative of this equation with respect to $\dot{q}_j$ is (recall we treat $\dot{q}_j$ as if it was fully independent of q_j, t, and all other $\dot{q}_k$, with $k \neq j$):

$$\frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_j} = \frac{\partial \boldsymbol{r}_i}{\partial q_j}. \tag{8.12}$$

So, we can re-write (8.10) to

$$\frac{\partial T_i}{\partial \dot{q}_j} = m_i \dot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}. \tag{8.13}$$

To obtain the second term of Lagrange's notation (8.4), we need to take the derivative with respect to time of (8.13). Again, we apply the product rule of differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T_i}{\partial \dot{q}_j} \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(m_i \dot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \right) \tag{8.14}$$

$$= m_i \left(\ddot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} + \dot{\boldsymbol{r}}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \boldsymbol{r}_i}{\partial q_j} \right) \right). \tag{8.15}$$

We can further show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \boldsymbol{r}_i}{\partial q_j} \right) = \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j}, \tag{8.16}$$

because (recall the position vector is a function of both generalized coordinates and time)

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \boldsymbol{r}_i}{\partial q_j} \right) = \sum_{l=1}^{n} \frac{\partial^2 \boldsymbol{r}_i}{\partial q_l \partial q_j} \dot{q}_l + \frac{\partial^2 \boldsymbol{r}_i}{\partial t \partial q_j} \tag{8.17}$$

is identical to the partial derivative of (8.11) with respect to q_j:

$$\frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\sum_{l=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_l}\dot{q}_l + \frac{\partial \boldsymbol{r}_i}{\partial t}\right) = \sum_{l=1}^{n} \frac{\partial^2 \boldsymbol{r}_i}{\partial q_j \partial q_l}\dot{q}_l + \frac{\partial^2 \boldsymbol{r}_i}{\partial q_j \partial t}. \tag{8.18}$$

Note that we used Schwarz' theorem (Karl Hermann Amandus Schwarz, 1843-1921), which states that the sequence of partial second derivatives does not matter as long as the derivatives are continuous. Therefore, the expressions $\frac{\partial^2}{\partial q_j \partial q_l}$ and $\frac{\partial^2}{\partial q_l \partial q_j}$ are equivalent.

So when we substitute (8.16) in (8.15), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_i}{\partial \dot{q}_j}\right) = m_i \ddot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} + \underbrace{m_i \dot{\boldsymbol{r}}_i \cdot \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j}}_{\frac{\partial T_i}{\partial q_j}}. \tag{8.19}$$

The second term is equal to the first part of Lagrange's notation: $\frac{\partial T_i}{\partial q_j}$, (8.8).

Finally, if we substitute (8.8) and (8.19) in (8.4), we obtain:

$$A_{ij} = \frac{\partial T_i}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_i}{\partial \dot{q}_j}\right) \tag{8.20}$$

$$= m_i \dot{\boldsymbol{r}}_i \cdot \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j} - \left(m_i \ddot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} + m_i \dot{\boldsymbol{r}}_i \cdot \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j}\right) \tag{8.21}$$

$$= -m_i \ddot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \quad \text{q.e.d..} \tag{8.22}$$

Therefore, because $A_j = \sum_{i=1}^{N} A_{ij}$ and $T = \sum_{i=1}^{N} T_i$, we find (8.1).

8.1.2 Lagrange's Equations with Conservative Forces

We can simplify (8.3) further for a system that has conservative forces acting on it. As derived in Section 2.2.5, the work done by a conservative force, such as gravity for example, can be expressed as a function of the start and end position of its point of application only. This saves integration work since the path the point of application takes from start to the end does not affect the work done on the system.

We will split the generalized forces Q_j into a conservative part Q_{cj} and a part Q'_j that is non-conservative (or for which it is unknown

whether or not it is conservative):

$$Q_j = Q_{\mathrm{c}j} + Q'_j. \tag{8.23}$$

We will now focus on the conservative part $Q_{\mathrm{c}j}$. As we discussed in Section 2.2.5, there must be an associated potential V for all conservative forces.

We found in (2.55) that a conservative force $\boldsymbol{F}$ in Cartesian space is equal to the negative gradient of an associated scalar function $V(\boldsymbol{r})$, the potential energy, which only depends on position. Now, we express a potential V as a function of the generalized coordinates, so $V = V(\boldsymbol{q})$, such that we can find an equivalent relationship in configuration space: The generalized conservative force $\boldsymbol{Q}_c = (Q_{\mathrm{c}1},\, Q_{\mathrm{c}2} \dots Q_{\mathrm{c}j} \dots Q_{\mathrm{c}n})^{\mathrm{T}}$ can be calculated from the N conservative forces in the system as

$$Q_{\mathrm{c}j} = \sum_{i=1}^{N} \left(F_{iX} \frac{\partial r_{iX}}{\partial q_j} + F_{iY} \frac{\partial r_{iY}}{\partial q_j} + F_{iZ} \frac{\partial r_{iZ}}{\partial q_j} \right), \tag{8.24}$$

where the i-th conservative force has the Cartesian force components F_{iX}, F_{iY}, and F_{iZ} and its point of application has the position vector $\boldsymbol{r}_i$ with components r_{iX}, r_{iY}, and r_{iZ}, all in an inertial coordinate system XYZ.

We can substitute components of the force using its associated potential function V_i:

$$F_{iX} = -\frac{\partial V_i(\boldsymbol{r}_i)}{\partial r_{iX}}, F_{iY} = -\frac{\partial V_i(\boldsymbol{r}_i)}{\partial r_{iY}}, F_{iZ} = -\frac{\partial V_i(\boldsymbol{r}_i)}{\partial r_{iZ}}, \tag{8.25}$$

such that (8.24) transforms into

$$Q_{\mathrm{c}j} = -\sum_{i=1}^{N} \left(\frac{\partial V_i(\boldsymbol{r}_i)}{\partial r_{iX}} \frac{\partial r_{iX}}{\partial q_j} + \frac{\partial V_i(\boldsymbol{r}_i)}{\partial r_{iY}} \frac{\partial r_{iY}}{\partial q_j} + \frac{\partial V_i(\boldsymbol{r}_i)}{\partial r_{iZ}} \frac{\partial r_{iZ}}{\partial q_j} \right). \tag{8.26}$$

The chain rule of differentiation enables simplification to

$$Q_{\mathrm{c}j} = -\sum_{i=1}^{N} \frac{\partial V_i(\boldsymbol{r}_i(q_1, ..., q_j, ..., q_n))}{\partial q_j}. \tag{8.27}$$

With the definition of a joint potential

$$V := \sum_{i=1}^{N} V_i \tag{8.28}$$

we finally arrive at:

$$Q_{cj} = -\frac{\partial V(q_1, ..., q_j, ..., q_n)}{\partial q_j}. \tag{8.29}$$

So, the generalized conservative force is equal to the negative gradient of the potential energy V in configuration space.

Problem 8.1 This Problem is about the tube with a particle in it of Problem 4.35 on page 142. The connection between the particle and point O is elastic, so L is not constant. The unstretched length of the connection is 0. Movements of both angles α and β are allowed.

Write down a potential energy function V for the system which includes all conservative forces as functions of α, $\dot{\alpha}$, β, $\dot{\beta}$, L and the given constants k, m and g (Remark: None of these terms has to necessarily appear in the requested function). ▶

Substituting (8.29) in (8.23) gives:

$$Q_j = -\frac{\partial V}{\partial q_j} + Q'_j. \tag{8.30}$$

Subsequently this entire term is substituted in (8.3):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} + Q'_j. \tag{8.31}$$

This equation can be written even more compactly by introducing the *Lagrangian* function $\mathcal{L}$. The Lagrangian is defined as the difference between kinetic and potential energy of the mechanical system:

$$\mathcal{L} := T - V. \tag{8.32}$$

Substituting $\mathcal{L} + V$ for T in (8.31) leads to

$$Q'_j = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(\mathcal{L}+V)}{\partial \dot{q}_j}\right) - \frac{\partial(\mathcal{L}+V)}{\partial q_j} + \frac{\partial V}{\partial q_j} \tag{8.33}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(\mathcal{L}+V)}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j}. \tag{8.34}$$

From the definition of V, we know that it depends on configuration only, so only on q_j and not on time derivatives $\dot{q}_j$. Therefore, $\frac{\partial V}{\partial \dot{q}_j} = 0$, and finally we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q'_j. \tag{8.35}$$

This set of n equations is sometimes called Lagrange's equations of the second kind or the *Lagrange's equations*. With these equations, we have yet another way to derive the equations of motion.

These expressions are *only* valid if the generalized coordinates q_j are independent. Furthermore, as the equations were derived from virtual work, there is no need to include any ideal constraint forces in Q'_j.

Example 8.1 This Example is about the tube with a particle in it of Problem 4.35 on page 142. Here, we assume a connection with infinite stiffness: $k \to \infty$, but movements of both angles α and β are allowed.

Given the Lagrangian

$$\mathcal{L} = \frac{1}{2}mL^2\left(\dot{\alpha}^2 + \dot{\beta}^2\sin^2(\alpha)\right) + mgL\cos(\alpha)\ , \tag{8.36E}$$

and no applied forces, calculate $\ddot{\alpha}$ as a function of α, $\dot{\alpha}$, β, $\dot{\beta}$ and the given constants g and L (Remark: None of these terms has to necessarily appear in the requested function).

Exemplary solution

The chosen generalized coordinates are independent, so we may apply Lagrange's equation for α in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\alpha}}\right) - \frac{\partial\mathcal{L}}{\partial\alpha} = Q'_\alpha = 0, \tag{8.37E}$$

where Q'_α is the generalized applied force associated with α, which is zero.

Taking the different derivatives gives:

$$\frac{\partial\mathcal{L}}{\partial\dot{\alpha}} = mL^2\dot{\alpha} \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\alpha}}\right) = mL^2\ddot{\alpha} \tag{8.38E}$$

and

$$\frac{\partial\mathcal{L}}{\partial\alpha} = mL^2\dot{\beta}^2\sin\alpha\cdot\cos\alpha - mgL\sin\alpha. \tag{8.39E}$$

Substituting this in (8.37E) provides

$$mL\left(L\ddot{\alpha} - L\dot{\beta}^2\sin\alpha\cdot\cos\alpha + g\sin\alpha\right) = 0 \tag{8.40E}$$

and finally the requested acceleration:

$$\ddot{\alpha} = \sin\alpha\cdot\left(\dot{\beta}^2\cos\alpha - \frac{g}{L}\right). \tag{8.41E}$$

To obtain the acceleration of the second generalized coordinate, the corresponding Lagrange equation would have to be evaluated.

◯ **Problem 8.2** Is it necessary to consider ideal constraint forces in Lagrange's equations if you have chosen independent generalized coordinates? Why (not)?

◯ **Problem 8.3** Is it necessary to consider internal forces in Lagrange's method if they are capable of doing work on a mechanical system? Provide examples. ►

◎ **Problem 8.4** This Problem is about the ship installing a monopile of Problem 4.21 on page 130. A designer wants to calculate the forces acting on the crane as well as the movement of the monopile during operation, using the method of Lagrange. To simplify the problem, the designer considers only 2-D monopile motion.

A Lagrangian for this movement is

$$\begin{aligned}\mathcal{L} = \tfrac{1}{2}m\Big(\, &\dot{Y}_D^2 + \dot{Z}_D^2 + l^2\dot{\alpha}^2 + \tfrac{1}{3}h^2\dot{\beta}^2 + lh\dot{\alpha}\dot{\beta}\cos(\alpha-\beta) \\ &+ 2l\dot{Y}_D\dot{\alpha}\cos\alpha + h\dot{Y}_D\dot{\beta}\cos\beta \\ &+ 2l\dot{Z}_D\dot{\alpha}\sin\alpha + h\dot{Z}_D\dot{\beta}\sin\beta \\ &- 2gZ_D + 2lg\cos\alpha + hg\cos\beta\,\Big)\,,\end{aligned} \tag{8.42P}$$

with four generalized coordinates $q_1 = Y_D$, $q_2 = Z_D$, $q_3 = \alpha$, and $q_4 = \beta$. The generalized force components associated with q_2 are Q_2 (applied by water) and C_2 (imposed by the cable). No further forces are applied.

Determine the equation of motion for the generalized coordinate $q_2 = Z_D$, solved for the generalized constraint force component C_2 (As a remark, note that the constraint imposed by the cable is not ideal, because we are considering only the monopile, and point D moves). ▷►

◎ **Problem 8.5** This Problem is about the ballerina doing a jumping pirouette of Problem 2.25 on page 67. Now, we consider a less simplified model, which consists of a symmetrical body of mass m_b and moment of inertia I_b about the symmetry axis, and two pointmasses, each of mass m_h, representing the hands (Figure 8.1). The symmetry axis is also the rotation axis, and it remains vertical throughout.

The distance between the center of mass S of the ballerina and her toes is l_1, the distance between the toes and the ground is d, the height of the shoulder above S is l_2. The ballerina holds her hands in a symmetric position, each at shoulder height and at a horizontal distance r away from the body.

Two coordinate systems are used: a body-fixed system xyz associated with a triad $\mathcal{B}$, and an inertial coordinate system XYZ, associated with triad $\mathcal{N}$. The origin of xyz is at the ballerina's center of mass S, and the origin of XYZ is on the ground. Triad $\mathcal{B}$ is rotated with respect to $\mathcal{N}$ by an angle θ about the Z-axis.

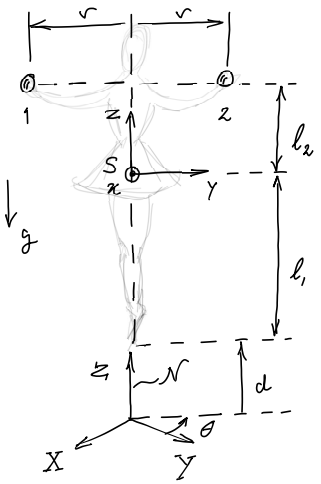


Figure 8.1: Jumping pirouette ballerina.

The ballerina pulls her arms in, by exerting a force on each mass m_h. The components of these forces in direction towards the axis of rotation each have magnitude F_h. Gravity with acceleration constant g acts in negative Z-direction.

In addition to gravity and the forces exerted by the ballerina, now a dancing partner applies three generalized forces Q_r, Q_θ and Q_d that act upon the generalized coordinates r, θ and d, respectively.

Use Lagrange's method to show that the equation for the second time derivative of the generalized coordinate r is given as[a]

$$\ddot{r} = \frac{-F_\mathrm{h} + 0.5 Q_r}{m_\mathrm{h}} + r\dot{\theta}^2. \tag{8.43P}$$

▶

[a] *Hint*: Use Virtual work to derive the generalized force associated with the generalized coordinate r.

Problem 8.6 This problem is about Problem 8.5, the ballerina performing a jumping pirouette. Use the Lagrangian method to show that the equations for the second time derivatives of the two other generalized coordinates are given as

$$\ddot{\theta} = \frac{Q_\theta - 4 m_\mathrm{h} r \dot{\theta} \dot{r}}{2 r^2 m_\mathrm{h} + I_\mathrm{b}}, \tag{8.44P}$$

$$\ddot{d} = \frac{Q_d}{2 m_\mathrm{h} + m_\mathrm{b}} - g. \tag{8.45P}$$

Problem 8.7 Use Lagrange's method to derive the equations of motion of a particle of mass m moving in 2-D, which is attached to a fixed point via a spring of stiffness k and resting length l_0. Use a set of generalized coordinates of your choice.

8.2 Lagrange for Dependent Coordinates

8.2.1 Re-Interpreting Constraints as Generalized Forces

Until now, we only used Lagrange's equations in a form that required independent virtual displacements of the generalized coordinates. However, it is not always possible to choose generalized coordinates such that this condition holds; for example a non-holonomic system forces us to choose more generalized coordinates than DoF exist in velocity space, and therefore the virtual displacements must be dependent. Also, sometimes it is simply more convenient to choose redundant generalized coordinates, even if it is not strictly necessary.

To enable using Lagrange's method also in the case of dependent generalized coordinates, we introduce additional generalized forces that perfectly enforce the constraints.

A constraint can be regarded as an ideal reaction force that makes sure a body cannot move in a certain direction. An example of such a *constraint force* is a normal force that keeps a particle on a surface.

Example 8.2 Consider a laptop on a table. This laptop is unilaterally constrained in vertical direction by the table, which means it cannot move downwards. The force that is needed to prevent movement in this direction is the normal force acting on the laptop. Recall that this is an *ideal* constraint force as it does no work; in other words, it is always exactly as large as needed to prevent movement that would violate the constraint.

If the constraints are not considered in the choice of the q_j, we have to consider them instead in the forces Q'_j in (8.35):

$$Q'_j = Q''_j + \sum_{i=1}^{m} C_{ij}, \tag{8.46}$$

with non-conservative forces Q''_j and m different constraint forces C_{ij}, where m is the difference between number of generalized coordinates n and number of DoF. In the following, we will seek expressions for these constraint forces from the given constraint functions.

8.2.2 Writing Constraint Equations as Scalar Products

Generally, any i-th constraint can be written as a function of the generalized coordinates and their derivatives, as well as the time t:

$$f_i(q_1,\, q_2 \ldots q_n,\, \dot{q}_1,\, \dot{q}_2 \ldots \dot{q}_n,\, t) = 0, \qquad i = 1 \ldots m. \tag{8.47}$$

First, we consider only holonomic constraints, where the constraint is expressed in a form without generalized coordinates derivatives, so

$$\Phi_i(q_1,\, q_2 \ldots q_n,\, t) = 0. \tag{8.48}$$

In order to obtain the instantaneous constraints on the virtual displacements δq_j, we can take the variation:

$$\delta\Phi_i = \sum_{j=1}^{n} \frac{\partial \Phi_i}{\partial q_j} \delta q_j = 0, \qquad i = 1 \ldots m. \tag{8.49}$$

Definition of the vector $\boldsymbol{a}_i$ with

$$\boldsymbol{a}_i := \nabla_{\boldsymbol{q}} \Phi_i = \left(\frac{\partial \Phi_i}{\partial q_1} \quad \frac{\partial \Phi_i}{\partial q_2} \quad \ldots \quad \frac{\partial \Phi_i}{\partial q_j} \quad \ldots \quad \frac{\partial \Phi_i}{\partial q_n} \right)^{\mathrm{T}}, \qquad i = 1 \ldots m, \tag{8.50}$$

allows re-writing (8.49) in more compact form, such that an equivalent representation of the i-th instantaneous constraint is

$$\boldsymbol{a}_i^{\mathrm{T}} \delta \boldsymbol{q} = 0, \qquad i = 1 \ldots m. \tag{8.51}$$

The vector $\boldsymbol{a}_i$ is the gradient of the holonomic constraint function with respect to the generalized coordinates, as defined in (8.50). The scalar product (8.51) shows that this vector is orthogonal to the *constraint surface*, which contains the allowed virtual displacements δq_j. Note that $\boldsymbol{a}_i$ can be constant or a function of generalized coordinates.

Problem 8.8 This Problem is about the ship installing a monopile of Problem 4.21 on page 130. The same nine generalized coordinates are chosen as in Problem 7.5. The mass of the ship is much larger than the mass of the monopile. Therefore, the movements of point D are considered externally imposed constraints. At a specific instant the ship is at rest, so $\dot{X}_D = 0$, $\dot{Y}_D = 0$, and $\dot{Z}_D = 0$. The vector $\boldsymbol{C}_D$ of generalized constraint forces associated with the above three constraints can be written as

$$\boldsymbol{C}_D = \lambda_X \hat{\boldsymbol{a}}_X + \lambda_Y \hat{\boldsymbol{a}}_Y + \lambda_Z \hat{\boldsymbol{a}}_Z. \tag{8.52P}$$

Calculate numerical values for the unit vectors $\hat{\boldsymbol{a}}_X$, $\hat{\boldsymbol{a}}_Y$, and $\hat{\boldsymbol{a}}_Z$. ▷►

Example 8.3 Consider a particle on a rigid surface. The particle's location is described by generalized coordinates equal to the particle's Cartesian coordinates.

The allowed virtual displacements are all parallel to the surface. If the rigid surface is flat, like a table, the vector $\boldsymbol{a}$ is constant and must always be normal to the surface, such that the scalar product with the vector of virtual generalized displacements $\delta\boldsymbol{q}$ is zero. If the surface is curved, like a slide on a playground, the vector $\boldsymbol{a}$ is also always normal to the constraint surface, in this case the slide. However, the direction of this normal changes with the location of the particle on the slide, so $\boldsymbol{a}$ is a function of configuration $\boldsymbol{q}$.

Example 8.4 A cart rides on a curved track in a 2-D xy-coordinate system. The track is parabolic and described by the function $y = cx^2$, where c is a constant. A pendulum is mounted on the cart via a hinge joint. The joint allows rotation of the pendulum by an angle φ. What is the number of DoF of this system? If we use x, y, and φ as generalized coordinates q_j, what is the vector $\boldsymbol{a}$ that characterizes the constraint?

Exemplary solution

The system has only two DoF, so there must be one constraint equation linking the three generalized coordinates. In this case, the constraint is holonomic and the constraint equation can be written as

$$\Phi(x,\, y,\, \varphi) = y - cx^2 = 0, \tag{8.53E}$$

which leads to

$$\boldsymbol{a} = \begin{pmatrix} \frac{\partial\Phi}{\partial x} & \frac{\partial\Phi}{\partial y} & \frac{\partial\Phi}{\partial\varphi} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} -2cx & 1 & 0 \end{pmatrix}^{\mathrm{T}}. \tag{8.54E}$$

Problem 8.9 For Example 8.4, draw the constraint surface (the track) and the vector $\boldsymbol{a}$ in configuration space for a few example configurations, using $c = 1\,\mathrm{m}^{-1}$. Is $\boldsymbol{a}$ indeed always orthogonal to the constraint surface?

Also for non-holonomic constraints, it is generally possible to find an expression of the form (8.51). For a particular constraint i, one can determine the associated vector $\boldsymbol{a}_i$ by establishing the constraint equation in terms of the generalized virtual displacements or in velocity space, and then comparing coefficients.

Example 8.5 For the cart from Example 8.4, the parabolic track imposes a holonomic constraint. However, but we can also re-write (8.53E) in velocity space:

$$\dot{y} - 2cx\dot{x} = 0, \tag{8.55E}$$

and by comparing coefficients with the equation

$$a_x\dot{x} + a_y\dot{y} + a_\varphi\dot{\varphi} = 0, \tag{8.56E}$$

we find the three components a_x, a_y, and a_φ of the vector $\boldsymbol{a}$ as $a_x = -2cx$, $a_y = 1$, and $a_\varphi = 0$. This is the same result as (8.54E).

8.2.3 Lagrange Multipliers

To find a relationship between the i-th constraint force $\boldsymbol{F}_{\mathrm{c}i}$ in Cartesian space and the associated generalized constraint force $\boldsymbol{C}_i$, we can again use equivalence of their virtual work, as in Section 7.2:

$$\delta W = \boldsymbol{F}_{ci}^{\mathrm{T}}\delta\boldsymbol{r} = \boldsymbol{C}_i^{\mathrm{T}}\delta\boldsymbol{q}. \tag{8.57}$$

We know that constraint forces cannot do work, so

$$\boldsymbol{C}_i^{\mathrm{T}}\delta\boldsymbol{q} = 0. \tag{8.58}$$

Both (8.51) and (8.58) are valid for any $\delta\boldsymbol{q}$ that complies with the constraints. This means that vectors $\boldsymbol{a}_i$ and $\boldsymbol{C}_i$ must both be orthogonal to the constraint surface in which the $\delta\boldsymbol{q}$ exist. Therefore, $\boldsymbol{a}_i$ and $\boldsymbol{C}_i$ must be linearly dependent (they point in the same direction).

Therefore, we can write them as:

$$\boldsymbol{C}_i = \lambda_i\boldsymbol{a}_i, \tag{8.59}$$

where the scalar multiplication factor λ_i is called the *Lagrange multiplier*.

In fact, if we choose $\boldsymbol{a}_i$ to be a unit vector (which is advised, but only possible if all generalized constraint forces have the same unit), we can interpret $\hat{\boldsymbol{a}}_i$ as the direction vector of the generalized constraint force $\boldsymbol{C}_i$, while λ_i is the magnitude.

With (8.59) we can write the equation for Q'_j (8.46) as

$$Q'_j = Q''_j + \sum_{i=1}^{m}\lambda_i a_{ij}. \tag{8.60}$$

Finally, we can write Lagrange's equations (8.35) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\right) - \frac{\partial\mathcal{L}}{\partial q_j} = Q''_j + \sum_{i=1}^{m}\lambda_i a_{ij}. \tag{8.61}$$

This yields n equations, so-called *Lagrange's equations with multipliers*, or also sometimes called Lagrange's equations of the first kind.

Together with the m constraint equations, we have sufficient equations to solve for all the unknowns. These unknowns are the second derivatives of the n generalized coordinates $\ddot{q}_j$ with respect to time and the m Lagrange multipliers λ_i.

Example 8.6 This Example is about the cart on a parabolic track of Example 8.4 on page 260. We are now looking to formulate the generalized constraint force vector $\boldsymbol{C}$ using a Lagrange multiplier λ.

For that purpose, we use the previously found vector $\boldsymbol{a}$, and we can write

$$\boldsymbol{C} = \lambda \begin{pmatrix} -2cx & 1 & 0 \end{pmatrix}^{\mathrm{T}}. \tag{8.62E}$$

Note that here, it was not possible to use a unit vector multiplied by λ, because the entries of $\boldsymbol{C}$ do not all have the same unit.

Problem 8.10 For the cart on a track from Example 8.4 and Example 8.6, what are the units of the three components of the vector $\boldsymbol{C}$? What is the unit of λ (see also Section 1.1.4)?

Problem 8.11 Assuming that the cart in Example 8.4 has mass m and the pendulum is a homogeneous slender rod with length l, mass M and moment of inertia of I_C about its center of mass, use Lagrange multipliers to derive the equations of motion for the given generalized coordinates.

Problem 8.12 This Problem is about the ballerina doing a jumping pirouette of Problem 8.5 on page 256. The equations of motion were given in Problem 8.5 and Problem 8.6.

Now, the ballerina is in point contact with the ground and still dancing with her partner. The ground is modeled as a constraint: $d = 0$. Use Lagrange multipliers to find the magnitude of the constraint force, as a function of the variables r, θ, d and their time derivatives and the parameters $m_{\mathrm{h}}, I_{\mathrm{b}}, g, m_{\mathrm{b}}$ (Remark: None of these terms has to necessarily appear in the requested function). Assume there are no further forces besides gravity, the forces exerted by the ballerina, and the ground reaction forces. ▶

Problem 8.13 How many Lagrange's equations and how many Lagrange multipliers do you obtain if you choose five generalized coordinates for a system, and there are two constraint equations linking them?

Problem 8.14 A large advantage of Lagrange's method is that ideal constraint forces can be neglected, because they do not do virtual work. Therefore, they do not appear in the equations. However, in case we do want to know certain constraint forces, what can we do to calculate them?

8.3 Summary

The necessary variables to calculate in Lagrange's method are

1. $\mathcal{L} = T - V$; the Lagrangian, which is a combination of kinetic and potential energy. If a force can be expressed in terms of potential energy, it is easier to include it in V instead of in the generalized applied forces. Examples are gravity or spring forces.
2. Q''_j: the applied forces that are not conservative (or, more precisely, which are not considered in V). These are all forces that can do work on the system, except the ones expressed in terms of potential energy. Ideal constraint forces do not need to be considered. We use virtual work to derive the generalized forces from the Cartesian forces and moments.
3. a_{ij}: the coefficients of the i-th constraint equation, written in the form $\sum_{j=1}^{n} a_{ij}\delta q_j = 0$. Alternatively, one can also find the coefficients from $\sum_{j=1}^{n} a_{ij}\dot{q}_j = 0$. If possible, it is convenient to choose $\boldsymbol{a}_i$ as a unit vector, to enable interpretation of λ_i as the magnitude of the generalized constraint force.

From n Lagrange's equations in combination with the m constraint equations, we then obtain

1. $\ddot{q}_j$, $j = 1 \dots n$: The second derivatives of the generalized coordinates with respect to time. These are generally functions of the generalized coordinates, their first derivatives with respect to time, and the applied forces.
2. λ_i, $i = 1 \dots N$: the Lagrange multipliers. The i-th multiplier determines the magnitude of the i-th generalized constraint force that has not been considered in the generalized coordinates choice.

The last points of both lists are only needed with dependent generalized coordinates, so when applying Lagrange's equations with multipliers.

8.4 Problems

8.4.1 Guided Problems

Problem 8.15 The slender rod AB depicted in Figure 8.2 is connected by a revolute joint to a rolling disk in point B, whereas point A of the rod slides without friction over the ground. The disk rolls without slipping over the ground. Its orientation is described by the angle φ with respect to the

vertical. A force F_C is applied at the center of the disk.

The slender rod of length l has a mass m, the solid disk has a rolling radius R and a mass M. Both bodies are of homogeneous material. Point B on the disk is at a distance r from the center C of the disk. Gravity with acceleration constant g acts in downward direction.

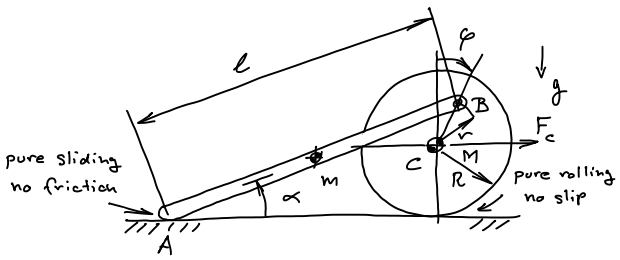


Figure 8.2: A slender rod connected to a rolling wheel.

We want to derive the equations of motion of this system, using the method of Lagrange.

a. Make a drawing to define an inertial coordinate system that you will use throughout this problem.
b. State and explain how many degrees of freedom this system has. Also provide at least three different possible sets of generalized coordinates.
c. We choose the generalized coordinate φ. Use virtual work to calculate $Q_{C,\varphi}$ as the contribution of F_C to the generalized force Q_φ. Apply both methods, the geometric method with drawings, and the analytic method using partial derivatives. Compare results.
d. To simplify calculations, assume $r = 0$. Calculate kinetic and potential energy of the system, as functions of φ, $\dot{\varphi}$, and the given constants R, g, l, m, M.
e. Still, assume $r = 0$. Apply the Lagrange equations to establish the equations of motion for this system in terms of the angle φ, the force F_C, and the given constants, in the form of $\ddot{\varphi} = f(\varphi, \dot{\varphi}, F_C)$.
We are interested in calculating the contact force F_A between the rod and the ground in point A, using Lagrange multipliers. To this end, we introduce an additional generalized coordinate α, the angle between the horizontal and the rod, so that $\boldsymbol{q} = (\varphi, \alpha)^{\mathrm{T}}$.
f. Now, $r \neq 0$. Write down a constraint equation that links the angles φ and α. [a]
g. For $r \neq 0$, write down the constraint equation in the form $\boldsymbol{a}^{\mathrm{T}}\dot{\boldsymbol{q}} = 0$. [b]
Check your result for plausibility, for example by setting $r = 0$.
h. Still, $r \neq 0$. Use virtual work to calculate the generalized force components $Q_{A,\varphi}$ and $Q_{A,\alpha}$ contributed by F_A, as functions of F_A and

the generalized coordinates. Verify your result by checking that the "vector" $\boldsymbol{Q}_A = (Q_{A,\varphi}, Q_{A,\alpha})^{\mathrm{T}}$ is a scalar multiple of $\boldsymbol{a}$.

i. To simplify calculations, assume again that $r = 0$. Set up the new Lagrange equations using φ and α as generalized coordinates, and consider the constraint force F_A in A via a Lagrange multiplier. Calculate F_A in the form of $F_A = f(\varphi, \dot{\varphi}, \ddot{\varphi}, F_C)$.

j. Now consider $r \neq 0$ and derive the equation of motion again, using Lagrange equations, using φ as the generalized coordinate for the system. You should do this part using the symbolic toolbox of MATLAB$^{\mathrm{TM}}$, or another program of your liking. [c]
Verify your result, for example by substituting $r = 0$ in the equation and comparing to the result you found above.

---

[a] *Hint*: Realize that the time derivative of a symbolic variable q_j will always be zero if you use the Matlab command `diff`, because Matlab has no idea about q_j being a function of time t, as in $q_j = q_j(t)$. Instead, use the basic definition of time derivative of a functional dependancy by taking partial derivatives of a function $r(\boldsymbol{q})$ with respect to the generalized coordinates $\boldsymbol{q}$, as in $\frac{\mathrm{d}r}{\mathrm{d}t} = \sum_{j=1}^{n} \frac{\partial r}{\partial q_j} \frac{\mathrm{d}q_j}{\mathrm{d}t}$, and introduce a new set of symbolic variables for the time derivatives of the generalized coordinates $\dot{\boldsymbol{q}} = \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t}$, say `qdot`. Then you can calculate $\frac{\mathrm{d}r}{\mathrm{d}t}$ in Matlab as `jacobian(r,q)*qdot`. Note that the coefficients in `jacobian(r,q)` are the coefficients a_j in $\sum_{j=1}^{n} a_j \frac{\mathrm{d}q_j}{\mathrm{d}t}$.

[b] *Hint*: Re-write your answer to the previous part of this question by taking the derivative with respect to time. Then, compare coefficients to determine the components of $\boldsymbol{a}$.

[c] *Hint*: Recognize that the height of point A above ground can be written as a function of φ and α.

Problem 8.16 The homogeneous semicylinder in Figure 8.3 rolls without slipping. As generalized coordinates, we choose the center of mass coordinates (X_C and Y_C), and the orientation ψ.

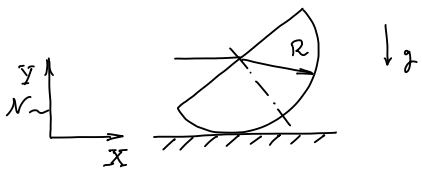


Figure 8.3: A homogenous semicylinder.

a. Derive the constraint equations that ensure roll without slip, in terms of the generalized coordinates.

b. Derive the equations of motion of the rolling semicylinder in terms of the generalized coordinates and the Lagrange multipliers pertaining to

the constraints.

c. Derive the equations of motion of the rolling semicylinder directly in terms of its only degree of freedom (ψ) and show that this result is identical to the result under part b.

8.4.2 Practice Problems

Problem 8.17 What is the meaning of the indices i, j, m, n, and N in this chapter? How are they related to each other and to the number of DoF? Note that i is used for two different things. Do you need to apply Lagrange multipliers if $m = 0$?

Problem 8.18 Use Lagrange's method to derive the equations of motion of a homogeneous ring of radius R and mass m rolling on level ground in 2-D, without slip. Use a set of generalized coordinates of your choice.

Problem 8.19 Use Lagrange's method and Lagrange multipliers to derive the equations of motion of a thin disk of radius R and mass m rolling on level ground in 3-D, without slip, under the influence of gravity g. Use a set of generalized coordinates of your choice.

Problem 8.20 This Problem is about the hoverboard of Problem 3.26 on page 95. The student made several simplifying assumptions for the system when it is moving only in 2-D, and calculated the following Lagrangian for this system:

$$\mathcal{L} = D\dot{C}_X^2 + E\dot{\gamma}^2 + G\dot{C}_X\dot{\gamma}\cos\gamma + H\cos\gamma, \tag{8.63P}$$

with generalized coordinates C_X and γ, and given constants D, E, G, H. Assume that the effect of gravity is already included in the Lagrangian, and that there are no further generalized applied forces.

Apply the method of Lagrange to derive one of the two equations of motion of the system in the implicit form $f_j(\ddot{C}_X, \dot{C}_X, C_X, \ddot{\gamma}, \dot{\gamma}, \gamma) = 0, j = 1, 2$. You may choose which one. ▶

Problem 8.21 This Problem is about the skateboard of Problem 1.24 on page 39. The rider is simplified as a single rigid body of mass m_p and center of mass P, with inertia tensor $\mathbf{I}_P$. We neglect the mass and inertia of the skateboard including the wheels.

Define a suitable set of generalized coordinates for the system and then calculate the Lagrangian $\mathcal{L}$ of the system, as a function of the generalized coordinates and their derivatives, and the given constants $m_\mathrm{p}, {}^{\mathcal{B}}\mathbf{I}_P, g, h, l, k$. Include also the effect of all conservative forces in $\mathcal{L}$. ▶

Problem 8.22 This Problem is about the skateboard of Problem 1.24 on page 39. Which statement(s) is/are true about the system? Check all that apply.

A. The virtual work of all ideal constraint forces is zero.
B. The constraint that the wheels do not slip laterally can be considered in the Lagrange approach using Lagrange multipliers.
C. To calculate the virtual work δW done by the resultant ground reaction force vector $\boldsymbol{F}$, one may use the equation $\delta W = \boldsymbol{F} \cdot \delta \boldsymbol{r}$, where $\delta \boldsymbol{r}$ is the virtual displacement of the system's center of mass.
D. None of the above

▷

Problem 8.23 This Problem is about the skateboard of Problem 1.24 on page 39. A set of generalized coordinates was chosen for the system, and γ was among these generalized coordinates. Also all conservative forces are now considered as part of the generalized forces, not in the Lagrangian.

Calculate the contribution of gravity to the generalized force Q_γ in γ-direction, as a function of $\gamma, m_\mathrm{p}, g, h, l$. Use the geometric approach. ▷

Problem 8.24 This Problem is about the biker of Problem 4.25 on page 131. Given the kinetic energy $T = a\dot{\gamma}^2 + b\dot{\psi}^2(1 - c\sin\gamma)^2$ of the body, with constants a, b, c, establish Lagrange's equations and determine the equations of motion of the body in the generalized coordinates ψ and γ, assuming that the wheels are currently rolling freely.

You do not need to decouple the equations of motion, so write them in implicit form as: $f_1(\psi,\, \gamma,\, \dot{\psi},\, \dot{\gamma},\, \ddot{\psi},\, \ddot{\gamma}) = 0$ and $f_2(\psi,\, \gamma,\, \dot{\psi},\, \dot{\gamma},\, \ddot{\psi},\, \ddot{\gamma}) = 0$. ▶

Problem 8.25 This Problem is about the point mass pendulum of Problem 7.16 on page 238. Use Lagrange multipliers to determine the force in the rod as a function of φ, $\dot{\varphi}$, $\ddot{\varphi}$, m, g, and l (Remark: It is possible that not all of these terms appear in the requested function). Clearly indicate whether your definition of positive force direction is associated with a compressive or a tensile force in the rod. ▶

Problem 8.26 This Problem is about the 3D rod pendulum of Problem 3.32 on page 98. The pendulum is a uniform slender rod of length l and mass m, and dimensions and mass of the joint itself are negligible. Constant gravity g acts in positive Z-direction.

Show that a possible Lagrangian function for this physical system equals: $\mathcal{L} = \frac{ml}{6}\left(l\dot{\beta}^2 + l\dot{\alpha}^2\sin^2(\beta) + 3g\cos(\beta)\right)$. In order to do so, express the kinetic energy of the pendulum using the origin O as reference point. Use the provided ${}^{\mathcal{B}}\boldsymbol{\omega}$ from Problem 4.30 in your calculation.

Problem 8.27 This Problem is about the 3-D rod pendulum of Problem 3.32 on page 98. Now, two motors create moments M_1 and M_2 on the rod in directions of Z and x', respectively. These moments translate to generalized forces Q_α and Q_β associated with the generalized coordinates α and β respectively.

Use Lagrange's equation associated with the generalized coordinate α, as provided in Problem 8.26, to show that the following expression for $\ddot{\alpha}$ holds: $\ddot{\alpha} = \frac{-2\dot{\alpha}\dot{\beta}}{\tan\beta} + \frac{3Q_\alpha}{ml^2\sin^2(\beta)}$.

Problem 8.28 This Problem is about the 3-D rod pendulum of Problem 3.32 on page 98.

As in Problem 8.27, two motors create generalized forces. Furthermore, Lagrange's equation associated with the generalized coordinate β provides the following expression for $\ddot{\beta}$:

$$\ddot{\beta} = \dot{\alpha}^2\sin(\beta)\cos(\beta) - \frac{3g}{2l}\sin(\beta) + \frac{3Q_\beta}{ml^2} \qquad \text{(You do not need to show this).} \tag{8.64P}$$

Now, the motors are controlled to enforce a constraint on the pendulum: Keeping the angle β constant, with $\beta = \frac{\pi}{4}$.

Calculate the generalized forces C_α, C_β that are necessary to enforce this constraint, as functions of $\dot{\alpha}$ and the parameters m, l, g (Remark: None of these terms has to necessarily appear in the requested functions). Use the two provided Lagrange's equations from Problem 8.26 and (8.64P) and apply Lagrange multipliers. The motors generate no further generalized forces apart from these constraint forces.

Problem 8.29 This Problem is about the governor of Example 2.2 on page 48. Derive the equation of motion for the spindle rotation ψ using the method of Lagrange. Do not substitute your results for A, B, C, and D as found in Example 2.5 and Example 2.9 but keep the variables and any nonzero partial and/or absolute derivatives thereof. Recall that A, B, C, D are only functions of constant parameters and θ. ▷▶

Problem 8.30 This Problem is about the governor of Example 2.2 on page 48. Compare the results for the variables Q and B calculated in Example 2.5 and Example 2.2 and interpret them as physical quantities. Using this comparison and interpretation, provide an explicit reasoning why the results you calculated for the two variables are plausible or not. ▶

9 Introduction to Numerical Integration

The previous chapters discussed different methods to derive the equations of motion of a mechanical system. These equations are in fact ordinary differential equations (ODEs). In this chapter, we will see how Euler's method can integrate the equations of motion numerically, in order to obtain the generalized coordinates of the system as functions of time. We will also see that as a prerequisite for this integration, we first need to transform the equations of motion into first-order explicit ODEs.

9.1 Explicit and Implicit Differential Equations

Ordinary differential equations (ODEs) contain only derivatives with respect to a single variable, in this case time t. A general set of n ODEs of N-th order in terms of the variable vector $\boldsymbol{q}(q_1, q_2 \dots q_j \dots q_n, t)$, with $(j = 1 \dots n)$, can be written in vector form:

$$\boldsymbol{g}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}} \dots \boldsymbol{q}^{(N)}, t) = 0 \qquad \text{(implicit)}. \tag{9.1}$$

This is called an *implicit* form, because all derivatives appear on one side of the equal sign. Solving an ODE means finding an expression for $\boldsymbol{q}(t)$ that satisfies the ODE.

There are many techniques and algorithms to solve ODEs, called *solvers*. However, they mostly require that one first transforms the ODE to a so-called *explicit* form, where the vector $\boldsymbol{q}^{(N)}$, which contains the highest-order derivatives of the q_j, is a function of the lower-order derivatives only:

$$\boldsymbol{q}^{(N)} = \boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}} \dots \boldsymbol{q}^{(N-1)}, t) \qquad \text{(explicit)}. \tag{9.2}$$

Example 9.1 Look at Figure 9.1 for an example system described by a first-order ($N = 1$) ODE: A train is driving along a track between two cities A and B.

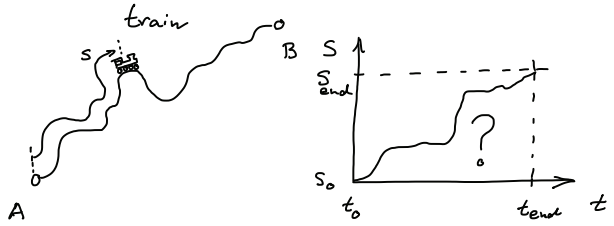


Figure 9.1: A train on a track, and its traveled distance as a function of time.

It leaves city A at time $t = t_0$. As the train has only one degree of freedom, only one generalized coordinate is needed to describe the train's configuration. We choose s, the distance traveled so far. We now assume that here are different velocity limits and recommendations at different locations on the track, and that the driver always drives according to the recommendations. Therefore, the speed $\dot{s}$ of the train, which is the time derivative of its location s, is a function of this location:

$$\dot{s} = f(s). \tag{9.3E}$$

This is a first-order explicit ODE in s. If we are interested in knowing the position of the train at a particular time t, so $s(t)$, or if we just want to know at which final time $t = t_e$ the train will arrive in city B, we need to solve the ODE.

9.2 Integration Algorithms

Many differential equations cannot be solved using analytic calculations, or the calculations would be too complex. This holds for many practical applications in mechanics. In those cases, we can use numerical integration methods to *approximate* the solution.

9.2.1 Euler's Method in 1-D

Consider a scalar function y, where we know its time derivative as a function of only y and t, so $\dot{y} = f(y, t)$. Euler's method is the simplest method of integrating such a first-order explicit ODE. Figure 9.2 illustrates this method. The figure shows that if the value y_n at the n-th time step is known, then the estimated value for y_{n+1} is found by simple linear extrapolation. This is equivalent to assuming that the function derivative remains constant over the entire time increment h and equal to the value $\dot{y}_n$ at t_n:

$$y_{n+1} \approx y_n + h\dot{y}_n. \tag{9.4}$$

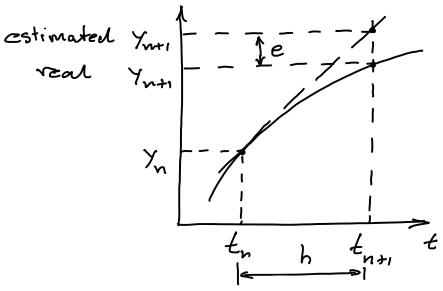


Figure 9.2: Visualization of Euler's method of numerical integration in 1D.

The derivative at t_n is given as $\dot{y}_n = f(y_n, t_n)$. So, Euler's method can be written as

$$y_{n+1} = y_n + hf(y_n, t). \tag{9.5}$$

Only if the function y is indeed linear, the result is correct. Otherwise, an error e will occur between the real value and the estimated value of y_{n+1}, also known as the *local truncation error*. These errors accumulate over time. The smaller the chosen h, the smaller the error becomes.

Example 9.2 For the train in Example 9.1, y would equal s. In case the speed of the train $\dot{s}$ remains constant over the interval h, the function is locally linear, and the error will be zero. However, as the figure shows, the speed is not constant, so integration errors will occur.

Problem 9.1 The acceleration of a ball that is thrown in the air is equal to the acceleration of gravity g in downward direction, so the equation of motion in terms of height y is trivial (assuming the y-axis points up): $\ddot{y} = -g$. When we integrate this twice in an analytic fashion, we obtain a second-order polynomial in time t for $y(t)$. Try to integrate numerically instead and give the integration error in position after one integration step as a function of step size h and initial velocity y_0, by comparing with the analytic solution.

9.2.2 Euler's Method in a Higher Dimensional State Space

Euler's method of numeric integration also applies to problems with higher dimension.

Instead of a scalar y, we now consider a *state vector* $\boldsymbol{y}$ for which we know its first derivative with respect to time as a vector function: $\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, t)$. To integrate the state vector from timestep n to the subsequent timestep $n+1$, with a time increment $h = t_{n+1} - t_n$, we can apply the same method as in (9.5):

$$\boldsymbol{y}_{n+1} \approx \boldsymbol{y}_n + h \cdot \boldsymbol{f}(\boldsymbol{y}_n, t_n). \tag{9.6}$$

Euler's method as in (9.6) can be applied to ODEs with arbitrarily many dimensions, so to arbitrarily high *state spaces*.

Example 9.3 Let us illustrate the integration of a vector in a three-dimensional Cartesian space $\boldsymbol{y} = (y_1, y_2, y_3)^{\mathrm{T}}$ (Note that this is not a state vector, as it only includes positions and not velocities). We assume that $\boldsymbol{y}$ equals the position vector $\boldsymbol{r}$ of a particle P, with Cartesian position coordinates $\boldsymbol{r} = (x, y, z)^{\mathrm{T}}$, and that we know the velocity $\dot{\boldsymbol{r}} = \dot{\boldsymbol{y}}$ of the particle as a function of the position $\boldsymbol{r}$ and time t.

Using Euler's method of numerical integration, we can estimate the position $\boldsymbol{y}_{n+1} = \boldsymbol{r}_{n+1}$ at the next instant in time t_{n+1}. In this example, $\dot{\boldsymbol{y}}_n$ is equal to $\dot{\boldsymbol{r}}_n$, namely the velocity of particle P at time t_n. So using Euler's method we would obtain the position at time instant t_{n+1} as:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} + h \begin{pmatrix} \dot{x}_n \\ \dot{y}_n \\ \dot{z}_n \end{pmatrix}, \tag{9.7E}$$

with time step $h = t_{n+1} - t_n$. So, if the position and velocity of P are known at a certain moment in time t_n, we can estimate the next position using Euler's method of numerical integration, as shown in Figure 9.3.

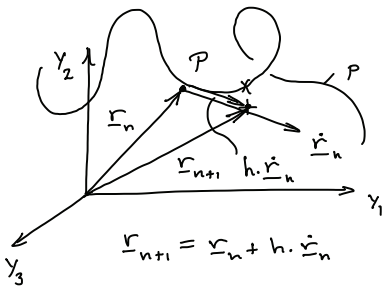


Figure 9.3: Visualization of Euler's method in 3-D.

Recalling Chapter 4, where we discussed natural coordinates, we remember that the velocity of a particle is always tangent to its path. Therefore, we see that Euler's method of integration is nothing else but a linear approximation of the particle's path in the neighborhood of its current location. In most cases, the smaller this neighborhood (so the smaller h), the more precise the solution.

In this example, the vector $\boldsymbol{y}$ had only three components, but as stated before, (9.6) can be applied to ODEs with arbitrarily many dimensions.

Problem 9.2 The equations of motion for a mechanical system are stated in the form $\dot{\boldsymbol{y}} = -k\boldsymbol{y}$, with a scalar constant k. Given this and the state vector $\boldsymbol{y}(t_a) = \boldsymbol{y}_a$ at time t_a, perform numerical integration with a step size of $h = 0.5\,\mathrm{s}$ to calculate the state vector $\boldsymbol{y}(t_c)$ at time $t_c = t_a + 1\,\mathrm{s}$. Express your answer as a function of $\boldsymbol{y}_a$ and k only. Make sure to use units correctly throughout. ▷►

9.2.3 Limitations of Euler's Method

Euler's method, being only a first-order method, only gives the precise result if the approximated function is linear. Otherwise, errors can grow, especially when the step size is too large. In most practical cases, the functions are not linear.

Note that there are also diverse other solvers. Many of these are available in the software package MATLAB™. Higher-order solvers can generally approximate nonlinear functions better. For example, the Trapezoidal Method, the Modified Euler Method (Heun's method), or Runge-Kutta of second or fourth order generally yield more precise results with the same step size. In case the function to approximate is a second-order or a fourth-order polynomial in time, then the second-order and fourth-order Runge-Kutta solvers even give the exact solutions, respectively. For more details and practical examples with MATLAB™, consult Chapter 14 in the second part of this book, or other work such as [27] or [71].

For most solvers, the requirement remains that the provided differential equations form a set of first-order ODEs, so that we can express the derivative $\dot{\boldsymbol{y}}$ exclusively as a function of $\boldsymbol{y}$ and t. In Example 9.1 and Example 9.3, we knew the particle's velocity as a function of its position and time.

However, in most cases we do *not* have such knowledge. Instead, all the methods we learned to derive the equations of motion normally deliver a set of *second*-order differential equations. For example, Newton's second law allows us to compute the acceleration as a function of forces acting on a particle. Acceleration is the second derivative of position with respect to time, not the first. Therefore, in order to apply numerical integration, we first need to find a way to transform a set of second-order ODEs to a set of first-order ODEs.

9.3 Obtaining Explicit, First-Order ODEs

There are many solvers for explicit, first-order ODEs available. However, with all the methods we discussed throughout this book, we will generally obtain implicit, second-order equations of motion for a mechanical system. Therefore, we need to transform a higher-order implicit ODE to a first-order explicit ODE.

9.3.1 Obtaining Explicit ODEs

The first step, transforming from implicit to explicit form, is mostly achieved by bringing all second derivatives $\ddot{q}_j$ of the generalized coordinates to the left side of the equation, and all other terms to the right side. For example, Lagrange's equations might have delivered a system of equations of motion that can be written in matrix form as

$$\mathbf{M}(\boldsymbol{q}) \cdot \ddot{\boldsymbol{q}} = \boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t). \tag{9.8}$$

Here, the matrix $\mathbf{M}$ (often called the *mass matrix*) and the vector $\boldsymbol{b}$ may be functions of the generalized coordinates $\boldsymbol{q}$, their first derivatives in $\dot{\boldsymbol{q}}$, and time t. We can invert (9.8) to

$$\ddot{\boldsymbol{q}} = \mathbf{M}^{-1}(\boldsymbol{q}) \cdot \boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t). \tag{9.9}$$

Since we are working with numerical methods, there is no need to calculate the inverse $\mathbf{M}^{-1}$ analytically, we could also do this numerically in each time step. This way, we have obtained a set of explicit ODEs. However, they are still of second order.

9.3.2 Obtaining First-Order ODEs

To transform a second-order ODE of the form

$$\ddot{\boldsymbol{q}} = \boldsymbol{g}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t), \tag{9.10}$$

in generalized coordinates $\boldsymbol{q}$, to a first-order ODE of the form

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, t), \tag{9.11}$$

with state vector $\boldsymbol{y}$, often a good choice is to define

$$\boldsymbol{y} := \begin{pmatrix} \boldsymbol{y}_1^{\mathrm{T}} & \boldsymbol{y}_2^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}, \qquad \text{with} \qquad \boldsymbol{y}_1 = \boldsymbol{q}, \qquad \boldsymbol{y}_2 = \dot{\boldsymbol{q}}. \tag{9.12}$$

With this definition, we will see that in fact two integration steps are performed simultaneously, because we can write

$$\dot{\boldsymbol{y}} = \begin{pmatrix} \dot{\boldsymbol{y}}_1 \\ \dot{\boldsymbol{y}}_2 \end{pmatrix} = \begin{pmatrix} \dot{\boldsymbol{q}} \\ \ddot{\boldsymbol{q}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{y}_2 \\ \boldsymbol{g}(\boldsymbol{q},\, \dot{\boldsymbol{q}},\, t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{y}_2 \\ \boldsymbol{g}(\boldsymbol{y}_1,\, \boldsymbol{y}_2,\, t) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{y},\, t). \quad (9.13)$$

Although this is not always the shortest or most elegant method, we can always find the set of first-order ODEs this way, as long as we have an expression for the second-order derivatives of the generalized coordinates q_j. Note that the state vector will have twice as many entries as there are generalized coordinates.

Example 9.4 A block of mass M is suspended from the ceiling by a spring with constant k (Figure 9.4).

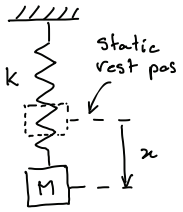


Figure 9.4: A rigid body of mass M is suspended from the ceiling by a spring with constant k.

The block has only one degree of freedom, so we need only one generalized coordinate. We choose $q = x$, the position x of its center of mass with respect to the static resting position of the system. This way, by choosing the coordinate with respect to the resting position, the equations of motion are independent of gravity. Then, the movement of the block in terms of x can be described by the second-order differential equation:

$$\ddot{x} = -\frac{k}{M}x. \quad (9.14\text{E})$$

In order to find $x(t)$ numerically, we have to transform (9.14E) to a first-order ODE and then use Euler's method to integrate. Following (9.12), we define a new 2-dimensional state vector $\boldsymbol{y}$:

$$\boldsymbol{y} := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}. \quad (9.15\text{E})$$

The first derivative of this new vector with respect to time is:

$$\dot{\boldsymbol{y}} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix}. \quad (9.16\text{E})$$

We know from the equation of motion (9.14E) that $\ddot{x} = -\frac{k}{M}x$. So, we can re-write $\dot{\boldsymbol{y}}$ as

$$\dot{\boldsymbol{y}} = \begin{pmatrix} \dot{x} \\ -\frac{k}{M}x \end{pmatrix}. \tag{9.17E}$$

Finally, the components can be written as functions of $\boldsymbol{y}$:

$$\dot{\boldsymbol{y}} = \begin{pmatrix} y_2 \\ -\frac{k}{M}y_1 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{y}). \tag{9.18E}$$

We see that there are no more derivatives higher than first order, which is what we intended.

If we know the position x_n and velocity $v_n = \dot{x}_n$ of the block at time instant t_n, we can apply Euler's method (9.6) to integrate the system (9.18E) numerically with respect to time:

$$\begin{pmatrix} y_{1,n+1} \\ y_{2,n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ v_n \end{pmatrix} + h \begin{pmatrix} v_n \\ -\frac{k}{M}x_n \end{pmatrix}. \tag{9.19E}$$

This way, we can find the position $x_{n+1} = y_{1,n+1}$ and the velocity $v_{n+1} = y_{2,n+1}$ at time instant t_{n+1}. Repeating this integration for many steps finally yields an approximation for the trajectory $x(t)$ and for $v(t)$. The smaller the step size, the closer these approximations come to the analytic (exact) solution.

Example 9.5 This Example is about the governor of Example 2.2 on page 48. The equations of motion have been found in the form $\ddot{\psi} - f(\theta, \dot{\psi}, \dot{\theta}) = 0$ and $\ddot{\theta} - h(\theta, \dot{\psi}, \dot{\theta}) = 0$, with given functions f and h. Transform this set of equations into a set of first-order explicit differential equations.

Exemplary solution

The first-order explicit differential equations are

$$\boldsymbol{x} = \begin{pmatrix} \theta \\ \dot{\psi} \\ \dot{\theta} \end{pmatrix}, \qquad \dot{\boldsymbol{x}} = \begin{pmatrix} \dot{\theta} \\ f(\theta, \dot{\psi}, \dot{\theta}) \\ h(\theta, \dot{\psi}, \dot{\theta}) \end{pmatrix}. \tag{9.20E}$$

Remark: One can also include ψ in the state vector, but it is not needed to perform the integration of the state vector; it has no influence on the dynamics. So, one can save the computational effort. This angle would also grow to large values.

Problem 9.3 Derive (9.14E) using five different methods: work and kinetic energy, conservation of energy, Newton's second law, virtual work, Lagrange, and show that with the chosen definition of x, the equation of motion is independent of gravity.

Problem 9.4 Conduct three time steps to integrate (9.19E) and to find the position x at time $t = 3h$, assuming the system starts from resting position at $t = 0$, but with initial speed v_0 in downward direction.

Problem 9.5 Show that the function $x(t) = A\sin(\omega t - \varphi)$, with constants A, ω, and φ satisfies the equations of motion of the mass-spring system (9.14E), and find the value for ω.

Problem 9.6 Compare the analytic solution and the numerical solution of the spring-mass system to each other (see previous two problems) and compute the error in the final position as a function of h.

Sometimes, it is more convenient to choose a different definition for $\boldsymbol{y}$ than the one in (9.12), particularly when the second derivatives of the generalized coordinates are not readily obtained. This is for example often the case if Euler angles are part of the generalized coordinates.

Example 9.6 To determine the equations of motion of a rigid body in 3-D that is not subjected to any constraints, a suitable choice of generalized coordinates often is the position vector $\boldsymbol{r}$ containing Cartesian positions X_C, Y_C, Z_C of the center of mass C in a fixed coordinate system, and a set of Euler angles, for example yaw ψ, pitch θ, and roll φ. A very compact way to write the equations of motion is to combine Newton's second law with Euler's second law using a body-fixed triad, particularly if that triad is aligned with the principal axes of inertia.

However, using Newton-Euler, we do not obtain the second derivatives of the Euler angles, as we would need for (9.13). Instead, we obtain angular acceleration $\dot{\boldsymbol{\omega}}$. It is possible to re-write the equations of motion in terms of the Euler angles, by using the relationship between angular velocity and Euler angle derivatives. It is also possible to find the equations of motion directly in terms of the generalized coordinates, using the method of Lagrange. However, in both cases we will see that the equations of motion are much more lengthy and complex.

Therefore, a better choice of $\boldsymbol{y}$ in this case it to choose

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}_1^{\mathrm{T}} & \boldsymbol{y}_2^{\mathrm{T}} & \boldsymbol{y}_3^{\mathrm{T}} & \boldsymbol{y}_4^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}, \tag{9.21E}$$

with

$$\boldsymbol{y}_1 = \begin{pmatrix} X_C & Y_C & Z_C \end{pmatrix}^{\mathrm{T}}, \tag{9.22E}$$

$$\boldsymbol{y}_2 = \begin{pmatrix} \psi & \theta & \varphi \end{pmatrix}^{\mathrm{T}}, \tag{9.23E}$$

$$\boldsymbol{y}_3 = \begin{pmatrix} \dot{X}_C & \dot{Y}_C & \dot{Z}_C \end{pmatrix}^{\mathrm{T}}, \tag{9.24E}$$

$$\boldsymbol{y}_4 = \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix}^{\mathrm{T}}, \tag{9.25E}$$

where ω_x, ω_y, and ω_z are the angular velocity components in the body-fixed triad, because then we can directly find

$$\dot{\boldsymbol{y}} = \begin{pmatrix} \dot{\boldsymbol{y}}_1^{\mathrm{T}} & \dot{\boldsymbol{y}}_2^{\mathrm{T}} & \dot{\boldsymbol{y}}_3^{\mathrm{T}} & \dot{\boldsymbol{y}}_4^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} = \boldsymbol{f}(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4) = \boldsymbol{f}(\boldsymbol{y}) \tag{9.26E}$$

using Newton's second law and Euler's second law of motion. If the body has mass m and inertia tensor $\mathbf{I}_C$ about its center of mass, we find that

$$\boldsymbol{f}(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{y}_3 \\ \mathbf{A}(\boldsymbol{y}_2) \cdot \boldsymbol{y}_4 \\ \frac{1}{m} \sum \boldsymbol{F}_i \\ \mathbf{I}_C^{-1} \left(\sum \boldsymbol{M}_{Ci} - \boldsymbol{y}_4 \times (\mathbf{I}_C \boldsymbol{y}_4) \right) \end{pmatrix}. \tag{9.27E}$$

Hereby, the matrix $\mathbf{A}$ transforms angular velocity to Euler angle derivatives. We can find this relationship using the techniques of Chapter 4. See Problem 9.8 for the case of yaw-pitch-roll Euler angles, and Problem 9.13 for precession-nutation-spin.

We recognize Newton's and Euler's second laws in the third and fourth entries in (9.27E), with the sum of external forces $\boldsymbol{F}_i$ and the sum of external moments about C, $\sum \boldsymbol{M}_{Ci}$.

Problem 9.7 This Problem is about the toy top of Example 6.4 on page 190. Derive a first-order explicit ODE for this system.

9.4 Summary

Numerical integration enables us to calculate the movement of a mechanical system as a function of time, without having to find the analytical solution to the equations of motion. One particularly simple and straightforward method is Euler's method, which approximates the mechanical system locally by a first-order system.

In order to apply simple integration methods, one first needs to transform the equations of motion into a set of first-order explicit ordinary differential equations. This requires choosing a state vector $\boldsymbol{y}$ such that

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, t). \tag{9.28}$$

Often, it is convenient to choose a concatenation of generalized coordinates and their first derivatives in the state vector. However, in case Euler angles are used as part of the generalized coordinates, it often leads to simpler equations if not the derivatives of Euler angles, but angular velocity is chosen instead as part of the state vector.

9.5 Problems

9.5.1 Guided Problems

Problem 9.8 We aim to create a simulation of the model plane of Problem 3.23 on page 92.

The plane's mass distribution is symmetric about the body-fixed axes, and the mass moments of inertia about these axes are I_{xx}, I_{yy}, I_{zz}. The center of mass is at C. Gravity g points in positive Z-direction.

a. **Visualization of Kinematics:** Draw a kinematic diagram to provide all necessary information on angular velocities, including Euler angles and axes labels. Use the Yaw-Pitch-Roll convention, with intermediate triads $\mathcal{N}$ (fixed), $\mathcal{F}$ (performs yaw), $\mathcal{G}$ (performs yaw and pitch), $\mathcal{B}$ (performs yaw, pitch, roll).

b. **Re-writing the equations of motion as first-order explicit differential equations:** The equations of motion of the plane have been derived in Example 6.8 on page 199.
Show that the state vector $\boldsymbol{y} := ({}^{\mathcal{N}}\boldsymbol{r}_C^{\mathrm{T}},\, \psi,\, \theta,\, \varphi,\, {}^{\mathcal{N}}\dot{\boldsymbol{r}}_C^{\mathrm{T}},\, {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathrm{p}}^{\mathrm{T}})^{\mathrm{T}}$ is a suitable choice to transform the system into the form $\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y},\, t)$. For the purpose of this part of the question, assume that you already know the mapping from the angular velocity vector ${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathrm{p}}$ to Euler angle derivatives, so $\dot{\psi} = h_1({}^{\mathcal{B}}\boldsymbol{\omega}_{\mathrm{p}})$, $\dot{\theta} = h_2({}^{\mathcal{B}}\boldsymbol{\omega}_{\mathrm{p}})$, $\dot{\varphi} = h_3({}^{\mathcal{B}}\boldsymbol{\omega}_{\mathrm{p}})$.

c. **Kinematics:** Express the plane's angular velocity vector ${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathrm{p}}$ as a vector function of the derivatives $\dot{\psi}$, $\dot{\theta}$, $\dot{\varphi}$ of the Euler angles ψ, θ, φ with respect to time, and of appropriate unit vectors of the rotation axes. For the purpose of this part of the question, assume that you know all rotation matrices from any triad to any other, you do not need to write these out. Then, determine the matrix $\mathbf{A}$ that allows re-writing this vector function into the form ${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathrm{p}} = \mathbf{A} \cdot (\dot{\psi},\, \dot{\theta},\, \dot{\varphi})^{\mathrm{T}}$. Note that the last term is not a true vector.

d. **Inverse Kinematics:** Implement a generic function in MATLAB$^{\mathrm{TM}}$ or a program of your choice that returns yaw-pitch-roll angular rates $\dot{\psi}$, $\dot{\theta}$, $\dot{\varphi}$ as functions of a triad $\mathcal{B}$'s angular velocity ${}^{\mathcal{B}}\boldsymbol{\omega}$, with components ω_x, ω_y, ω_z.
Use your implemented function to show that the generic symbolic equations that map angular velocity to Euler angle derivatives are

$$\dot{\psi} = \frac{1}{\cos\theta}(\omega_y \sin\varphi + \omega_z \cos\varphi), \tag{9.29P}$$

$$\dot{\theta} = \omega_y \cos\varphi - \omega_z \sin\varphi, \tag{9.30P}$$

$$\dot{\varphi} = \omega_x + \dot{\psi} \sin\theta. \tag{9.31P}$$

e. **Simulation:** Choose numerical values for all parameters and initial conditions and numerically integrate the equations of motion of the plane, using Euler's method.
f. **Check results by varying parameters:** Create an animation and visually inspect the physics and the behavior for special choices of parameters or initial conditions (e.g. stiffer or softer spring, very high inertia about some axes, etc.). Is this plausible behavior?
g. **Check results using energy:** Use your results from Problem 5.32 to check the simulated movement using energy principles. You can do this by creating a function that takes as inputs a parameter `struct` and the matrix of states provided by the numerical integration (Using a `struct` for parameters, you can more easily change parameters across your entire code), and that outputs different energy terms[a].

[a] *Hint*: Since this is a conservative system, the sum of potential and kinetic energy should remain constant. Alternatively, calculate the power of the spring force and compare it to the rate of change of the plane's kinetic energy.

9.5.2 Practice Problems

Problem 9.9 Explain the difference between Cartesian space, configuration space, and state space. Which vectors in this chapter are associated with which space? How are the dimensions of the spaces related to each other?

Problem 9.10 This Problem is about the ship installing a monopile of Problem 4.21 on page 130. Given is the equation of motion for the ship's vertical (heave) motion:

$$\ddot{Z}_K = -\Omega^2 Z_K. \tag{9.32P}$$

All other ship translations and rotations (surge, sway, yaw, pitch, roll) are zero. The constant heave frequency Ω follows from a simplified ocean wave with a time period of 7 s such that $\Omega = 0.9\,\mathrm{rad/s}$.

Use the state vector $\boldsymbol{x} = (Z_K,\, \dot{Z}_K)^{\mathrm{T}}$ to perform one step of numerical integration with Euler's method starting at $t = 0\,\mathrm{s}$, with the initial conditions

$$Z_K(t = 0\,\mathrm{s}) = 0.5\,\mathrm{m}, \tag{9.33P}$$

$$\dot{Z}_K(t = 0\,\mathrm{s}) = 0\,\mathrm{m/s}, \tag{9.34P}$$

in order to calculate the numerical values of the components of $\boldsymbol{x}(t = 0.2\,\mathrm{s})$.

Be sure to include correct units throughout your calculation. ▷►

Problem 9.11 This Problem is about the tube with a particle in it of Problem 4.35 on page 142.

Here, we assume an infinitely stiff connection ($k \to \infty$) and a restriction of movement in β, with $\beta = 0, \dot{\beta} = 0, \ddot{\beta} = 0$, such that we are in fact considering a pendulum in 2-D. The equation of motion is as follows:

$$\ddot{\alpha} = -\frac{g}{L}\sin(\alpha). \tag{9.35P}$$

Use numerical integration with Euler's method in vector notation to obtain the angular velocity $\dot{\alpha}$ at time $t = \frac{\pi}{4}\,\mathrm{s}$. Use initial angle $\alpha(t=0) = 0\,\mathrm{rad}$, initial angular velocity $\dot{\alpha}(t=0) = 2\,\mathrm{rad/s}$ and timestep $h = \frac{\pi}{8}\,\mathrm{s}$. Give $\dot{\alpha}(t = \frac{\pi}{4}\,\mathrm{s})$ as a function of g and L. Note the consistent handling of units throughout the calculation, as explained in Section 1.1.4. ▶

Problem 9.12 This Problem is about the ballerina doing a jumping pirouette of Problem 8.5 on page 256. Convert the equations of motion, as given in Problem 8.5 and Problem 8.6, into a system of first-order equations. Then, use the Euler integration scheme to numerically integrate the system for one time step of ΔT and provide numerical values for the generalized coordinates r, θ, and d after this time interval.

Assume Q_r, Q_θ and Q_d are zero, and use the below initial conditions and parameter values:

$r = 1\,\mathrm{m}$	$\theta = 0\,\mathrm{rad}$	$d = 0.2\,\mathrm{m}$
$\dot{r} = 0\,\mathrm{m/s}$	$\dot{\theta} = 1\,\mathrm{rad/s}$	$\dot{d} = 0\,\mathrm{m/s}$
$m_\mathrm{h} = 0.1\,\mathrm{kg}$	$I_\mathrm{b} = 0.2\,\mathrm{kg\,m^2}$	$m_\mathrm{b} = 60\,\mathrm{kg}$
$g = 10\,\mathrm{m/s^2}$	$\Delta T = 0.1\,\mathrm{s}$	$Q_r = 0\,\mathrm{N}$
$Q_\theta = 0\,\mathrm{N\,m}$	$Q_d = 0\,\mathrm{N}$	$F_\mathrm{h} = 0$

Make sure to include units correctly throughout (see Section 1.1.4). ▶

Problem 9.13 Use the MATLAB$^{\mathrm{TM}}$ symbolic toolbox to show that the relationships between body-fixed angular velocity components $\omega_x, \omega_y, \omega_z$ and the derivatives of type II-Euler angles (precession-nutation-spin) are:

$$\dot{\varphi} = \frac{1}{\sin\theta}(\omega_x \sin\psi + \omega_y \cos\psi), \tag{9.36P}$$

$$\dot{\theta} = \omega_x \cos\psi - \omega_y \sin\psi, \tag{9.37P}$$

$$\dot{\psi} = \omega_z - \dot{\varphi}\cos\theta. \tag{9.38P}$$

Part II

Multibody Dynamics

Overview Part II

Multibody system dynamics is a branch of mechanics that studies the motion, under the action of forces, of mechanical systems which consist of bodies interconnected by joints, the so-called *multibody systems* (MBSs). The bodies can be deformable or rigid, and the same holds for the joints, which can be idealized or deformable. This book only treats rigid bodies connected by idealized joints, sometimes described by "rigid bodies with constraints."

In order to determine how the multibody system will move, we need to derive the equations of motion (EoM). If we are only interested in the motion, then we can numerically integrate these equations of motion, given some initial conditions. However, the equations of motion can also be used to find equilibrium configurations or steady motions of the system, and to analyze their stability.

In this part II of the book, we will focus on the generation and analysis of the motion, and therefore the subject of numerical integration is an important chapter. At the end of this part, the reader will be able to derive the equations of motion for complex 3-D multibody system which consist of rigid bodies with constraints, and to simulate the motion by means of numerical integration.

In the first chapters, we make a gentle start by first applying the basic techniques to two-dimensional (2-D) systems, mainly because the treatment of rotation in the three-dimensional (3-D) world is a tricky subject and distracts from the basic concepts of multibody dynamics. In the 2-D treatment, we start with a pencil-and-paper approach for setting up the equations of motion for the MBS. In Chapter 11, a more systematic approach is developed, which builds on the concepts of virtual power and Lagrangian multipliers, to incorporate the constraints. The result is a set of coupled differential and algebraic equations (DAE). Some extra attention is given to the incorporation of passive and active elements, like springs, dampers, and motors, in the system. The previously derived equations of motion are the basis for the derivation of the impact equations for a general multibody system. This is followed by the introduction of the concept of independent generalized coordinates, in order to eliminate the constraint equations from the set of system equations. With these generalized coordinates, the so-called Lagrange's equations of motion are derived in Chapter 8. Although Lagrange's equations look very elegant and compact, a more general method for deriving the equations of motion in terms of generalized

constraints based on the principle of virtual power, the inclusion of d'Alembert forces, and the transformation of the center of mass coordinates in terms of generalized coordinates, the so-called TMT method, is introduced in Chapter 13. In order to analyze the motion in time of the multibody system, the concept of numerical integration of ordinary differential equations is introduced in Chapter 14. We describe a number of methods, pay attention to accuracy and stability of the solutions, and apply the methods to the equations of motion. In case we cannot use independent generalized coordinates, we are left with a system of coupled differential and algebraic equations. The numerical integration of these types of coupled equations is treated in Chapter 15. The 2-D part ends with the treatment of idealized rolling contact in the form of non-holonomic constraints and incorporating them into to the system equations, and the numerical integration of these systems of coupled differential and algebraic equations.

From Chapter 17 onward, we start treating the kinematics and general dynamics of a 3-D rigid body. Here, much attention is given to the description of orientation in 3-D. We first revisit Euler angles, which were already treated in part I, and then introduce Euler parameters. The basic concept of Euler angles by means of "cans-in-series" is demonstrated. The problems at the end of Chapter 18 bring together all concepts treated in the chapters before.

The book ends by combining the concepts from the 2-D part, setting up the equations of motion and solving them, with the 3-D part. With this knowledge, complex 3-D multibody dynamics systems with holonomic and non-holonomic constraints can now be described and analyzed. As a last example, the complete set of coupled differential and algebraic and kinematic equations for some complex 3-D MBS with holonomic and non-holonomic constraints are derived.

There already exists a vast amount of literature in the field of Multibody System Dynamics (MSD), initiated by the introduction of small and inexpensive computers in the 1970s. The first classical book on computational MSD is from 1977 by Wittenburg [77], after which books by Kane & Levinson [35], Roberson & Schwertassek [58], Nikravesh [50] and Haug [29] appeared. Then in 1990 an overview of methods and computer codes in MBD was published by Schiehlen [61]. Real-time applications were treated by Garcia de Jalon & Bayo [20] whereas the application to deformable MSD was treated by Gèrardin & Rixen [23]. Numerical aspects were addressed by Eich-Soellner & Führer [14]. After which books by Shabana [64], Moon [46], Baruh [3] appeared. Flexible

MSD was treated more in-depth in Gèradin & Cardona [24]. The recent book by Nikavesh [51] gives a nice overview of planar multibody dynamics with many examples and MATLAB$^{\text{TM}}$ programs. Of course all this literature builds on classical mechanics and dynamics books mentioned in part I.

Concerning notations, we tried to use the same conventions and practices as in part I of this book, however some small differences may occur. One notable difference is that in part II, we will make little use of the uppercase X, Y, and Z for an inertial coordinate system, but rather stick to the usual lowercase x, y, and z. Whenever there is room for confusion, we will use the uppercase symbols, like in Chapter 18 on 3-D rotation.

10 Newton-Euler Equations and Constraints, Free-Body Diagrams

In this chapter, we will derive the equations of motion for a 2-D system of rigid bodies connected by idealized joints, by hand. The "pencil-and-paper" method is very basic and straightforward; its essence is to "divide and conquer".

First, all rigid bodies are identified, after which we cut all joints and introduce the joint forces. Then, for the bodies cut free, the *free-body diagrams* (FBDs) are drawn and the Newton-Euler equations of motion for each individual body are established. This results in the differential equations of motion. However, we are not able to solve these, because we have more unknowns than equations. To be specific, the unknowns are the linear and angular accelerations of the *center of mass* (CoM) of the rigid bodies, together with the constraint forces.

To return from the loose bodies to the original MBS, we have to "glue" the system back. As "glue" we have the constraint equations, which represent the joints. Since these equations are as many as the unknown constraint forces, we obtain the same number of equations as there are unknowns. Given the initial conditions on the positions and velocities of the rigid bodies, together with the applied forces, we can solve for the unknown accelerations and constraint forces.

This "pencil-and-paper" method is demonstrated in the next sections by means of an example: a double pendulum under the action of gravity.

10.1 Free-Body Diagrams for Multibody Systems

For each individual rigid body in the system, a separate FBD is drawn. This requires cutting at all joints between the rigid bodies, and introducing joint forces (and possibly moments) in those directions where the connections previously prevented relative motion of the bodies with respect to each other.

Joint forces always come in pairs, as stated in Newton's third law of motion: "To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts." In other words, a joint force is never alone. When restoring the cuts and joining the bodies, both joint forces cancel, and their contribution to the net force on the system is zero.

Example 10.1 The double pendulum system presented in Figure 10.1 consists of two rigid beams connected by two revolute joints. The applied forces are due to gravity, which in this example acts in the x direction.

One end of the first beam is connected by a revolute joint to the ground, and on the other end is connected to the second beam. The rigid beams each have a length l_i, mass m_i, and a mass moment of inertia I_i, at the CoM$_i$ ($i = 1, 2$), which is at the middle of each beam.

The position of each beam is described by the x_i- and y_i-coordinates of the centers of mass, and the orientation of each beam is described by the angle φ_i, measured counter-clockwise relative to the x-axis. The O-xy coordinate system is an inertial system.

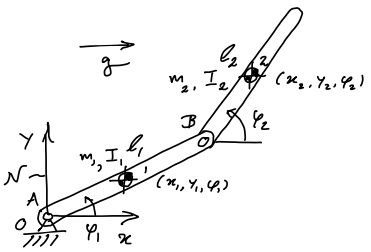


Figure 10.1: Double pendulum system.

To draw the free-body diagrams of the individual rigid bodies, we start with cutting the joints, see Figure 10.2.

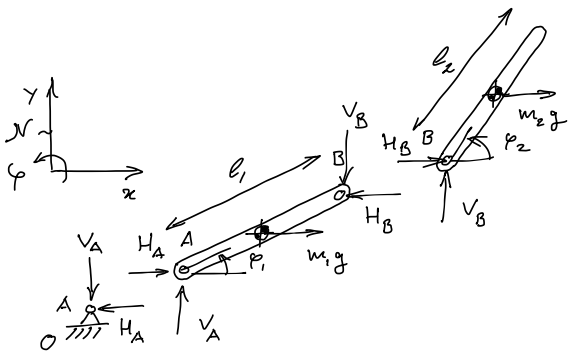


Figure 10.2: Two FBDs: one for each of the rigid bodies of the double pendulum.

For each cut, we introduce a pair of action-reaction joint forces, resulting in a set of unknown joint forces: (H_A, V_A, H_B, V_B). Note that also the support in A is drawn, with forces acting on it. Strictly speaking, this is not a free-body diagram, but it clearly shows joint forces always come in pairs.

10.2 Equations of Motion with Newton-Euler

In 2-D, each rigid body has three *degrees of freedom* (DoF), two translations and one rotation. So, for a number b of rigid bodies, there will be $n = 3b$ unknown accelerations.

Recall that Newton states that the sum of the applied forces equals mass times acceleration in the two possible directions. Moreover, Euler states that (in 2-D) the sum of the applied moments with respect to the CoM equals the mass moment of inertia at the CoM times the angular acceleration. This results in n equations of motion.

Example 10.2 The free-body diagrams for the system of Example 10.1 enable us to write down the equations of motion for each individual body. Every rigid body has three DoF, the position of the CoM (x_i, y_i) together with the orientation angle of body φ_i with respect to the x-axis.

So, the Newton-Euler equations of motion for body 1 are,

$$\begin{aligned} H_A + m_1 g - H_B &= m_1 \ddot{x}_1, \\ V_A - V_B &= m_1 \ddot{y}_1, \\ (H_A + H_B)\,\tfrac{1}{2} l_1 \sin\varphi_1 - (V_A + V_B)\,\tfrac{1}{2} l_1 \cos\varphi_1 &= I_1 \ddot{\varphi}_1, \end{aligned} \tag{10.1E}$$

and for body 2,

$$\begin{aligned} H_B + m_2 g &= m_2 \ddot{x}_2, \\ V_B &= m_2 \ddot{y}_2, \\ H_B \tfrac{1}{2} l_2 \sin\varphi_2 - V_B \tfrac{1}{2} l_2 \cos\varphi_2 &= I_2 \ddot{\varphi}_2. \end{aligned} \tag{10.2E}$$

10.3 Constraints

To solve for the n unknown accelerations as well as the m constraint forces, we need m more equations, which will come from assembling the free bodies back in their joints, the "glue". In technical terms, we call this glue the constraint equation, because the joint between the two bodies constrains the motion of the two bodies.

Example 10.3 In the six equations of motion derived in Example 10.2, we have ten unknowns: the $n = 6$ accelerations of the two bodies ($\ddot{x}_1$, $\ddot{y}_1$,

$\ddot{\varphi}_1$, $\ddot{x}_2$, $\ddot{y}_2$, $\ddot{\varphi}_2$) and the $m = 4$ joint forces (H_A, V_A, H_B, V_B). To solve for these ten unknowns, we need four more equations from the constraints.

In A, body 1 is connected by a revolute joint to the fixed world, and in B, body 1 is connected to body 2 also by a revolute joint. The first two constraints then describe that A is fixed in the origin O in the x and y direction. Furthermore, the two other constraints arise from the connection between the bodies at B, which restricts separate movement of the two bodies at that point in the x and y direction:

$$\begin{aligned} x_1 - \tfrac{1}{2} l_1 \cos\varphi_1 &= 0, \\ y_1 - \tfrac{1}{2} l_1 \sin\varphi_1 &= 0, \\ x_1 + \tfrac{1}{2} l_1 \cos\varphi_1 &= x_2 - \tfrac{1}{2} l_2 \cos\varphi_2, \\ y_1 + \tfrac{1}{2} l_1 \sin\varphi_1 &= y_2 - \tfrac{1}{2} l_2 \sin\varphi_2. \end{aligned} \tag{10.3E}$$

Note that while in the free-body system we have 6 degrees of freedom, 2×3 for each rigid body, the 4 constraint equations on the 6 CoM coordinates result in $6 - 4 = 2$ degrees of freedom for the double-pendulum system. Describing the kinematics and dynamics of the system in terms of the degrees of freedom will not be treated in this chapter, but further on in the book in Chapter 12 and Chapter 13.

10.4 Differential – Algebraic Equations (DAEs)

We are now able to form the complete set of constrained equations of motion. These are the combination of the Newton-Euler equations of motion and the constraint equations. Since the former are differential equations and the latter are algebraic equations[1], we call this the system of coupled differential and algebraic equations, the DAEs.

However, in this form we are not yet able to solve for the unknown accelerations and the unknown forces, because the constraint equations are expressed in terms of the CoM coordinates.

Therefore, we will express the constraints on the accelerations of the CoM coordinates by differentiating the constraint equations twice with respect to time.

Example 10.4 For the double pendulum example, the unknowns are the six accelerations $\ddot{x}_1$, $\ddot{y}_1$, $\ddot{\varphi}_1$, $\ddot{x}_2$, $\ddot{y}_2$, and $\ddot{\varphi}_2$ and the four joint forces H_A, V_A, H_B, and V_B, while the constraint equations are expressed in (x_1, y_1, φ_1,

[1] The term algebraic equation need not denote an equation that only contains algebraic functions, but can have an extended meaning and contain also transcendental functions; it is merely used to distinguish it from a differential equation in the respect that it does not contain derivatives.

x_2, y_2, φ_2).

Therefore, first the constraint equations are differentiated twice with respect to time. With some rearranging of terms in order to write all second-order time derivatives on the left hand side, we obtain

$$\begin{aligned}
-\ddot{x}_1 - \tfrac{1}{2}l_1\ddot{\varphi}_1 \sin\varphi_1 &= \tfrac{1}{2}l_1\dot{\varphi}_1^2 \cos\varphi_1,\\
-\ddot{y}_1 + \tfrac{1}{2}l_1\ddot{\varphi}_1 \cos\varphi_1 &= \tfrac{1}{2}l_1\dot{\varphi}_1^2 \sin\varphi_1,\\
\ddot{x}_1 - \ddot{x}_2 - \tfrac{1}{2}l_1\ddot{\varphi}_1 \sin\varphi_1 - \tfrac{1}{2}l_2\ddot{\varphi}_2 \sin\varphi_2 &= \tfrac{1}{2}l_1\dot{\varphi}_1^2 \cos\varphi_1 + \tfrac{1}{2}l_2\dot{\varphi}_2^2 \cos\varphi_2,\\
\ddot{y}_1 - \ddot{y}_2 + \tfrac{1}{2}l_1\ddot{\varphi}_1 \cos\varphi_1 + \tfrac{1}{2}l_2\ddot{\varphi}_2 \cos\varphi_2 &= \tfrac{1}{2}l_1\dot{\varphi}_1^2 \sin\varphi_1 + \tfrac{1}{2}l_2\dot{\varphi}_2^2 \sin\varphi_2.
\end{aligned} \tag{10.4E}$$

Combination of the equations of motion (10.1E) and (10.2E), and the constraint equations (10.4E) leads to the set of DAEs, as in:

$$\begin{pmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \ddot{\boldsymbol{x}} \\ \boldsymbol{F}_\mathrm{c} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F}_\mathrm{g} \\ \boldsymbol{a}(\boldsymbol{x},\, \dot{\boldsymbol{x}}) \end{pmatrix}, \tag{10.5E}$$

with the matrices

$$\begin{aligned}
\mathbf{M} &= \mathrm{diag}\,(m_1,\, m_1,\, I_1,\, m_2,\, m_2,\, I_2)\,,\\
\mathbf{A} &= \begin{pmatrix}
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
-\frac{1}{2}l_1 \sin\varphi_1 & \frac{1}{2}l_1 \cos\varphi_1 & -\frac{1}{2}l_1 \sin\varphi_1 & \frac{1}{2}l_1 \cos\varphi_1 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -\frac{1}{2}l_2 \sin\varphi_2 & \frac{1}{2}l_2 \cos\varphi_2
\end{pmatrix},\\
\mathbf{B} &= \begin{pmatrix}
-1 & 0 & -\frac{1}{2}l_1 \sin\varphi_1 & 0 & 0 & 0 \\
0 & -1 & \frac{1}{2}l_1 \cos\varphi_1 & 0 & 0 & 0 \\
1 & 0 & -\frac{1}{2}l_1 \sin\varphi_1 & -1 & 0 & -\frac{1}{2}l_2 \sin\varphi_2 \\
0 & 1 & \frac{1}{2}l_1 \cos\varphi_1 & 0 & -1 & \frac{1}{2}l_2 \cos\varphi_2
\end{pmatrix},
\end{aligned} \tag{10.6E}$$

and vectors

$$\begin{aligned}
\ddot{\boldsymbol{x}} &= \begin{pmatrix} \ddot{x}_1 & \ddot{y}_1 & \ddot{\varphi}_1 & \ddot{x}_2 & \ddot{y}_2 & \ddot{\varphi}_2 \end{pmatrix}^\mathrm{T},\\
\boldsymbol{F}_\mathrm{c} &= \begin{pmatrix} H_A & V_A & H_B & V_B \end{pmatrix}^\mathrm{T},\\
\boldsymbol{F}_\mathrm{g} &= \begin{pmatrix} m_1 g & 0 & 0 & m_2 g & 0 & 0 \end{pmatrix}^\mathrm{T},\\
\boldsymbol{a}\,(\boldsymbol{x},\, \dot{\boldsymbol{x}}) &= \begin{pmatrix}
\frac{1}{2}l_1\dot{\varphi}_1^2 \cos\varphi_1 \\
\frac{1}{2}l_1\dot{\varphi}_1^2 \sin\varphi_1 \\
\frac{1}{2}l_1\dot{\varphi}_1^2 \cos\varphi_1 + \frac{1}{2}l_2\dot{\varphi}_2^2 \cos\varphi_2 \\
\frac{1}{2}l_1\dot{\varphi}_1^2 \sin\varphi_1 + \frac{1}{2}l_2\dot{\varphi}_2^2 \sin\varphi_2
\end{pmatrix}.
\end{aligned} \tag{10.7E}$$

With given initial conditions at $t = t_0$ for the CoM positions and velocities, $\boldsymbol{x}(t_0)$, and $\dot{\boldsymbol{x}}(t_0)$, these equations (10.5E) can be solved for the accelerations of the CoM's $\ddot{\boldsymbol{x}}$, together with the joint or constraint forces $\boldsymbol{F}_\mathrm{c}$.

Upon close inspection of the system of constrained equations of motion (10.5E), we see in (10.6E) that $\mathbf{A}^{\mathrm{T}} = \mathbf{B}$. This is not really a coincidence, which will be demonstrated in the next chapter, were we develop an even more systematic approach for deriving the constrained equations of motion.

A specific set of initial conditions enables solving the constrained EoM for accelerations and constraint forces. By choosing special cases as initial conditions, plausibility of the solution can be investigated.

Example 10.5 For the MBS described earlier in Example 10.1 to Example 10.4 we are looking at the initial motion for the initial vertical position (Figure 10.3), $\varphi_1 = \varphi_2 = \frac{\pi}{2}$ and initially from rest, that is $\dot{\varphi}_1 = \dot{\varphi}_2 = 0$. For this example, we use the following values for the physical parameters:

$$m_1 = m_2 = 0.10\,\mathrm{kg}, \tag{10.8E}$$

$$l_1 = l_2 = 0.50\,\mathrm{m}, \tag{10.9E}$$

$$I_1 = \tfrac{1}{12} m_1 l_1^2, \tag{10.10E}$$

$$I_2 = \tfrac{1}{12} m_2 l_2^2. \tag{10.11E}$$

With all these values (10.5E) can be solved for the unknown accelerations of the CoMs $\ddot{\boldsymbol{x}}$ and joint or constraint forces $\boldsymbol{F}_{\mathrm{c}}$, for example by using MATLAB$^{\mathrm{TM}}$ or some other programming language:

$$\ddot{\boldsymbol{x}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\varphi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} \frac{9}{14} \\ 0 \\ \frac{-18}{7}\frac{1}{\mathrm{m}} \\ \frac{15}{14} \\ 0 \\ \frac{6}{7}\frac{1}{\mathrm{m}} \end{pmatrix} g, \quad \boldsymbol{F}_{\mathrm{c}} = \begin{pmatrix} H_A \\ V_A \\ H_B \\ V_B \end{pmatrix} = \begin{pmatrix} -\frac{1}{35} \\ 0 \\ \frac{1}{140} \\ 0 \end{pmatrix} g \cdot \mathrm{kg}. \tag{10.12E}$$

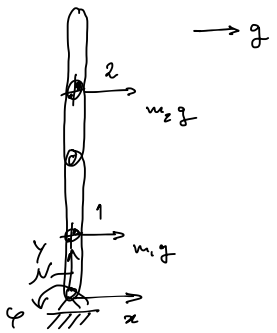


Figure 10.3: Vertical position of the double pendulum.

If we interpret the results, we see that initially $\ddot{x}_1 < g$ and $\ddot{x}_2 > g$. The first pendulum is thus obstructed in its motion, because it is hinged to the

world at one end. On the other hand, the CoM of the second pendulum is accelerated faster than gravity. This is because the first pendulum is pushing the second pendulum in joint B, $H_B > 0$.

Problem 10.1 Implement a function with the name `DAEdp`, which takes as inputs specific numerical values for time `t`, configuration vector `x`, and derivative vector `xd`, and which provides as outputs the vector of accelerations `xdd` and the vector of constraint forces `Fc`.

In order to obtain the motion of the system as a function of the time, $\boldsymbol{x}(t)$, we have to integrate numerically the equations of motion in a step-by-step procedure. This is the subject of Chapter 14.

Example 10.6 Here we demonstrate the basic concept of numerical integration by means of a simple MATLAB^{TM}-type script applied to (10.5E), and making use of the function `DAEdp` as derived in Problem 10.1.

```
% initialize variables
% initial position and velocities
x  = [0 l/2 pi/2 0 3*l/2 pi/2];
xd = [0 0 0 0 0 0];
% plot the initial configuration.
t  = 0;   %(s)
dt = 0.1; %(s)
% acc. and cons. forces in initial state
[xdd,Fc] = DAEdp(t,x,xd);
% loop over time
for tindex = 2:10
  % increment time and estimate new position and new velocity
  % by an Euler step
  t = t + dt;
  x  = x + xd.*dt;
  xd = xd + xdd.*dt;
  % acc. and cons. forces in new state
  [xdd,Fc] = DAEdp(t,x,xd);
end
```

The resulting configuration for the first four steps is shown in Figure 10.4. We see in the left figure that the double pendulum moves, but that the segments are flying apart in the joints.

Even with smaller integration time steps, as shown in the right part of the figure, the bodies still separate at the joints. Indeed, even if we make the time step smaller and smaller, they still will fly apart, tough less and less. In the forthcoming chapters we will address and solve this problem.

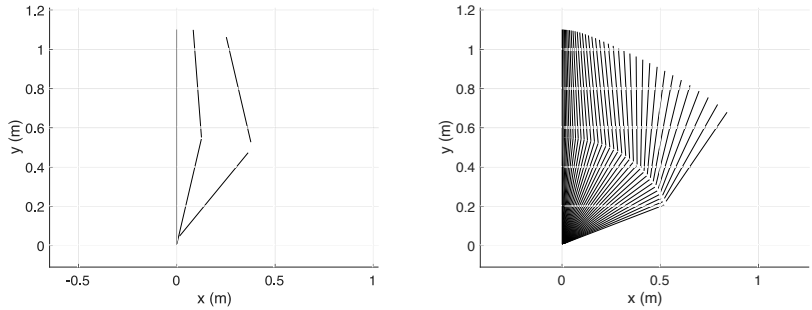


Figure 10.4: Simulation of the motion of the double pendulum by numerical integration of the constrained equations of motion. Left with $\Delta t = 0.1\,\mathrm{s}$, and right with $\Delta t = 0.01\,\mathrm{s}$.

10.5 Problems

Problem 10.2 The model of a double pendulum consists of two slender rigid bars made out of steel ($\rho = 7850\,\mathrm{kg/m^3}$), each with length $l = 0.60\,\mathrm{m}$ and with a rectangular cross section of $w = 50\,\mathrm{mm}$ by $t = 3\,\mathrm{mm}$. The first bar is connected to the ground by a cylindrical joint in A, the left side of the bar. The left end of the second bar, B, is connected to the right end of the first bar by a second cylindrical joint. We use an inertial xy-coordinate system with A in the origin, and we assume a gravitational field operating in the positive x-direction with a field strength of $g = 9.81\,\mathrm{N/kg}$.

a. Derive the equations of motion together with the joint constraint equations for this system:

Calculate for each of the following initial conditions the linear and angular accelerations of the center of mass of the bodies, together with the forces of interaction in the joints (joint constraint forces).

b. Both bars along the positive y-axis and with zero speeds.
c. Both bars along the positive x-axis and with zero speeds.
d. Both bars along the positive x-axis and with an initial angular speed of $\omega = 60\,\mathrm{rpm}$ anti-clockwise on both bars. Also calculate the initial velocities of the center of mass of the two bodies in this state.

Note: Be sure to present your results by both a value *and* a dimension, like $5\,\mathrm{m/s}$ or $18\,\mathrm{N}$. ▶

Problem 10.3 Figure 10.5 shows the model of a double pendulum. Model the bars as uniform very slender bars, where the dimensions of the

cross sections are neglected with respect to the length. The CoM is *not* in the middle of each bar. Assume that in the initial configuration, the first bar is horizontal and the second bar is vertical up. Gravity is acting vertically down ($-y$ direction). The system is released from this position with zero velocities. Anti-clockwise rotation is assumed positive.

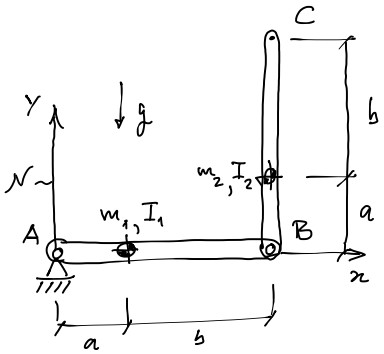


Figure 10.5: Double pendulum.

Determine the angular acceleration of the first bar, $\ddot{\varphi}_1$, just after the system is released.[a] For parameters use: $a = 0.25\,\mathrm{m}, b = 0.5\,\mathrm{m}, m_1 = m_2 = 3\,\mathrm{kg}, I_1 = I_2 = 0.3\,\mathrm{kg\,m^2}, g = 9.81\,\mathrm{N/kg}$. ▷

[a] *Hint*: Remember that gravity is pointing down. In this configuration and loading case, you expect no angular acceleration for bar 2. When formulating the constraints, take into account that the CoM is not in the middle of the bar. For instance, the horizontal constraint in joint B should read something like this: $(x_2 - a\cos(\varphi_2)) - (x_1 + b\cos(\varphi_1)) = 0$.

Problem 10.4 Consider the model of a two-bar crank-slider mechanism as shown in Figure 10.6. The crank rotates about the origin of xy in A, and the slider in C slides along the x-axis.

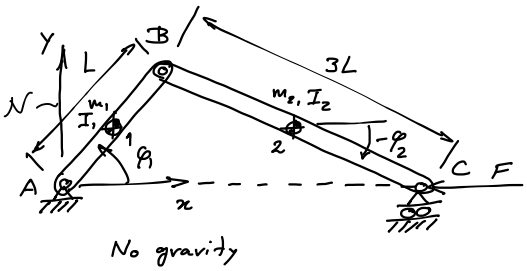


Figure 10.6: Crank-slider mechanism.

Model the bars as uniform very slender bars, where the dimensions of the cross sections are neglected with respect to the length. The CoM is in the middle of each bar. The length of bar 1 is L and the length of bar 2 is $3L$. The initial configuration is with all bars at rest, that is zero velocities, and bar 1 encloses an angle of $\varphi_1 = 45^\circ$ with the positive x-axis. A force F is applied at point C in the negative x-direction. Anti-clockwise rotation is assumed positive. Calculate the angular acceleration of the bar 1, $\ddot{\varphi}_1$, just after the force F is applied[a]. For parameters use: $L = 0.25\,\mathrm{m}$, $m_1 = 3\,\mathrm{kg}$, $I_1 = 0.3\,\mathrm{kg\,m^2}$, $m_2 = 5\,\mathrm{kg}$, $I_2 = 4.1\,\mathrm{kg\,m^2}$, $F = 10\,\mathrm{N}$. ▷

[a] *Hint*: This is like the double pendulum problem, but with an added constraint in C, such that $y_C = 0$. Remember to transfer the force F in C to the CoM of bar 2, by introducing an applied force $f_{x2} = -F$ and an applied torque $M_2 = F \cdot 3L/2 \sin(\varphi_2)$. You can check your calculations for the special case where bar 1 is vertical up ($\varphi_1 = 90^\circ$), in which case the following results should all be zero: vertical acceleration of bar 1, vertical acceleration of bar 2, and angular acceleration of bar 2. Now set φ_1 back to 45° and determine the angular acceleration of the bar 1 just after the force F is applied

11 Virtual Power and Lagrange Multipliers

In this chapter we will derive the equations of motion for a 2-D system of rigid bodies connected by idealized joints by means of a systematic approach. Here we will use the concept of virtual power together with the so-called d'Alembert forces. The idealized joint constraints on the system will be included by means of the method of Lagrangian multipliers. Special attention is given to the physical interpretation of this mathematical technique. We will demonstrate that for this systematic approach we only need to define the rigid bodies and formulate the constraints. Having defined these ingredients, deriving the constrained equations of motion is merely applying an algorithm, like a recipe. We will dedicate special attention to including passive and active elements like springs and motors in the system.

For calculating systems with intermittent contact, an idealized approach to describe the transition from a state just before to a state just after contact is by means of the impact equations. These impact equations for the system of rigid bodies connected by idealized constraints, undergoing single or multiple instantaneous impulsive contacts, will be derived in the same systematic way as the equations of motion.

11.1 Systematic Approach to Derive DAEs

A systematic approach to derive the DAEs of motion of a constrained MBS is by the application of the principle of virtual power. The principle of virtual power is very similar to the principle of virtual work, so wherever one reads "power", one can also think "work". However, here we have a slight preference for virtual power.

The virtual power of a force on a system is the scalar product of the force vector $\boldsymbol{F}$ with the virtual velocity vector $\delta\dot{\boldsymbol{x}}$ of the point of application of that force:

$$\delta P = \boldsymbol{F}^{\mathrm{T}} \delta\dot{\boldsymbol{x}}, \tag{11.1}$$

where $\delta\dot{\boldsymbol{x}}$ is the vector containing the virtual velocity components of the point of application. We use the common notation δ to denote "virtual"; note that it has nothing to do with the "d" from differentiation. We could also have written $P_\mathrm{v} = \boldsymbol{F}^\mathrm{T}\dot{\boldsymbol{x}}_\mathrm{v}$, or $\hat{P} = \boldsymbol{F}^\mathrm{T}\dot{\hat{\boldsymbol{x}}}$, with virtual velocities $\dot{\boldsymbol{x}}_\mathrm{v}$ or $\dot{\hat{\boldsymbol{x}}}$. Virtual velocities are just finite velocities[1], where virtual is opposed to the momentary real velocity distribution.

Example 11.1 A simple pendulum consists of a particle on a massless rod rotating about a hinge. The constraint force vector the rod exerts on the particle can never generate virtual power, because it is always orthogonal to any virtual velocity of the particle.

For the virtual power in a mechanical system, we use the following basic principle in mechanics:

The virtual power of all forces acting on a system is the sum of the individual virtual powers generated by every force.

With the help of the concept of virtual power of the forces acting on a system, we can make a statement on the (static or dynamic) equilibrium of that system. This proposition reads,

A mechanical system is in equilibrium if the virtual power is zero for all virtual velocities that satisfy the constraints.

The *dynamic equilibrium* of a system yields the equations of motion. In contrast to static equilibrium, dynamic equilibrium includes inertial effects in system. We include these inertia terms by using the so-called *d'Alembert forces*, $\boldsymbol{F}_\mathrm{in} = -m\ddot{\boldsymbol{x}}$. The concept of d'Alembert forces has already been explained in part I, Section 7.2.4

Note that while any physically possible system will always be in dynamic equilibrium, it does not need to be in static equilibrium.

Example 11.2 The pendulum mentioned in Example 11.1 can be at rest if pointing perfectly upwards or downwards. These are special cases where it is in static equilibrium and the virtual power generated by gravity (the only applied force) must be zero. In *all* cases, however, the pendulum will be in dynamic equilibrium, meaning that the sum of virtual power generated by gravitational forces *and* inertial forces is zero.

Velocities that satisfy the constraints are sometimes called *kinematically admissible velocities*.

[1] Note the difference between virtual displacements, which are usually infinitely small, and virtual velocities, which are finite.

◯ **Example 11.3** In the pendulum of Example 11.1, the particle's path must be circular, and all kinematically admissible velocities are tangential to this circle. Thus, gravity cannot generate virtual power when the pendulum is in perfectly upright or downward configuration. Centrifugal d'Alembert forces cannot generate virtual power either; only the particle's tangential acceleration components cause inertial forces that can generate virtual power.

Integrating over all infinitesimal mass elements $\mathrm{d}m$ and associated d'Alembert forces $\mathrm{d}\boldsymbol{F}_{\mathrm{in}} = -\ddot{\boldsymbol{x}}\,\mathrm{d}m$, we obtain a virtual power expression for the complete system,

$$\delta P = \int_m (\mathrm{d}\boldsymbol{F} + \mathrm{d}\boldsymbol{F}_{\mathrm{in}}) \cdot \delta\dot{\boldsymbol{x}} = \int_m (\mathrm{d}\boldsymbol{F} - \ddot{\boldsymbol{x}}\,\mathrm{d}m) \cdot \delta\dot{\boldsymbol{x}}. \tag{11.2}$$

Integration of this virtual power over the masses of all bodies greatly simplifies, because the bodies are rigid. Using the principle of statically equivalent force systems and the definition of the inertia tensor, we can discretize the system by centroidal properties of each individual body. More specifically, this requires replacing the acting forces and moments by equivalent force systems, i.e. a single force vector acting at each body's center of mass in combination with a free moment vector, and calculating the inertia tensor with respect to the body's CoM.

For the 2-D case, and using a global, inertial O-xy coordinate system, this leads to the discrete form of the virtual power expression for a system of N rigid bodies,

$$\delta P = \sum_{k=1}^{N} (\boldsymbol{F}_k - \mathbf{M}_k\ddot{\boldsymbol{x}}_k) \cdot \delta\dot{\boldsymbol{x}}_k, \tag{11.3}$$

with for every rigid body k the coordinates x_k and y_k of the center of mass and the body's orientation angle φ_k with respect to a fixed reference orientation subsumed in $\boldsymbol{x}_k = (x_k,\, y_k,\, \varphi_k)^{\mathrm{T}}$, the applied forces and moment at the center of mass $\boldsymbol{F}_k = (F_{xk},\, F_{yk},\, M_k)^{\mathrm{T}}$, and the diagonal mass matrices $\boldsymbol{M}_k = \mathrm{diag}(m_k,\, m_k,\, I_k)$.

To shorten the notation and make things clearer, from here on we will use index notation with Einstein's summation convention and comma-denoted partial derivatives. This method of notation is explained in Appendix D. Splitting all vectors and the matrix in the virtual power expression (11.3) into components, this notation reads:

$$\delta P = (F_i - M_{ij}\ddot{x}_j) \cdot \delta\dot{x}_i, \qquad i,j = 1 \ldots 3N, \tag{11.4}$$

where the summation is over the repeated indices i and j. Here we have introduced a vector x_i which contains the CoM coordinates and orientation of each rigid body in sequence, as in $x_i = (x_1, y_1, \varphi_1, x_2, y_2, \varphi_2, ...)$. We will often refer to this vector x_i as the vector of CoM coordinates, which strictly speaking should be the vector of generalized CoM coordinates, since it also includes the orientation of the bodies. The vector of applied force is now defined as $F_i = (fx_1, fy_1, M_1, fx_2, fy_2, M_2, ...)$.

Joint constraints are equations which can always be written in a zero equation form ($... = 0$), either in terms of a vector function $\boldsymbol{C}$ or its components, the scalar functions C_k,

$$\boldsymbol{C}(\boldsymbol{x}) = \mathbf{0}, \quad \text{or} \quad C_k\left(x_i\right) = 0, \tag{11.5}$$

where $k = 1 \ldots m$, with m constraints and $i = 1 \ldots n$, with n the total number of coordinates of the center of mass of the rigid bodies, usually $n = 3N$. The symbol C is used, because it refers to constraints.

To determine the velocities that satisfy the constraints, the kinematically admissible velocities, we first differentiate the constraints (11.5) with respect to time,

$$\dot{C}_k = \frac{\partial C_k\left(x_l\right)}{\partial x_i}\dot{x}_i = C_{k,i}\dot{x}_i = 0, \qquad k = 1 \ldots m, \quad i, l = 1 \ldots n, \tag{11.6}$$

and then replace the real velocities $\dot{x}_i$ by the virtual velocities $\delta\dot{x}_i$, as,

$$\delta\dot{C}_k = C_{k,i}\delta\dot{x}_i = 0, \tag{11.7}$$

which are the kinematic constraints on the virtual velocities. Note that the only free index, that is the index that is left after the summation has been done over the repeated index, is k, and therefore, these are again m equations.

These subsidiary conditions on the virtual velocities can be included in the virtual power expression by means of a mathematical technique called the method of the *Lagrange multipliers* [39][2]. The basic idea is that each subsidiary condition $\delta\dot{C}_k = 0$ is multiplied with an unknown multiplier λ_k, the so-called Lagrange multiplier, and added to the original virtual power expression (11.4), leading to the modified version

$$\delta P^* = (F_i - M_{ij}\ddot{x}_j) \cdot \delta\dot{x}_i - \lambda_k C_{k,i} \cdot \delta\dot{x}_i. \tag{11.8}$$

[2] besides Lanczos's book 'The variational principles of mechanics' [39], which is really a must to read, see also Bussotti's article [6] for some more background on Lagrange multipliers.

The "addition" is done with a minus sign for sign convention, which will become clear further on. Note that if the constraints are satisfied we just have added zeros to the virtual power expression.

Having added the subsidiary conditions we can make the statement about the dynamic equilibrium of the system, namely that the system is in dynamic equilibrium if $\delta P^* = 0$ for arbitrary virtual velocities $\delta\dot{x}_i$. The fact that it must hold for arbitrary $\delta\dot{x}_i$ allows breaking down the scalar equation into n separate equations,

$$F_i - M_{ij}\ddot{x}_j - C_{k,i}\lambda_k = 0, \qquad i,j = 1\dots n, \quad k = 1\dots m. \tag{11.9}$$

These are the constrained equations of motion: a set of differential equations that describe the dynamic equilibrium of the system, expressed in the unknown CoM accelerations $\ddot{x}_j$ and the unknown Lagrange multipliers λ_k.

Next, we add the constraint equations, which are the glue that glues the individual rigid bodies back into the original system. Since the unknowns are the accelerations, we have to differentiate the constraint equations twice with respect to time,

$$\ddot{C}_k = C_{k,j}\ddot{x}_j + C_{k,ij}\dot{x}_i\dot{x}_j = 0, \qquad i,\, j = 1\dots n, \quad k = 1\dots m. \tag{11.10}$$

Finally, we combine the equations of motion (11.9) with the acceleration constraints (11.10) into the set of DAEs, written in index notation and a semi-matrix-vector form,

$$\begin{pmatrix} M_{ij} & C_{k,i} \\ C_{k,j} & 0_{kk} \end{pmatrix} \begin{pmatrix} \ddot{x}_j \\ \lambda_k \end{pmatrix} = \begin{pmatrix} F_i \\ -C_{k,ij}\dot{x}_i\dot{x}_j \end{pmatrix}, \tag{11.11}$$

or in matrix-vector form,

$$\begin{pmatrix} \mathbf{M} & \mathbf{C}_{,\boldsymbol{x}}^{\mathrm{T}} \\ \mathbf{C}_{,\boldsymbol{x}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \ddot{\boldsymbol{x}} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F} \\ -\mathbf{C}_{,\boldsymbol{xx}}\dot{\boldsymbol{x}}\dot{\boldsymbol{x}} \end{pmatrix}, \tag{11.12}$$

from which we can solve the unknown accelerations $\ddot{x}_j$ together with the Lagrange multipliers λ_k.

If we compare this with the pencil-and-paper method from the previous chapter, we identify the Lagrange multipliers λ_k as the constraint forces and moments. This shows us that the systematic approach frees us from the burden to cut the system into rigid bodies, draw the free-body diagrams, introduce the joint forces, and write down the individual

Newton–Euler equations for each individual body. Instead, in the systematic approach, we only have to identify the rigid bodies and applied forces and write down the constraints. Then, to derive the constrained equations of motion, we just apply the algorithm.

The method of adding constraints by means of Lagrange multipliers feels like magic, but it can be illuminated by the following reasoning. We add a quantity $\lambda_k \delta \dot{C}_k$ to the virtual power expression, and if the dimension of $\dot{C}_k$ is either a linear velocity or an angular velocity, then the nature of the Lagrange multiplier λ_k must be a generalized force or moment. Moreover, since the quantity must be zero, we either enforce the constraint $\dot{C}_k = 0$ or we release the joint, and have $\lambda_k = 0$.

Example 11.4 If the constraint is a horizontal distance between two bodies, then the Lagrange multiplier is the horizontal joint force acting on the two bodies. If it regards a difference in angles, then the multiplier is a joint moment acting on the two bodies.

Example 11.5 We will apply the systematic approach to the double pendulum from Example 10.1. The constraints should be in the zero-delimited form. All constraints combined in one vector C_k are

$$\boldsymbol{C} = C_k = \begin{pmatrix} x_1 - \frac{1}{2} l_1 \cos\varphi_1 \\ y_1 - \frac{1}{2} l_1 \sin\varphi_1 \\ -x_1 + x_2 - \frac{1}{2} l_1 \cos\varphi_1 - \frac{1}{2} l_2 \cos\varphi_2 \\ -y_1 + y_2 - \frac{1}{2} l_1 \sin\varphi_1 - \frac{1}{2} l_2 \sin\varphi_2 \end{pmatrix} = \boldsymbol{0}. \tag{11.13E}$$

The partial derivatives or Jacobian matrix is

$$C_{k,j} = \begin{pmatrix} 1 & 0 & \frac{1}{2} l_1 \sin\varphi_1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} l_1 \cos\varphi_1 & 0 & 0 & 0 \\ -1 & 0 & \frac{1}{2} l_1 \sin\varphi_1 & 1 & 0 & \frac{1}{2} l_2 \sin\varphi_2 \\ 0 & -1 & -\frac{1}{2} l_1 \cos\varphi_1 & 0 & 1 & -\frac{1}{2} l_2 \cos\varphi_2 \end{pmatrix}, \tag{11.14E}$$

and the convective acceleration terms are

$$C_{k,jl} \dot{x}_j \dot{x}_l = \begin{pmatrix} \frac{1}{2} l_1 \dot{\varphi}_1^2 \cos\varphi_1 \\ \frac{1}{2} l_1 \dot{\varphi}_1^2 \sin\varphi_1 \\ \frac{1}{2} l_1 \dot{\varphi}_1^2 \cos\varphi_1 + \frac{1}{2} l_2 \dot{\varphi}_2^2 \cos\varphi_2 \\ \frac{1}{2} l_1 \dot{\varphi}_1^2 \sin\varphi_1 + \frac{1}{2} l_2 \dot{\varphi}_2^2 \sin\varphi_2 \end{pmatrix}. \tag{11.15E}$$

These can be compared with the results (10.7E) and (10.6E).

For a clear interpretation of the Lagrange multipliers, we take a closer look at the constrained equations of motion (11.11), and for simplicity consider the static case, where velocities and accelerations are zero. In that case, the equations should express static equilibrium of the system.

We know that for static equilibrium, the sum of applied forces and moments must be zero. That requirement is not directly obvious from the equations. However, we will demonstrate that the equations indeed express static equilibrium. The static equilibrium equations are

$$F_i = C_{k,i}\lambda_k. \tag{11.16}$$

To check that these forces are indeed equilibrium force systems, we determine the forces for each individual Lagrange multiplier λ_k by giving it a nonzero value while setting all remaining multipliers equal to zero.

Example 11.6 Substitution of a position where both pendula are in the $+y$ direction ($\varphi_1 = \varphi_2 = \pi/2$) in the equations of motion of the double pendulum Example 11.5 results in

$$\begin{pmatrix} F_{x1} \\ F_{y1} \\ M_1 \\ F_{x2} \\ F_{y2} \\ M_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ \frac{1}{2}l_1 & 0 & \frac{1}{2}l_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2}l_2 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}. \tag{11.17E}$$

Now, we determine the forces for each of the four individual Lagrange multipliers λ_k by making that the only multiplier with a nonzero value. The free-body diagrams for these four cases are shown in Figure 11.1.

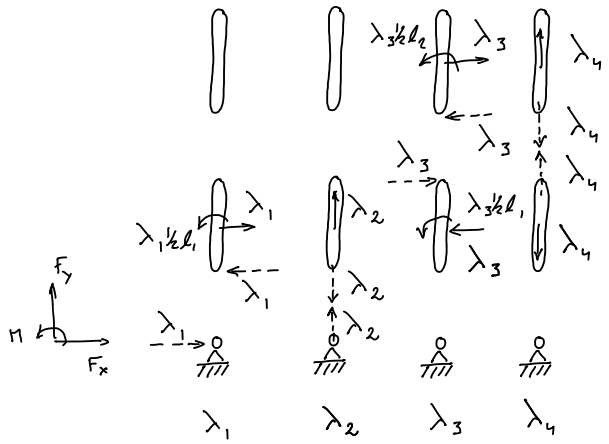


Figure 11.1: Four static equilibrium force systems for the double pendulum system, where all λ's are zero, expect λ_i, $i = 1, 2, 3, 4$. The applied forces are represented by the solid arrows, and the reaction forces (=Lagrange multipliers) are represented by dashed arrows.

From the figures in the above example, we conclude that the columns of

the Jacobian matrix associated with $C_{k,i}$ represent applied forces on the center of mass of the bodies for which the system is in equilibrium. We can of course combine these force vectors (four in the example) by taking different values for λ_k. All other force vectors, the kernel of $C_{k,i}$ or the orthogonal complement, can change the motion of the system. So, in the example, two independent load systems remain for which the system will accelerate, or start moving from rest. Thus, the system has two independent degrees of freedom, or in general: $d = \dim(\boldsymbol{F}) - \dim(\boldsymbol{\lambda}) = n - m$, where d is the number of degrees of freedom.

Example 11.7 Until now, the constraints have all been expressed in terms of the coordinates of the center of mass of the rigid bodies and some geometrical parameters. Another type of constraint is where we express idealized pure rolling or non-slipping. Such constraints are expressed in terms of the velocities and the coordinates of the center of mass of the rigid bodies, and in 3-D are non-holonomic constraints. As an example, Figure 11.2 shows an upright wheel which rolls, without slip, on the xy-plane.

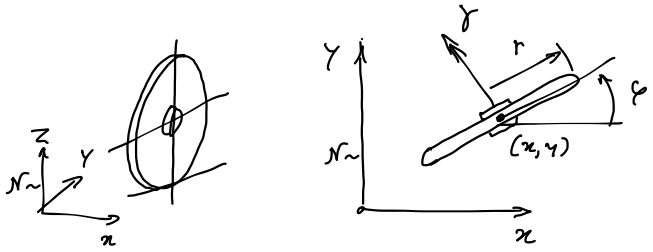


Figure 11.2: An upright rolling wheel with rolling radius r and rotation γ about the axis of rotation.

The configuration of the wheel can be described by four coordinates: the projection of the center of the wheel on the plane $(x,\, y)$, the wheel's orientation φ with respect to the z-axis, and the wheel's rotation γ about its axis of rotation. The condition for pure rolling in the driving direction is

$$\dot{\gamma} r = \dot{x} \cos\varphi + \dot{y} \sin\varphi. \tag{11.18E}$$

This is a non-holonomic constraint, which we can also write as a slip velocity S_1, which should be zero for pure rolling, as in

$$S_1 = \dot{x} \cos\varphi + \dot{y} \sin\varphi - \dot{\gamma} r = 0. \tag{11.19E}$$

This velocity constraint can be written in terms of a pseudo-Jacobian matrix

that is multiplied by the velocities, as in

$$S_1 = S_{1i}\dot{x}_i = 0, \tag{11.20E}$$

with the velocity vector $\dot{x}_i = (\dot{x},\, \dot{y},\, \dot{\varphi},\, \dot{\gamma})$, and the pseudo-Jacobian matrix,

$$S_{1i} = (\cos\varphi,\, \sin\varphi,\, 0,\, -r). \tag{11.21E}$$

We write pseudo, because this Jacobian matrix is not really the result from differentiating a function with respect to time. That function just does not exist. In 3-D, we can only express pure rolling in terms of velocities and not in coordinates only. This becomes clear when we determine the constraints on the acceleration level, by differentiating the velocity constraint:

$$\dot{S}_1 = S_{1i}\ddot{x}_i + S_{1i,j}\dot{x}_i\dot{x}_j = 0, \tag{11.22E}$$

with the matrix

$$S_{1i,j} = \begin{pmatrix} 0 & 0 & -\sin\varphi & 0 \\ 0 & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{11.23E}$$

This matrix $S_{1i,j}$, which expresses the second-order terms, is not symmetric. Or in other words, $S_{1i,j} \neq S_{1j,i}$, which proves that the function S_{1i} is not the result of a partial derivative of a function S_1.

Example 11.8 Figure 11.3 shows a particle constrained to move in 2-D along a parabola of the form $y = kx^2$, with constant k. Derive the particle's constrained equations of motion and interpret the Lagrange multiplier λ.

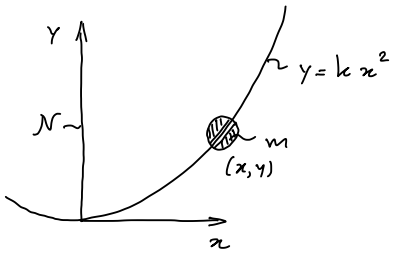


Figure 11.3: A particle in 2-D moving along the line of a parabola.

Exemplary solution

The virtual power expression is

$$\delta P = (F_x - m\ddot{x})\,\delta\dot{x} + (F_y - m\ddot{y})\,\delta\dot{y} = 0, \quad \forall\left\{\delta\dot{x},\, \delta\dot{y}\,|\,\text{constraint}\, y = kx^2\right\}, \tag{11.24E}$$

and the constraint and its derivatives are

$$C = y - kx^2, \quad \dot{C} = \dot{y} - 2kx\dot{x}, \quad \delta\dot{C} = \delta\dot{y} - 2kx\delta\dot{x}. \tag{11.25E}$$

Incorporating this constraint in the virtual power expression leads to

$$\begin{aligned} \delta P^* &= (F_x - m\ddot{x})\,\delta\dot{x} + (F_y - m\ddot{y})\,\delta\dot{y} - \lambda\delta\dot{C} = 0, && (11.26E)\\ &= (F_x - m\ddot{x})\,\delta\dot{x} + (F_y - m\ddot{y})\,\delta\dot{y} - \lambda\,(\delta\dot{y} - 2kx\delta\dot{x}) = 0, && (11.27E)\\ &= (F_x - m\ddot{x} + 2kx\lambda)\,\delta\dot{x} + (F_y - m\ddot{y} - \lambda)\,\delta\dot{y} = 0. && (11.28E) \end{aligned}$$

Of course, $\delta\dot{x}$ and $\delta\dot{y}$ are dependent. However, we have an extra variable, λ. We choose λ such that $(F_y - m\ddot{y} - \lambda) = 0$, then $\delta\dot{y}$ can be anything. So:

$$\lambda = F_y - m\ddot{y}. \tag{11.29E}$$

Now, $\delta\dot{x}$ is free, and the remaining non-zero term in the virtual power expression is

$$\delta P^* = (F_x - m\ddot{x} + 2kx\lambda)\,\delta\dot{x} = 0, \qquad \forall\,\delta\dot{x}. \tag{11.30E}$$

Since this must hold for arbitrary $\delta\dot{x}$, this leads to

$$F_x - m\ddot{x} + 2kx\lambda = 0. \tag{11.31E}$$

Substituting the assumed value of λ results in the equation of motion,

$$F_x - m\ddot{x} + 2kx\,(F_y - m\ddot{y}) = 0. \tag{11.32E}$$

Still there are two unknowns, $\ddot{x}$ and $\ddot{y}$. From the second derivate of the constraint equation, we can express $\ddot{y}$ in $\ddot{x}$ and remaining terms, as in

$$\ddot{C} = \ddot{y} - 2kx\ddot{x} - 2k\dot{x}^2 = 0 \qquad \Rightarrow \qquad \ddot{y} = 2kx\ddot{x} + 2k\dot{x}^2. \tag{11.33E}$$

Substituting $\ddot{y}$ from (11.33E) in the equation of motion (11.32E) results in

$$(1 + 4k^2x^2)m\ddot{x} = F_x + 2kxF_y - 4k^2mx\dot{x}^2. \tag{11.34E}$$

From this equation of motion, we can solve for the acceleration $\ddot{x}$ as a function of the state, x and $\dot{x}$, and the applied forces F_x and F_y. With $\ddot{x}$, we can determine $\ddot{y}$ and finally the value of the Lagrange multiplier λ.

Now, we interpret this Lagrange multiplier λ. We start by drawing the particle on the parabola and the derivative y' along the path (see Figure 11.4).

The derivative is calculated from

$$C = y - kx^2 = 0 \;\Rightarrow\; \mathrm{d}C = \mathrm{d}y - 2kx\,\mathrm{d}x = 0 \;\Rightarrow\; y' = \frac{\mathrm{d}y}{\mathrm{d}x} = 2kx. \tag{11.35E}$$

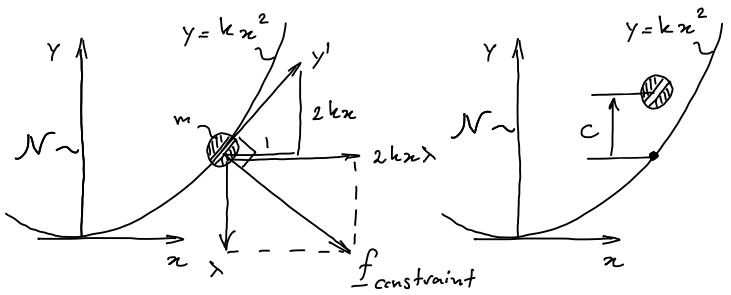


Figure 11.4: Lagrange multiplier for a point mass on a parabola and geometrical interpretation of the constraint distance.

Until now we have not written out the constrained equations of motion, in full form. They are

$$M_{ij}\ddot{x}_j = F_i - C_{,i}\lambda, \tag{11.36E}$$

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} F_x \\ F_y \end{pmatrix} \underbrace{- \begin{pmatrix} -2kx \\ 1 \end{pmatrix} \lambda}_{\substack{\boldsymbol{F}_c \\ \perp \text{curve } y=kx^2}}. \tag{11.37E}$$

The constraint forces, represented by λ, are $F_{ci} = -C_{,i}\lambda$. This force vector is clearly perpendicular to the tangent of the path, and this constraint force is also in accordance with postulate A from Lanczos [39] (p.76), which reads,

> "*The virtual work [power] of the force of reaction is always zero for any virtual displacement [velocity] which is in harmony with the given kinematic constraint.*"

Or in mathematical terms,

$$F_{ci}\delta\dot{x}_i = 0, \tag{11.38E}$$

$$C_{,i}\lambda\delta\dot{x}_i = 0, \tag{11.39E}$$

$$\lambda\delta\dot{C} = 0. \tag{11.40E}$$

Problem 11.1 The double pendulum from Example 10.1 is considered again, but now an extra constraint is added. This constraint allows only vertical sliding of the end of the second bar, point C, at a horizontal distance c from the y-axis, as shown in Figure 11.5. The sliding constraint in this point can be expressed in terms of the CoM coordinates of body 2,

$$C_5 = x_2 + \tfrac{1}{2}l_2\cos\varphi_2 - c = 0. \tag{11.41P}$$

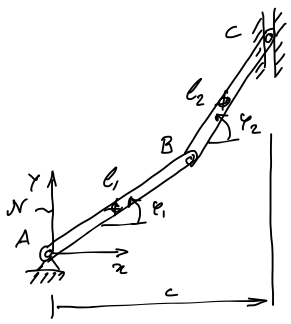


Figure 11.5: Double pendulum with added slide constraint in C.

This extra constraint also introduces an extra Lagrange multiplier λ_5. Can you physically interpret λ_5?[a]

Yet, the best way to interpret λ_k is from the static equilibrium equation $F_i = C_{5,i}\lambda_5$, as already demonstrated in Example 11.6.

[a] *Hint*: Assume that C_5 is positive, point C is to the right of the vertical slider, as shown in Figure 11.6. You can also think of a spring-like force λ_5 which has the task to close the gap.

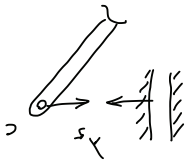


Figure 11.6: Joint constraint forces in the sliding joint at C, expressed in terms of the Lagrange multiplier λ_5.

11.2 Passive and Active Elements

Passive and active elements are non-rigid body components like springs, dampers or motors, and can be added to the system via the virtual power equation. The distinction between passive and active elements is usually based on the energy consumption and active control. When we supply energy by means of such an element to the system we call it active, whereas an element with stores or dissipates energy, is usually called passive. However, one can think of an actuated damper, which then would classify as an active element. We simply add the virtual power of such an element, the product of an element force σ and a virtual element velocity $\delta\dot{\varepsilon}$, to the virtual power equation of the system. The virtual power for one element is $\delta P = -\sigma\delta\dot{\varepsilon}$, where we use a minus

sign, because this is the absorbed power by the elements. Think for instance of a linear elastic spring, where both the spring elongation and the spring tensile force are defined positive. With a total of e active and passive elements, this becomes

$$\delta P = -\sigma_l \delta \dot{\varepsilon}_l, \qquad l = 1 \dots e. \tag{11.42}$$

Such an element displacement ε_l is always a relative displacement or a relative rotation which can be calculated from absolute coordinates and orientations x_i. Think of the elongation of a spring or hydraulic cylinder, or the relative rotation between the stator and the rotor of an electric motor. To generalize the approach we define these element displacements and rotations in the same form as the, constraints

$$\varepsilon_l(x_i) = C_l(x_i), \quad l = 1 \dots e,\ i = 1 \dots n, \tag{11.43}$$

with e the number of passive and active elements and n the total number of CoM coordinates of the rigid bodies. However, now these element constraints need not be zero. The non-zero value expresses the relative displacement or rotation within the element. The rate of change of these element displacements and rotations $\dot{C}_l$ describes the relative motion of these passive or active element, and is obtained by differentiating the element constraints,

$$\dot{C}_l = C_{l,i}(x_j)\dot{x}_i, \quad l = 1 \dots e,\ i, j = 1 \dots n. \tag{11.44}$$

Then the virtual power of these passive and active elements becomes,

$$\delta P = -\sigma_l C_{l,i} \delta \dot{x}_i. \tag{11.45}$$

Adding these terms to the virtual power expression of the system results in

$$\delta P^* = \left(F_i - M_{ij}\ddot{x}_j - \sigma_l C_{l,i}\right) \delta \dot{x}_i \tag{11.46}$$

Next we add the regular constraints $C_k(x_i) = 0$ in the same manner as described in (11.9) to the virtual power equation,

$$\delta P^{**} = \left(F_i - M_{ij}\ddot{x}_j - \sigma_l C_{l,i} - \lambda_k C_{k,i}\right) \delta \dot{x}_i. \tag{11.47}$$

The system is in equilibrium if this virtual power δP^{**} is zero for arbitrary virtual velocities, $\delta \dot{x}_i$, resulting in the following equations of motion,

$$\begin{aligned} &M_{ij}\ddot{x}_j + \lambda_k C_{k,i} = F_i - C_{l,i}\sigma_l, \\ &i, j = 1 \dots n,\ l = 1 \dots e,\ k = e+1 \dots e+m, \end{aligned} \tag{11.48}$$

CoM coordinates, e the number of passive and active elements, and m the number of kinematic constraints. Note that we have written the contribution of the extra elements on the right-hand side of the equation, because these are known terms. The forces in these elements, σ_l, can be calculated from the element state C_l and $\dot{C}_l$, and can even be a function of time t, as in $\sigma_l = \sigma_l(t, C_l, \dot{C}_l)$. We call this the constitutive behavior of the element. Think of a linear elastic spring, where the constitutive behavior is $\sigma_l = kC_l$, or a linear viscous damper with $\sigma_l = c\dot{C}_l$, or an electric motor where the behavior is more complex. Adding the second derivatives of the kinematic constraints completes the the set of constrained equations of motion,

$$\begin{pmatrix} M_{ij} & C_{k,i} \\ C_{k,j} & 0_{kk} \end{pmatrix} \begin{pmatrix} \ddot{x}_j \\ \lambda_k \end{pmatrix} = \begin{pmatrix} F_i - C_{l,i}\sigma_l \\ -C_{k,ij}\dot{x}_i\dot{x}_j \end{pmatrix}. \tag{11.49}$$

The passive and active elements show up as an additional force term on the right-hand side of the equations of motion. Note that the Jacobian matrix of the element constraints, $C_{l,i}$, transforms the element forces σ_l to forces at the CoM coordinates. This will become clear in the following example.

Example 11.9 A spring is added to the double-pendulum system from Example 10.1, as shown in Figure 11.7.

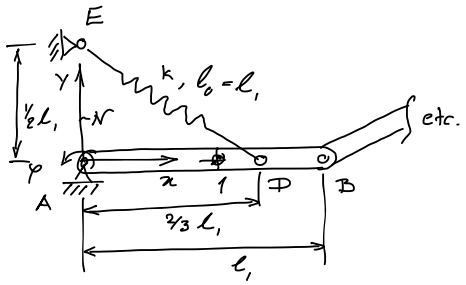


Figure 11.7: Double pendulum system with added spring.

The first pendulum, with length l_1, is hinged to the fixed world in A. In D on the body, a spring is connected. The other side of the spring is connected to the fixed world in E. The spring has an unstretched length of $l_0 = l_1$ and a linear stiffness k. The spring's elongation is the element constraint,

$$C_\mathrm{s} = l_\mathrm{s} - l_0 \tag{11.50E}$$

$$= \sqrt{(x_D - x_E)^2 + (y_D - y_E)^2} - l_0 \tag{11.51E}$$

The coordinates of point D can be expressed in terms of the CoM coordinates of body 1. Together with the coordinates for point E, this results in

$$C_{\mathrm{s}}(x_i) = \sqrt{\left(x_1 + \tfrac{1}{6}l_1\cos\varphi_1 - 0\right)^2 + \left(y_1 + \tfrac{1}{6}l_1\sin\varphi_1 - \tfrac{1}{2}l_1\right)^2} - l_0. \quad (11.52\mathrm{E})$$

The partial derivatives with respect to the $x_i = (x_1, y_1, \varphi_1, x_2, y_2, \varphi_2)^{\mathrm{T}}$, the system coordinates, are

$$C_{\mathrm{s},i} = \frac{1}{l_{\mathrm{s}}}\begin{pmatrix} x_1 + \frac{1}{6}l_1\cos\varphi_1 \\ y_1 + \frac{1}{6}l_1\sin\varphi_1 - \frac{1}{2}l_1 \\ -\frac{1}{12}l_1\left(l_1\cos\varphi_1 + 2x_1\sin\varphi_1 - 2y_1\cos\varphi_1\right) \\ 0 \\ 0 \\ 0 \end{pmatrix}^{\mathrm{T}}. \quad (11.53\mathrm{E})$$

The constitutive behavior for the linear-elastic spring force is

$$\sigma_{\mathrm{s}} = kC_{\mathrm{s}}. \quad (11.54\mathrm{E})$$

With the initial configuration, $(x_1, y_1, \varphi_1) = (\frac{1}{2}l_1, 0, 0)$, as depicted in Figure 11.7, this yields

$$C_{\mathrm{s}} = -\tfrac{1}{6}l_1, \quad C_{\mathrm{s},i} = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & -\frac{1}{10}l_1 & 0 & 0 & 0 \end{pmatrix}. \quad (11.55\mathrm{E})$$

The force in the spring is $\sigma_{\mathrm{s}} = kC_{\mathrm{s}} = -\frac{1}{6}kl_1$, which is a compressive force. This force is applied at point D on body 1, and it is transformed by $C_{\mathrm{s},i}$ to forces at the body's CoM. The values $\frac{4}{5}$ and $-\frac{3}{5}$ correspond to the projections of this spring force in the x- and y-direction, whereas the factor $-l_1/10$ is the moment arm of the spring force at the CoM of body 1 (check this yourself). Indeed, the Jacobian matrix $C_{\mathrm{s},i}$ transforms the spring force σ_{s} to forces acting at the CoM of body 1. We get this transformation automatically thanks to the virtual power approach.

Example 11.10 An example of an active element is a rotational motor. Here we consider an asynchronous three-phase AC squirrel cage electric motor shown in Figure 11.8, commonly used in many industrial machines.

The relative rotation between the stator and the rotor is the element displacement and is written here in the form of this element constraint,

$$C_l(x_i) = \varphi_{\mathrm{rotor}} - \varphi_{\mathrm{stator}}. \quad (11.56\mathrm{E})$$

The angular speed of the motor ω is the derivative of this relative rotation,

$$\dot{C}_l = \dot{\varphi}_{\mathrm{rotor}} - \dot{\varphi}_{\mathrm{stator}} = \omega. \quad (11.57\mathrm{E})$$

Often the rotational speed ω is given by n, and expressed in revolutions per minute (rpm), so be aware of these units and remember to convert to SI-units if necessary, where $1\,\mathrm{rpm} = 2\pi/60\,\mathrm{rad/s}$.

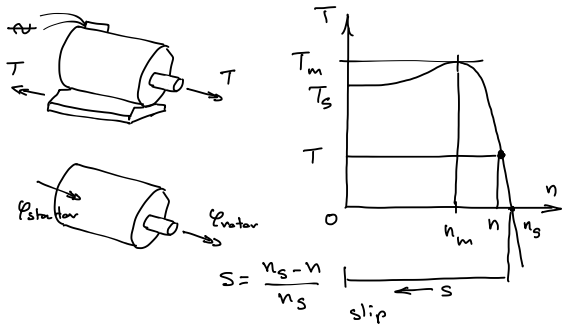


Figure 11.8: Asynchronous three-phase AC squirrel cage electric motor together with torque-speed motor characteristic.

The element force σ_l is minus the motor torque T, because the motor supplies energy to the system, as opposed to a spring which stores energy from the system. The Jacobian $C_{l,i}$ transforms this motor torque to applied action-reaction torques,

$$M_i = -C_{l,i}\sigma_l, \tag{11.58E}$$

$$\begin{pmatrix} M_{\text{stator}} \\ M_{\text{rotor}} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} T. \tag{11.59E}$$

The motor torque characteristic of such a asynchronous three-phase squirrel cage electric motor is characterized by a stiff behavior around the synchronous speed n_s, see Figure 11.8. First we introduce the dimensionless slip, $s = (n_\mathrm{s} - n)/n_\mathrm{s}$, which describes the offset of the speed n from the synchronous speed n_s. Then motor torque T of this type of electric engine is determined by

$$T = 2\frac{s/s_\mathrm{m}}{1 + (s/s_\mathrm{m})^2} T_\mathrm{m}, \tag{11.60E}$$

with the maximum torque T_m at dimensionless slip speed $s_\mathrm{m} = 1 - n_\mathrm{m}/n_\mathrm{s}$. Such an electric engine usually operates around the synchronous speed n_s, where large variations in the torque demand only give rise to small changes in the rotational speed n. Therefore we call this three-phase alternating current motor an asynchronous stiff electric engine. Opposed to this is a direct current electric motor, where the torque characteristic is approximately a straight line, sloped down, like in $T(n) = T_s(n - n_\mathrm{s})$. Whatever engine is used, the motor characteristic, or constitutive behavior of this active element, can always be determined as a function of the element state variables, as in $\sigma_l = \sigma_l(t,\, C_l,\, \dot{C}_l,\, p)$, and some control parameters p, like the applied voltage.

11.3 Prescribed Motion

Opposed to a prescribed element force, we sometimes want to model an active element by a prescribed motion. The element displacement or rotation is now a kinematic constraint, where this constraint can be some general function of the CoM coordinates but also of time. An example is a rotational motor between body 1 and 2 with a constant speed ω. The kinematic constraint then has the form $C_l(x_i, t) = (\varphi_2 - \varphi_1) - \omega t = 0$. We incorporate such a motion constraint just like a kinematic constraint in the equations of motion as described in Section 11.1, but note that the time derivative of such a general time dependent kinematic constraint $C_l(x_i, t)$ has additional terms,

$$\dot{C}_l = C_{l,i}\dot{x}_i + C_{l,t}, \tag{11.61}$$

$$\ddot{C}_l = C_{l,i}\ddot{x}_i + C_{l,ij}\dot{x}_i\dot{x}_j + 2C_{l,it}\dot{x}_i + C_{l,tt}, \tag{11.62}$$

where the comma subscript t denotes differentiation with respect to time. The element force σ_l is now an unknown, just like the unknown Lagrange multiplier λ_k from a regular kinematic constraint, and follows from the solution of the constrained equations of motion.

11.4 Impact Equations

The impact equations for a constrained multibody system can easily be derived from the constrained equations of motion. An impact is characterized by a high force for a short time period. When the time interval decreases the forces will increase. However, the sum of the product of these two, the impulse, will be finite. We idealize an impact by defining the impulse p as the integral of the contact force F_c with respect to time, for the limit case where the time right before the impact t^- approaches the time right after the impact t^+,

$$p = \lim_{t^- \to t^+} \int_{t^-}^{t^+} F_\mathrm{c}\,\mathrm{d}t. \tag{11.63}$$

Example 11.11 For our double pendulum system of Example 10.1, an impact might occur when one of the end points of the pendula comes into contact with a hard surface such as depicted in Figure 11.9.

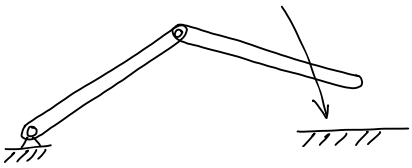


Figure 11.9: Impact of the end point of bar 2 of the double pendulum with a hard surface.

The linkage has a velocity and when hitting the hard surface the motion is almost instantaneously reversed by the impact. The accelerations are almost infinite and in effect the contact force is very high, whereas the time period of contact is very small.

When there is no energy loss during the collision, we call it fully elastic. In that case the contact force as a function of relative displacement during the collision is as depicted in the left figure of Figure 11.10. However, in reality there are usually energy losses during the impact, as depicted in the figure on the right, and we call such an impact partly or fully plastic.

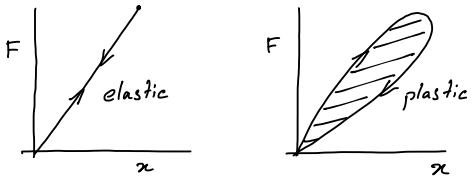


Figure 11.10: Compressive contact force during collision as a function of the relative displacement, left elastic, right partly plastic.

To understand the mechanics of an impact in an MBS we first look at a simple example of two point masses colliding, a classic problem[3]. Two point masses m_1 and m_2 collide at time t, as shown in Figure 11.11. Right before collision, the masses have velocities of v^- and u^-, respectively, with mass 1 moving faster than mass 2.

[3]see for instance: Descartes, Marcus Marci, Huygens, and Newton

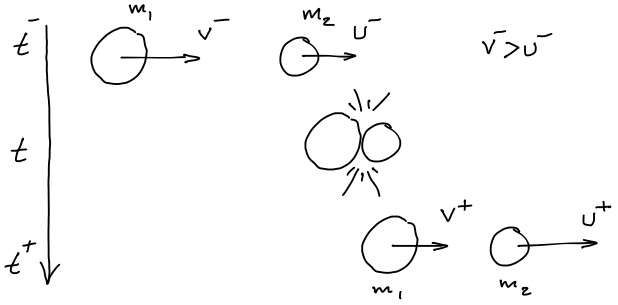


Figure 11.11: Two point masses colliding.

After the collision, the velocities of the masses have changed almost instantaneously into v^+ and u^+, respectively. The velocity and acceleration of each of the two masses over time is qualitatively shown in Figure 11.12.

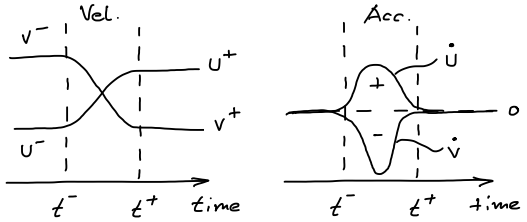


Figure 11.12: Velocity and acceleration of the two masses colliding as a function of time.

During the short impact time, $t^- \to t^+$, a high impact force F is present that of course follows the acceleration pattern, as shown qualitatively in Figure 11.13.

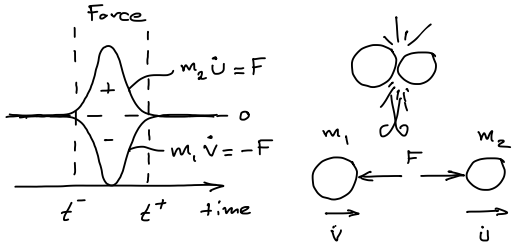


Figure 11.13: Mass times acceleration as a function of time for the collision of the two masses, and free-body diagrams of the two masses with the action-reaction contact force F.

To analyze the impact behavior we start by drawing the free-body diagrams of the two masses, shown in Figure 11.13, and write down the

equation of motion for both masses,

$$-F = m_1 \dot{v}, \quad \text{and} \quad F = m_2 \dot{u}. \tag{11.64}$$

Because of the high impact force and the short duration of the impact, we integrate these equations over time and take the limit case of $t^- \to t^+$, as in

$$\lim_{t^- \to t^+} \int_{t^-}^{t^+} -F \, \mathrm{d}t = \lim_{t^- \to t^+} \int_{t^-}^{t^+} m_1 \dot{v} \, \mathrm{d}t, \tag{11.65}$$

$$\lim_{t^- \to t^+} \int_{t^-}^{t^+} F \, \mathrm{d}t = \lim_{t^- \to t^+} \int_{t^-}^{t^+} m_1 \dot{u} \, \mathrm{d}t. \tag{11.66}$$

Next we introduce a contact impulse

$$p = \lim_{t^- \to t^+} \int_{t^-}^{t^+} F \, \mathrm{d}t, \tag{11.67}$$

positive when compressive, and the limit of the integral of the accelerations results in the jumps in the velocities. The resulting two impact equations, one for each mass, are

$$-p = m_1 v^+ - m_1 v^-, \tag{11.68}$$

$$p = m_2 u^+ - m_2 u^-. \tag{11.69}$$

From these equations we have to solve for the unknown velocities after the impact, v^+ and u^+, and the contact impulse p. Clearly we need an extra equation. Or in other words, we need more information on the nature of the impact.

Among the first who investigated the mechanics of impact was Newton. In his monumental work on mechanics, "Philosophiae Naturalis Principia Mathematica" [49], he discusses the impact of "hard" bodies and formulates the following mechanical impact law, which relates the relative velocity before and after impact by a constant factor:

> "*... and makes the bodies to return one from the other with a relative velocity, which is in a given ratio to that relative velocity with which they met.*"

This Newton impact law can be formulated in mathematical terms as

$$\frac{\dot{x}_{\mathrm{rel}}^+}{\dot{x}_{\mathrm{rel}}^-} = -e. \tag{11.70}$$

The amount of dissipated energy is related to an empirical constant e. For $e = 1$ we have mechanical energy preservation, which is a fully elastic impact. On the other hand, for $e = 0$, all impact energy is lost into dissipation and plastic deformation, and we speak of a fully inelastic or fully plastic impact, where the bodies stick together after impact ($\dot{x}^+_{\text{rel}} = 0$). It's quite interesting to read the part where Newton introduces this impact law. He discusses the literature and gives praise to colleague researchers, like Dr. Wallis, Sir Christopher Wren, Mr. Huygens [Christiaan] and Mr. Mariotte. Then he explains the experiments he did with impacting hard bodies of various materials and discusses the results. He ends the paragraph by saying,

> "*And thus the third Law, so far as it regards percussions and reflexions, is proved by a theory exactly agreeing with experience.*"

With Newton's impact law we are now able to solve the impact equations for our two mass impact problem. Applying Newton's impact law to our problem results in

$$\dot{x}^+_{\text{rel}} = -e\dot{x}^-_{\text{rel}}, \qquad \Rightarrow \qquad \left(v^+ - u^+\right) = -e\left(v^- - u^-\right). \tag{11.71}$$

Combining the two impulse equations and the impact law results in the following set of linear equations,

$$\begin{pmatrix} m_1 & 0 & 1 \\ 0 & m_2 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} v^+ \\ u^+ \\ p \end{pmatrix} = \begin{pmatrix} m_1 v^- \\ m_2 u^- \\ -e\left(v^- - u^-\right) \end{pmatrix}, \tag{11.72}$$

from which we clearly can solve the velocities after impact v^+ and u^+, together with the contact impulse p. Notice that the structure of these equations looks a lot like the structure of the constrained equations of motion, as in (11.12). Indeed, the systematic derivation of the impact equations will show this.

Problem 11.2 Prove that the general solution of the two mass impact problem (11.72) for $e = 1$ results in no kinetic energy loss, i.e. $T^+ = T^-$.

For the systematic derivation of the impact equations for a constrained multibody system, we start from the constrained equations of motion. Because at the instant of impact there is full contact and consequently a contact or constraint force, we just add the c contact conditions,

$$C_l(x_i) = 0, \quad i = 1 \ldots n,\, l = m + 1 \ldots m + c. \tag{11.73}$$

like any other constraint to the constrained equations of motion via the virtual power method. This results in the constrained equations of motion,

$$\begin{aligned} F_i - M_{ij}\ddot{x}_j &= C_{k,i}\lambda_k + C_{l,i}\lambda_l, \\ i,j &= 1\dots n,\ k = 1\dots m,\ l = m+1\dots m+c, \end{aligned} \tag{11.74}$$

with m regular kinematic constraints and c contact conditions. The joint forces are determined by the Lagrange multipliers λ_k and the contact forces by the Lagrange multipliers λ_l. Next, we introduce the concept of high impulsive forces at the contacts during a short impact time, and integrate over time, and take the limit case $t^- \to t^+$,

$$\lim_{t^-\to t^+} \left(\int_{t^-}^{t^+} (F_i - M_{ij}\ddot{x}_j)\ \mathrm{d}t = \int_{t^-}^{t^+} (C_{k,i}\lambda_k + C_{l,i}\lambda_l)\ \mathrm{d}t \right). \tag{11.75}$$

We define the external applied impulses,

$$s_i = \lim_{t^-\to t^+} \int_{t^-}^{t^+} F_i\,\mathrm{d}t, \quad i = 1\dots n, \tag{11.76}$$

which are like hitting the bodies with a hammer, and are usually zero. We define the joint constraint impulses,

$$\rho_k = \lim_{t^-\to t^+} \int_{t^-}^{t^+} \lambda_k\,\mathrm{d}t, \quad k = 1\dots m, \tag{11.77}$$

and the contact impulses,

$$\rho_l = \lim_{t^-\to t^+} \int_{t^-}^{t^+} \lambda_l\,\mathrm{d}t, \quad l = m+1\dots m+c. \tag{11.78}$$

Substitution of these terms in 11.75 results in

$$s_i - M_{ij}\dot{x}_j|_{t^-}^{t^+} = C_{k,i}\rho_k + C_{l,i}\rho_l, \tag{11.79}$$

and reordering of terms gives us the impact equations for a constrained multibody system,

$$\underbrace{M_{ij}\dot{x}_j^+}_{\substack{\text{momentum}\\ \text{after}\\ \text{impact}}} + \underbrace{C_{k,i}\rho_k}_{\substack{\text{reaction}\\ \text{impulse}}} + \underbrace{C_{c,i}\rho_c}_{\substack{\text{contact}\\ \text{impulse}}} = \underbrace{s_i}_{\substack{\text{applied}\\ \text{impulse}}} + \underbrace{M_{ij}\dot{x}_j^-}_{\substack{\text{momentum}\\ \text{before}\\ \text{impact}}}. \tag{11.80}$$

To complete the system equations we add the kinematic constraints on the velocity level,

$$C_{k,j}\dot{x}_j^+ = 0, \qquad k = 1 \dots m. \tag{11.81}$$

The relative velocities at the contacts follow Newton's impact law,

$$\dot{C}_c^+ = -e\dot{C}_c^- \quad \Rightarrow \quad C_{l,j}\dot{x}_j^+ = -eC_{l,j}\dot{x}_j^-, \quad l = m+1 \dots m+c. \tag{11.82}$$

Combination of the impact equations (11.80), the kinematic constraints (11.81), and the impact law (11.82), results in the impact equations for a constrained multibody system,

$$\begin{pmatrix} M_{ij} & C_{k,i} & C_{l,i} \\ C_{k,j} & 0 & 0 \\ C_{l,j} & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_j^+ \\ \rho_k \\ \rho_l \end{pmatrix} = \begin{pmatrix} s_i + M_{ij}\dot{x}_j^- \\ 0 \\ -eC_{l,j}\dot{x}_j^- \end{pmatrix}, \tag{11.83}$$
$$i, j = 1 \dots n, \quad k = 1 \dots m, \quad l = m+1 \dots m+c.$$

We can write this set of equations even shorter, because the kinematic constraints all have a zero constraint rate, also before impact, $C_{k,j}\dot{x}_j^- = 0$, for $k = 1 \dots m$, resulting in the generalized impact equations,

$$\begin{pmatrix} M_{ij} & C_{k,i} \\ C_{k,j} & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_j^+ \\ \rho_k \end{pmatrix} = \begin{pmatrix} s_i + M_{ij}\dot{x}_j^- \\ -eC_{k,j}\dot{x}_j^- \end{pmatrix}, \tag{11.84}$$
$$i, j = 1 \dots n, \quad k = 1 \dots m+c.$$

11.5 Problems

Problem 11.3 Redo Problem 10.2 items a–d, but now by using the systematic approach by means of the application of the virtual power method and Lagrangian multipliers. In doing this you will have to define, among other things, the constraints according to $C_k(x_i) = 0$ and differentiate these. You should take extra care in this differentiation. Be sure that your expressions are correct, preferably use symbolic software like the Symbolic Toolbox in MATLAB™. Consider the following cases:

a. Derive the equations of motion and the joint constraint equations.

b, c, d. Solve for the accelerations of the center of mass of the two bodies together with the Lagrange multipliers λ_k in the three initial condition cases from Problem 10.2. Check your results and interpret the nature and value of the Lagrange Multipliers. ▶

Problem 11.4 Add a constraint to the double pendulum from Problem 10.2, such that the right end of the second bar, C, moves along the y-axis (the line $x = 0$). Gravity still acts in $+x$-direction. Use the systematic approach to calculate the accelerations of the CoM of the two bodies together with the Lagrange multipliers, and interpret the Lagrange multipliers (in particular the new one) for these initial conditions:

a. Both bars in the $+y$-direction and zero speeds.
b. Both bars in the $+y$-direction and with an initial angular speed of $\omega = 60\,\text{rpm}$ clockwise on bar 1.
c. Both bars along the x-axis, that is point C is at the origin, with zero speeds, but an additional applied force of $F_y = 100\,\text{N}$ in B, that is at the end of bar 1 in the $+y$-direction. Here you will have trouble finding a unique solution. Can you explain why?[a]

▶

[a] *Hint*: Investigate the rank of $C_{k,i}$, the Jacobian of the constraint equations, and think of what the null vectors v_i from the null space $C_{k,i}v_i = 0$ represent. How many DoFs does this system have in exactly this configuration? Does this comply with the number of coordinates for the rigid bodies and the number of constraints?

Problem 11.5 Add a spring to the model from Problem 10.2. The spring is attached to the ground in D, with coordinates $(-l/2, 0)$. The other side of the spring is connected to bar 1 in point E, being at $2/3$ of the length measured from point A. The free length of the spring is $l_0 = 2l/3$ and the linear stiffness is $k = 30mg/l$.

Us the systematic approach to calculate the the accelerations of the center of mass of the two bodies together with the Lagrangian multipliers for the case where the two bars are in line along the positive y-axis and at zero speed. ▶

Problem 11.6 Add a rotary motor to the double pendulum model from Problem 10.2. The motor is placed in the revolute joint between bar 1 and bar 2 in point B. We assume the motor to have a *constant* angular speed of $\omega = 30$ rpm, such that angular speed of body 2 with respect to body 1 is ω, where all angles and angular speeds are counter-clockwise positive.

Calculate the accelerations of the center of mass of the two bodies together with the Lagrangian multipliers for the case where the two bars are in line along the positive y-axis and with an initial angular speed of $-\omega$ on bar 1.

What is the torque exerted by the motor in the revolute joint in B on bar 1 and bar 2, and what is the power that the motor supplies to the mechanical system? ▶

Problem 11.7 Add an impulsive contact to the double pendulum model from Problem 10.2. We assume that the right end of body 2, point C, hits a flat wall. The wall surface is the y-axis where all material is at

$x < 0$. We assume that at the moment of impact both bars are both along the y-axis and we assume that the contact will occur at the end of bar 2 in point on the wall with coordinates $(0, 2l)$, furthermore we assume an angular speed of $\omega = 30$ rpm (counter-clockwise) on both bars. Consider the case were the coefficient of restitution e is respectively 0, 0.9 and 1. Calculate for these different values:

1. The velocities after impact of the center of mass of the two bars together with the contact impulse and the joint constraint impulses.
2. The difference in the kinetic energy of the system before and after the impact and the amount of work done by the contact impulse.

▶

Problem 11.8 Consider the model for a bipedal walking machine, Figure 11.14. The model consist of two long slender legs of length l and with evenly distributed mass m, and a point mass M at the hip joint in B. The stance leg is fixed to the ground by means of a revolute joint in A.

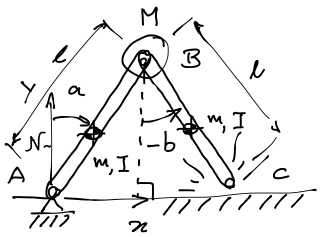


Figure 11.14: Model for a bipedal walking machine.

The configuration of this system can then be described by means of the generalized coordinates, the angles a and b, see figure (note: angle a for leg 1 and angle b for leg 2 are both clockwise positive). The swing leg BC is about to hit the floor. We assume the normal impact (y-direction) to be fully plastic ($e = 0$), and no impact in the x-direction, so the leg can move freely in that direction over the ground. Determine the velocity of point C in the x-direction just after impact[a].

The walker's initial conditions just before impact are: $a = \pi/6\,\text{rad}, b = -\pi/6\,\text{rad}, \dot{a} = 1.2\,\text{rad/s}, \dot{b} = 0.2\,\text{rad/s}$.

For parameters use: $l = 0.75\,\text{m}, m = 3\,\text{kg}, M = 5\,\text{kg}$.

▷

[a] *Hint*: For the fully plastic impact op point C in the vertical direction with the ground you have to add a constraint to the system impulse equations where you write that the speed of point C in the y-direction after impact is zero. The y-position of point C can be expressed in terms of the generalized coordinates a and b as $y_C = l\cos(a) - l\cos(-b)$. Check that after impact the rotational speeds of the legs should be equal in size and opposite in direction, that is $\dot{a} = -\dot{b}$.

Problem 11.9 A mechanical system consists of two bodies: a cart 1, with mass M, which is restricted to move along the x-axis and a very slender pendulum 2, with length l and mass m, which is connected to cart 1 by a revolute joint in A, see Figure 11.15. The configuration of the system can be described by two generalized coordinates, the distance s of the cart with respect to the origin O and the pendulum's angle α with respect to the vertical. Gravity acts in the negative y-direction with field strength g.

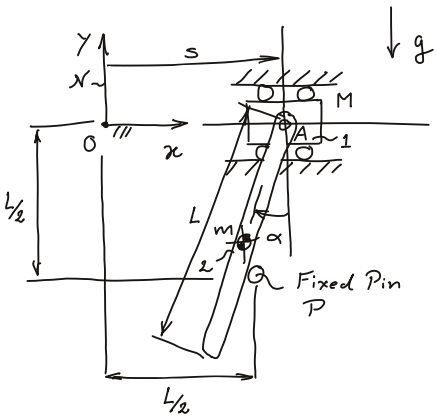


Figure 11.15: Pendulum on a moving cart system.

While the system is in motion, the pendulum is about to hit the inertially fixed pin P. The pin is very small compared to the length of the pendulum, and is located at $(x, y) = (l/2, -l/2)$. We assume the impact to be partly plastic, with a coefficient of restitution e for Newton's impact law, and no friction impact, so the pendulum can slide freely along the pin. Determine the loss of kinetic energy in N m due to the impact[a].

Conditions before impact are: $\alpha = \pi/6\,\text{rad}$, $\dot{\alpha} = -1.2\,\text{rad/s}$, $\dot{s} = 0.9\,\text{m/s}$, and parameters: $l = 0.8\,\text{m}$, $m = 3\,\text{kg}$, $M = 5\,\text{kg}$, $e = 0.5$, $g = 9.81\,\text{N/kg}$. ▷

[a] *Hint*: At impact, the two coordinates are dependent and one can determine the distance s given the angle α from $s = l/2 + l/2\tan(\alpha) = 0.6309\,\text{m}$. The impact of the pendulum with the pin is in the direction normal to the pendulum. For the partly plastic impact, add a constraint to the system impulse equations stating that the normal speed at the contact point after impact is $-e$ times the one before impact. The normal speed at the contact point results from differentiation of the normal distance between the pendulum and P with respect to time. The normal to the pendulum is the vector $\boldsymbol{u}_p = (-\cos(\alpha), \sin(\alpha))^\mathsf{T}$ and the normal distance is the dot product of $\boldsymbol{r}_{A/P}$ and $\boldsymbol{u}_p$. For the given parameter set, the normal speed before impact is $-1.3337\,\text{m/s}$. The partly plastic impact will always dissipate energy, thus the loss in kinetic energy is always > 0.

Problem 11.10 Prove that the kinetic energy loss $-\Delta T$ after an impact in a constrained multibody dynamics system can be written as the sum of the product of the contact impulses times the relative velocities before impact:

$$-\Delta T = \tfrac{1}{2}\,(1-e)\,\rho_k \dot{C}_k^-\,, \quad k = 1 \ldots c. \tag{11.85P}$$

►

Problem 11.11 The impulsive equations for the constrained multibody system are usually written as a set of linear equations in the unknown velocities of the center of mass after impact $\dot{x}_i^+$, and the contact impulses and constraint impulses ρ_k. Instead of the velocities $\dot{x}_i^+$ we could use the velocity jumps $\Delta\dot{x}_i = \dot{x}_i^+ - \dot{x}_i^-$ as unknowns. Rewrite the impulsive equations in terms of these unknown velocity jumps $\Delta\dot{x}_i$ and the contact and constraint impulses ρ_k and discuss the computational pros and cons.

12 Lagrange's Equations

This chapter first offers a motivation for minimal sets of generalized coordinates, as well as an alternative derivation for Lagrange's equations to the one presented in Chapter 8. Then, the equations are extended to include also active and passive elements as well as constraints, starting from the concept of virtual power. In addition to Chapter 8, also impacts are considered.

12.1 Motivation for Generalized Coordinates

When we derived our equations of motion for a multibody system which consists of rigid bodies with kinematic constraints, we ended up with the following set of equations:

$$\begin{pmatrix} \mathbf{M} & \mathbf{C}^{\mathrm{T}} \\ \mathbf{C} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \ddot{\boldsymbol{x}} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F} \\ -\mathbf{C}_2(\boldsymbol{x}, \dot{\boldsymbol{x}}) \end{pmatrix}, \tag{12.1}$$

with an $n \times n$ mass matrix $\mathbf{M}$ and an $m \times n$ Jacobian constraint matrix $\mathbf{C}$. In the case of many rigid bodies, $n \gg 1$, and many constraints, $m \gg 1$ (or connections) we have a large, $n + m \gg 2$, set of linear equations to solve, whereas, we could describe the motion of the system uniquely in terms of a much smaller set, namely in terms of the *degrees of freedom*, where the number of degrees of freedom (DoF) is $d = n - m$.

Lagrange was among the first to do this and he introduced the term *generalized coordinates* for a set of coordinates describing the motion of the system. He uses the word "generalized", because they are usually a mixture of absolute and relative coordinates, distances, and angles. Usually the set of generalized coordinates is a minimal set of coordinates to describe the kinematics, but not necessarily. Let us look at some examples of generalized coordinates.

Example 12.1 When constrained to be on the surface of the Earth (see Figure 12.1), a point has two degrees of freedom. The *latitude* and *longitude* form an example of a set of generalized coordinates that describe a location

on the surface of the Earth:

$$(52.01,\, 4.36) \qquad \Rightarrow \qquad \underbrace{52^\circ 1' \mathrm{N}}_{\text{Latitude}},\ \underbrace{4^\circ 22' \mathrm{E}}_{\text{Longitude}} .$$

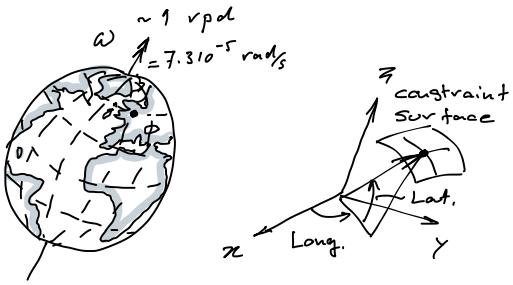


Figure 12.1: Earth and its constraint surface.

It is interesting to think of where the terms longitude and latitude came from. The Earth, being a sphere, has in every direction the same "longitude". One hypothesis goes back to the first navigators, who sailed the Mediterranean Sea (see Figure 12.2), which in the East-West direction is much longer than in the North-South direction.

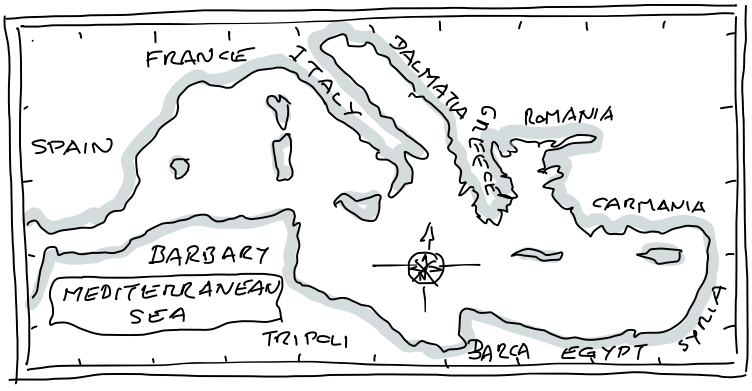


Figure 12.2: Mediterranean Sea.

Example 12.2 Some other examples of generalized coordinates. The four-bar mechanism with rolling support in Figure 12.3 has two degrees of freedom. Therefore, two generalized coordinates can be chosen to describe the configuration of the system, for example: $(\alpha,\, \beta)$, $(\alpha,\, s)$, $(\beta,\, s)$, etc.

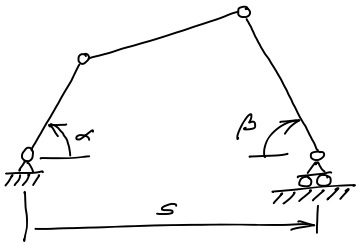


Figure 12.3: Four-bar mechanism with roller support.

However, the choice of generalized coordinates is not trivial. Let us look at the double pendulum system in Figure 12.4, which has two degrees of freedom. So, we have to select at least two generalized coordinates to describe the configuration of the system, for example: (α, β), (α, v), (u, β), (u, v), etc.

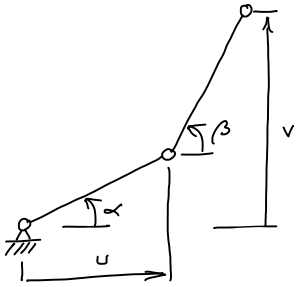


Figure 12.4: Double pendulum with some generalized coordinates.

One is free to choose the set that one "likes" best. However, some sets are better than others. The kinematics problem is highly nonlinear and therefore the mapping of the chosen generalized coordinates to the CoM coordinates of the rigid bodies can suffer from non-uniqueness. For instance, the two configurations depicted in Figure 12.5 have identical generalized coordinates (u, v), so these coordinates are in this case less convenient, leading to singularities and bifurcations, which gives ill-conditioned problems.

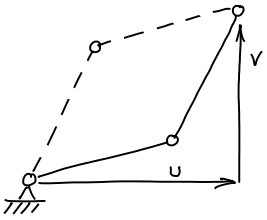


Figure 12.5: Double pendulumg with two configurations for the same u and v.

12.2 Lagrange's Equations of Motion

Joseph-Louis Lagrange (Turin 1736 – Paris 1813), see Figure 12.6, became the founder of Multibody dynamics, by his 1788 monumental work "Méchanique Analitique" [38]. The preface to the book is very interesting, reading:

> "*On ne trouvera point de Figures dans cet Ouvrage. Les méthodes que j'y expose ne demandent ni constructions, ni raisonnemens géométriques ou méchaniques, mais seulement des opérations algébriques, assujetties à une marche réguliere et uniforme. Ceux qui aimant l'Analyse, verront avec plaisir la Méchanique en devenir une nouvelle branche, et me sauront gré d'en avoir étendu ainsi le domaine.*"

In the English version of this preface, from the excellent translation of Lagrange's work by Auguste Boissonnade and Victor N. Vagliente, this reads:

> "*No figures will be found in this work. The methods I present require neither constructions nor geometrical or mechanical arguments, but solely algebraic operations subject to a regular and uniform procedure. Those who appreciate mathematical analysis will see with pleasure mechanics becoming a new branch of it and hence, will recognize that I have enlarged its domain.*"

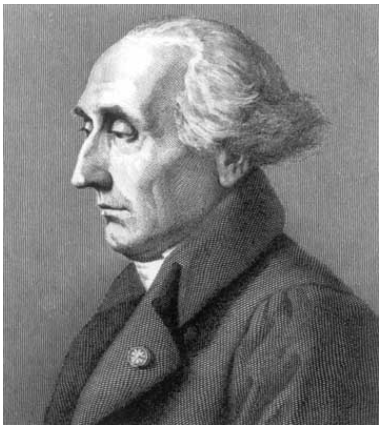


Figure 12.6: Joseph-Louis Lagrange, Turin 1736 – Paris 1813.

So what does Lagrange tell us? That we should buy his book, although there are no pictures? But then, "What is the use of a book without pictures or conversations?" says Alice. Thus, "no pictures" that is not really an advertisement. However, in this case it is! Before Lagrange, most mechanics was done by arguments, geometric reasoning with proportions and drawing pictures, as one can read for instance in the works by Simon Stevin, Christiaan Huygens and Isaac Newton. Lagrange shows us that one only needs algebra. We saw this already in the previous chapters, when we moved from the pencil-and-paper method, with drawing free-body diagrams, to the method of virtual power, to derive the equations of motion for a constrained multibody system.

In deriving Lagrange's equations of motion, we will make use of the concept of energy, which is the integral of power over time. For a single force applied at a rigid body in the one-dimensional case, the mechanical power exerted by the force on the system is

$$P = F\dot{x}, \tag{12.2}$$

and the energy applied by that force to the system is

$$E = \int P \,\mathrm{d}t = \int F\dot{x} \,\mathrm{d}t. \tag{12.3}$$

The motion of the system is described by Newton's equation of motion,

$$F = m\ddot{x}. \tag{12.4}$$

Multiplying by the velocity and integration over time results in the energy equation for the body,

$$\int F\dot{x} \,\mathrm{d}t = \int m\ddot{x}\dot{x} \,\mathrm{d}t, \tag{12.5}$$

$$\int F\frac{\mathrm{d}x}{\mathrm{d}t} \,\mathrm{d}t = \int m\frac{\mathrm{d}\dot{x}}{\mathrm{d}t}\dot{x} \,\mathrm{d}t, \tag{12.6}$$

$$\int F \,\mathrm{d}x = \int m\dot{x} \,\mathrm{d}\dot{x}. \tag{12.7}$$

Let us assume, for simplicity here, that the applied force is a constant force, then we can evaluate the integrals as,

$$Fx = \tfrac{1}{2}m\dot{x}^2 + \text{constant}. \tag{12.8}$$

Next we introduce the potential energy function V as a solution of the equation

$$\partial V/\partial x = -F, \tag{12.9}$$

which, for a constant force F, results in, $V = -Fx + \text{constant}$. The kinetic energy T of the mass in motion is

$$T = \tfrac{1}{2}m\dot{x}^2. \tag{12.10}$$

Then for such a conservative system (this means that all forces can be derived from a potential energy function) we conclude that the sum of these two energies is constant,

$$T + V = \text{constant}. \tag{12.11}$$

This conservation of energy is very useful if one is not interested in the time-history of a motion, but just in some event. It can easily be shown that the principle is valid for any conservative force, not just for constant forces.

Problem 12.1 Derive (12.11) from (12.7) without assuming that F is constant, only assuming that it is conservative. Use again the two definitions (12.9) and (12.10) to evaluate the integrals.

Example 12.3 If you want to know the speed by which a cannonball fired with an initial speed from a castle wall hits the ground, and you are not interested in the path it takes in the air (see Figure 12.7), you can use the principle of energy conservation, under the assumption that there are no energy losses during flight.

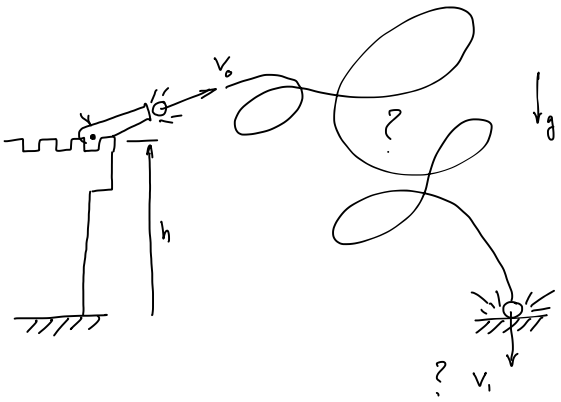


Figure 12.7: Firing a cannonball from a castle wall and hitting the enemy on the ground.

Energy conservation tells us that the sum of the potential and the kinetic energy is constant. At t_0 the cannon is fired and the cannonball has an initial speed v_0. The wall is at a height h above the ground, the moment in time when the ball hits the ground (the enemy) is t_1, then the speed v_1 can be

determined from

$$T_0 + V_0 = T_1 + V_1, \tag{12.12E}$$

$$\tfrac{1}{2}mv_0^2 + mgh = \tfrac{1}{2}mv_1^2 + 0, \tag{12.13E}$$

$$v_1 = \sqrt{v_0^2 + 2gh}. \tag{12.14E}$$

Next, we generalize these energy concepts from the simple one rigid-body case to a general multibody system of rigid bodies with constraints. Then the kinetic energy for the system is

$$T = \tfrac{1}{2}\dot{x}_i M_{ij} \dot{x}_j, \tag{12.15}$$

with the CoM coordinates of the rigid bodies and a diagonal mass matrix as

$$\dot{x}_i = (\dot{x}_b,\, \dot{y}_b,\, \dot{\varphi}_b), \quad b = 1 \ldots N, \quad i, j = 1 \ldots n, \tag{12.16}$$

$$M_{ij} = \mathrm{diag}(m_b,\, m_b,\, I_b), \tag{12.17}$$

with N the number of rigid bodies and n the number of coordinates. All conservative forces acting on the system are assumed to be collected in a potential energy function V. The remaining, non-conservative resultant force are denoted by F_i. Then, how do we get back from the scalar energy conservation equation, $T + V =$ constant, to the equations of motion? We obtained the energy equation by integration of the equations of motion with respect to CoM coordinate displacements, $\mathrm{d}x_i = \dot{x}_i \,\mathrm{d}t$. Therefore we will obtain the equations of motion when we differentiate the energy equation with respect to these coordinates,

$$-\frac{\partial V}{\partial x_i} + F_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}_i}\right). \tag{12.18}$$

On the left-hand side the applied forces reappear, and on the right-hand side we obtain the inertia terms. The partial differentiation of the kinetic energy is in complete accordance with the change we made in the integration from displacements to velocities times time.

However, instead of expressing the equations of motion in terms of the CoM coordinates (position and orientation) and their derivatives, we want to derive the equations of motion in generalized coordinates and their derivatives. We assume that all CoM coordinates x_i of the rigid bodies can be written in terms of the generalized coordinates q_j, as in

$$x_i = x_i(q_j), \quad i = 1 \ldots n, \quad j = 1 \ldots d, \tag{12.19}$$

where d is the number of degrees of freedom of the system. Note that these are in general highly nonlinear functions. Then the velocities of the CoM coordinates in terms of the generalized coordinates and their rates are

$$\dot{x}_i = \frac{\partial x_i}{\partial q_j}\dot{q}_j. \tag{12.20}$$

With these expressions we are able to write the kinetic and the potential energy as functions of the generalized coordinates q_j and their rates $\dot{q}_j$,

$$T = T(\dot{x}_i) = T(q_j, \dot{q}_j), \tag{12.21}$$
$$V = V(x_i) = V(q_j). \tag{12.22}$$

By introducing generalized coordinates q_j we have, implicitly, introduced generalized forces Q_j. These generalized forces are defined as the energetic dual of the generalized coordinate rates $\dot{q}_j$, such that the virtual power exerted by the generalized forces Q_j is $\delta P = Q_j \delta\dot{q}_j$. With this definition we are now able to transform the non-conservative forces F_i to the generalized forces Q_j, as in

$$F_i \delta\dot{x}_i = Q_j \delta\dot{q}_j \qquad \forall\, \delta\dot{x}_i = \frac{\partial x_i}{\partial q_j}\delta\dot{q}_j, \tag{12.23}$$

$$F_i \frac{\partial x_i}{\partial q_j}\delta\dot{q}_j = Q_j \delta\dot{q}_j \qquad \forall\, \delta\dot{q}_j \tag{12.24}$$

$$\Rightarrow \quad Q_j = \frac{\partial x_i}{\partial q_j} F_i = x_{i,j} F_i. \tag{12.25}$$

With the application of the principle of virtual power we are now able to transform the equations of motion (12.18) from the CoM forces to generalized forces,

$$\begin{aligned} Q_j &= \frac{\partial x_i}{\partial q_j}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}_i}\right) + \frac{\partial V}{\partial x_i}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x_i}{\partial q_j}\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x_i}{\partial q_j}\right)\frac{\partial T}{\partial \dot{x}_i} + \frac{\partial V}{\partial q_j} \end{aligned} \tag{12.26}$$

From the definition of the velocities, $\dot{x}_i = x_{i,j}\dot{q}_j$, we conclude that the partial derivatives for the coordinates and the velocities are identical,

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j}. \tag{12.27}$$

The time derivative of the partial derivatives can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x_i}{\partial q_j}\right) = \frac{\partial}{\partial q_k}\left(\frac{\partial x_i}{\partial q_j}\right)\dot{q}_k, \tag{12.28}$$

and since the second-order derivatives are symmetric,

$$\frac{\partial}{\partial q_k}\left(\frac{\partial x_i}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{\partial x_i}{\partial q_k}\right), \tag{12.29}$$

we can substitute (12.27) and (12.29) in (12.28) and conclude that the time derivative of the partial derivatives equals the partial derivatives of the velocities,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x_i}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{\partial x_i}{\partial q_k}\right)\dot{q}_k = \frac{\partial \dot{x}_i}{\partial q_j}. \tag{12.30}$$

Substituting this result in the transformed equations of motion (12.26) yields Lagrange's equations of motion,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j, \quad j = 1 \ldots d. \tag{12.31}$$

Lagrange's equations of motion are a very compact and elegant method to determine the equations of motion for a system expressed in terms of generalized coordinates and generalized forces. We demonstrate the ease of the method on a small example.

Example 12.4 We like to model the load-swing in an overhead track container crane as depicted in Figure 12.8 (for now we are not interested in hoisting).

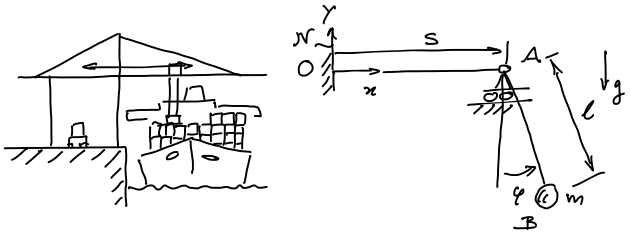


Figure 12.8: Overhead track container crane and model for load swing.

We can model this system as a point-mass pendulum on a moving base, as shown in the right of Figure 12.8. In order to find the equations of motion with Lagrange's equations, we first require expressions for the kinetic energy T and the potential energy V in terms of the generalized coordinates. There

is only one point mass in the system at point B, with CoM coordinates

$$x_i = \begin{pmatrix} x_B & y_B \end{pmatrix}^{\mathrm{T}}. \tag{12.32E}$$

In this two-degree-of-freedom model, the two generalized coordinates are chosen to be the horizontal displacement s of the moving support with respect to some inertial position, and the angle φ of the pendulum with respect to the downward vertical,

$$q_j = \begin{pmatrix} s & \varphi \end{pmatrix}^{\mathrm{T}}. \tag{12.33E}$$

Next, we need the CoM coordinates (12.32E) as functions of the generalized coordinates (12.33E), $x_i(q_j)$, and their derivatives:

$$x_B = s + l\sin\varphi, \qquad \dot{x}_B = \dot{s} + l\dot{\varphi}\cos\varphi, \tag{12.34E}$$

$$y_B = -l\cos\varphi, \qquad \dot{y}_B = l\dot{\varphi}\sin\varphi. \tag{12.35E}$$

The kinetic energy in the system is the kinetic energy of the point mass in B,

$$\begin{aligned} T &= \tfrac{1}{2}m\left(\dot{x}_B^2 + \dot{y}_B^2\right), \\ &= \tfrac{1}{2}m\left(\left(\dot{s} + l\dot{\varphi}\cos\varphi\right)^2 + \left(l\dot{\varphi}\sin\varphi\right)^2\right), \\ &= \tfrac{1}{2}m\left(\dot{s}^2 + 2\dot{s}l\dot{\varphi}\cos\varphi + \left(l\dot{\varphi}\right)^2\right), \end{aligned} \tag{12.36E}$$

and the potential energy of the point mass in the gravity field is

$$V = -(-mg)y_B = -mgl\cos\varphi. \tag{12.37E}$$

Substituting these expressions in the Lagrange's equations (12.31) and expanding the terms leads automatically to the system's equations of motion expressed in terms of independent generalized coordinates. Moreover, this can be done easily and almost error-free with help of symbolic software like MAPLE$^{\mathrm{TM}}$ or MATLAB$^{\mathrm{TM}}$. Here we illustrate the derivation step-by-step:

$$\frac{\partial T}{\partial \dot{q}_j}: \qquad \begin{pmatrix} \frac{\partial T}{\partial \dot{s}} \\ \frac{\partial T}{\partial \dot{\varphi}} \end{pmatrix} = \begin{pmatrix} m\left(\dot{s} + l\dot{\varphi}\cos\varphi\right) \\ m\left(l\dot{s}\cos\varphi + l^2\dot{\varphi}\right) \end{pmatrix} \tag{12.38E}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right): \qquad \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{s}}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) \end{pmatrix} = \begin{pmatrix} m\left(\ddot{s} + l\ddot{\varphi}\cos\varphi - l\dot{\varphi}^2\sin\varphi\right) \\ m\left(l\ddot{s}\cos\varphi - l\dot{s}\dot{\varphi}\sin\varphi + l^2\ddot{\varphi}\right) \end{pmatrix} \tag{12.39E}$$

$$\frac{\partial T}{\partial q_j}: \qquad \begin{pmatrix} \frac{\partial T}{\partial s} \\ \frac{\partial T}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ m\left(-l\dot{s}\dot{\varphi}\sin\varphi\right) \end{pmatrix} \tag{12.40E}$$

$$\frac{\partial V}{\partial q_j}: \qquad \begin{pmatrix} \frac{\partial V}{\partial s} \\ \frac{\partial V}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ mgl\sin\varphi \end{pmatrix} \tag{12.41E}$$

Substitution of these terms in the Lagrang's equations results in the equations of motion,

$$j=1: \quad m\ddot{s}+ml\ddot{\varphi}\cos\varphi-ml\dot{\varphi}^2\sin\varphi \qquad -0 \qquad +0 \qquad =Q_s \tag{12.42E}$$

$$j=2: \quad ml\ddot{s}\cos\varphi-ml\dot{s}\dot{\varphi}\sin\varphi+ml^2\ddot{\varphi} \quad +ml\dot{s}\dot{\varphi}\sin\varphi \quad +mgl\sin\varphi \quad =Q_\varphi \tag{12.43E}$$

Rearranging terms[a], yields the the equations of motion expressed in terms of the generalized coordinates,

$$\begin{pmatrix} m & ml\cos\varphi \\ ml\cos\varphi & ml^2 \end{pmatrix}\begin{pmatrix} \ddot{s} \\ \ddot{\varphi} \end{pmatrix}=\begin{pmatrix} Q_s+ml\dot{\varphi}^2\sin\varphi \\ Q_\varphi-mgl\sin\varphi \end{pmatrix} \tag{12.44E}$$

Note the symmetry of the mass matrix, and that neither s nor $\dot{s}$ appear in the equations of motion.

[a] Note the two canceling terms in the second equation of motion!

Example 12.5 For the equations of motion of the load swing model (12.44E) to have a unique solution for the accelerations of the generalized coordinates, the determinant of the mass matrix should be non-zero. Check if this is the case, and if not explain what the physical problem is.

Exemplary solution

To have a unique solution for the set of linear equations, the leading matrix, here the generalized mass matrix, must be regular or, in other words, have a non-zero determinant. The determinant of this mass matrix is

$$\det(M_{ij})=m^2l^2\left(1-\cos^2\varphi\right).$$

This determinant is zero if $m=0$, $l=0$, or $(1-\cos^2\varphi)=0\Rightarrow\varphi=0+k\pi$, with integer k. The zero mass case is trivial, no mass in the presence of forces gives infinite accelerations. Likewise the case where the pendulum length is zero, then the mass moment of inertia is zero and an applied generalized torque Q_φ will give infinite angular accelerations. The last case, $\varphi=0+k\pi$, with integer k, is an interesting one. This means that even in the static equilibrium condition, when the pendulum is hanging down, we are not able to solve the accelerations, given some applied forces. From our linear algebra course we remember that such a non-regular matrix will have a non-zero null space. In physical terms; non-zero acceleration results in zero applied force. The null space of the mass matrix for the case where $\varphi=0+k\pi$ is found as,

$$\begin{pmatrix} m & ml \\ ml & ml^2 \end{pmatrix}\begin{pmatrix} \ddot{s} \\ \ddot{\varphi} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} \ddot{s} \\ \ddot{\varphi} \end{pmatrix}=c\begin{pmatrix} 1 \\ -1/l \end{pmatrix} \tag{12.45E}$$

If we draw this motion in the figure of the model, we identify that for this non-zero motion of the generalized degrees of freedom, the point mass in B has zero accelerations (see Figure 12.9 for an exemplary value of $c = 1\,\mathrm{m/s^2}$).

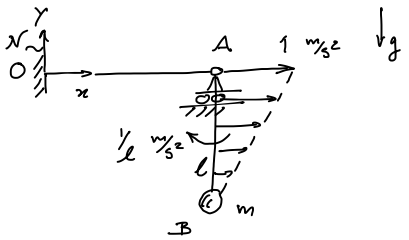


Figure 12.9: Accelerating motion from the null space of the mass matrix imposed on the load swing model.

So we have an infinitesimal motion which is not coupled to any mass. Clearly, such a system is not solvable.

12.3 Passive and Active Elements

The method for adding passive and active elements into the Lagrange equations is again by the concept of virtual power. The virtual power exerted by applied generalized forces and inertial forces in terms of generalized coordinates is

$$\delta P = \delta\dot{q}_j \Bigg(\underbrace{Q_j - \frac{\partial V}{\partial q_j}}_{\substack{\text{applied}\\ \text{forces}}} - \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j}}_{\substack{\text{“inertia}\\ \text{forces”}}} \Bigg), \tag{12.46}$$

with applied forces Q_j not derivable from a potential energy, and applied forces $-\frac{\partial V}{\partial q_j}$. The virtual power of additional passive or active elements is

$$\delta P = -\sigma_l \delta \dot{C}_l, \quad l = 1 \ldots e, \tag{12.47}$$

with the generalized forces σ_l and generalized displacement rate $\dot{C}_l$ of the additional elements. The minus sign is just a definition, where we assume that the extra element for positive σ_l and positive $\dot{C}_l$ subtracts energy from the system, like in a linear spring. The distinction between passive and active elements is usually based on the energy consumption and active control. When we supply energy by means of such an element to the system we call it active, whereas an element with stores or dissipates energy, is usually called passive. However, one can think of an actuated damper, which then would classify as an active element.

For specific active and passive elements we can describe the generalized element displacement and element force with the expressions as shown in Table 12.1.

Example 12.6 The concept of a passive element can even be further stretched from a simple one-dimensional spring to a 3-D elastic finite element, as used in the finite element method (FEM). This paves the way towards dynamics of elastic multibody systems. As an example, we demonstrate this approach for a 3-D triangular thin-plate finite element. The element, shown in Figure 12.10, has three 3-D nodes and therefore nine coordinates x_i. These coordinate are strictly speaking no CoM coordinates, but for now we use them as such. Later we have to fix this by deriving a consistent mass matrix for the d'Alembert inertia forces.

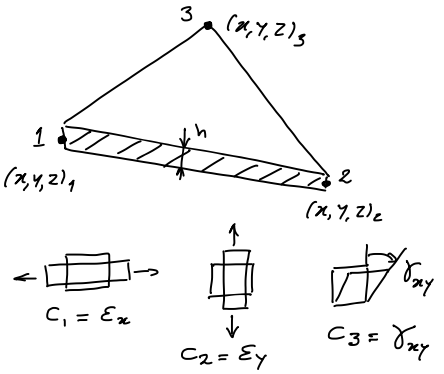


Figure 12.10: A 3-D thin-plate FEM element, together with the generalized deformations.

With nine coordinates and six degrees of freedom as a rigid body, this element can have three $(= 9 - 6)$ independent deformation modes. We will define these generalized deformations in terms of non-zero generalized constraint functions as,

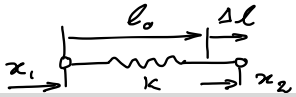
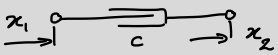
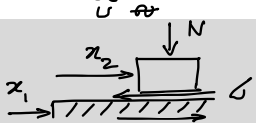
$$C_j = C_j(x_i), \qquad j = 1 \dots 3,\ i = 1 \dots 9. \tag{12.48E}$$

To be specific, for this element we take the in-plane stretching ε_x and ε_y as the first two generalized deformations, and the shear deformation γ_{xy} as the third. If we assume an elastic behavior, we can express the element forces (generalized stresses) in terms of the generalized deformations,

$$\sigma_i = S_{ij} C_j, \qquad i,\, j = 1 \dots 3, \tag{12.49E}$$

with stiffness $S_{ij}(E,\, \nu,\, h,\, A)$ depending on Young's modulus E, Poisson's ratio ν, and element thickness h and area A. For those interested in the FEM, a very good and complete course, with a variational approach to FEM (virtual work and virtual power), is freely offered by Carlos A. Felippa [16].

Table 12.1: Element generalized displacement constraints C and forces σ.

Spring		$C = \Delta l = (x_2 - x_1) - l_0$ $\sigma = k\,C$
Dashpot		$C = \Delta l = (x_2 - x_1) - l_0$ $\sigma = c\,\dot{C}$
Motor		$C = \Delta\varphi = \varphi_2 - \varphi_1$ $\sigma = -\text{Torque}(C,\, \dot{C},\, U)$
Friction		$C = \Delta x = x_2 - x_1$ $\sigma = \mu\,\lvert N\rvert\,\text{sgn}(\dot{C})$

We continue from the generalized element constraints

$$C_l = C_l(x_i(q_j)) = C_l(q_j) \tag{12.50}$$

and determine the virtual rates,

$$\delta\dot{C}_l = \frac{\partial C_l}{\partial q_j}\delta\dot{q}_j = C_{l,j}\delta\dot{q}_j. \tag{12.51}$$

Adding the virtual power of these extra elements to the virtual power expression results in

$$\delta P^* = \delta P - \sigma_l \delta\dot{C}_l, \tag{12.52}$$

and substituting (12.46) and (12.47) as well as (12.51) into (12.52), results in the complete virtual power equation,

$$\delta P^* = \left(Q_j - \frac{\partial V}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j}\right)\delta\dot{q}_j - \sigma_l C_{l,j}\delta\dot{q}_j = 0 \quad \forall\,\delta\dot{q}_j. \tag{12.53}$$

Since this must hold for arbitrary $\delta\dot{q}_j$ we obtain j Lagrange equations of motion, with added active and passive elements,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j - C_{l,j}\sigma_l, \quad j = 1\ldots d, l = 1\ldots e, \tag{12.54}$$

with e extra elements that are not in the potential energy function V. Conservative forces with a potential energy function, defined by

$$\frac{\partial V}{\partial x_i} = -F_i, \tag{12.55}$$

can be implemented either via V, or by including them in $C_{l,j}\sigma_l$. However, be aware not to include the same force in both terms.

Example 12.7 We will look at two types of force fields for which we can derive a potential energy function.

The first is a constant force such as gravity, as depicted in Figure 12.11.

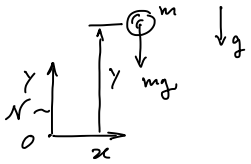


Figure 12.11: Point mass in a gravity field.

Then the constant force is $F_y = -mg$, and from the definition of the potential energy function, we derive the potential energy function V as,

$$\frac{\partial V}{\partial y} = -F_y \qquad \left(\text{and } \frac{\partial V}{\partial x} = 0\right) \tag{12.56E}$$

$$= -\left(-mg\right) \tag{12.57E}$$

$$\Rightarrow \quad V = mgy + c, \tag{12.58E}$$

with integration constant c.

The second example is a linear elastic spring, here the force depends on the location in the field $F(x)$. The behavior of such a linear elastic spring with rest length x_0 and spring stiffness k, is depicted in Figure 12.12.

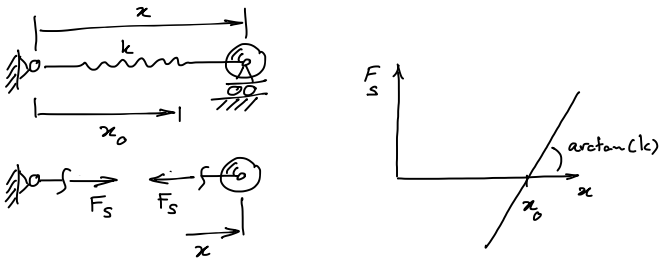


Figure 12.12: Linear elastic spring.

If we stretch the spring over a distance x, the tensile force in the spring will be $F_\mathrm{s} = k(x - x_0)$, and the applied force by the spring at the mass location in the direction of x is then $F(x) = -F_\mathrm{s}$. Then the potential V of this spring can be derived as

$$\frac{\partial V}{\partial x} = -F \tag{12.59E}$$

$$= -(-k\left(x - x_0\right)) \tag{12.60E}$$

$$\Rightarrow \quad V = \tfrac{1}{2}k\left(x - x_0\right)^2 + c. \tag{12.61E}$$

12.4 Adding Constraints

If a prescribed motion cannot directly be expressed in terms of a generalized coordinate like $q_j = q_j(t)$, then we include it via a general kinematic constraint. One of the advantages of this approach is that we can also obtain the reaction force which creates this motion. Assume we want to add m kinematic constraints of the form,

$$C_k(x_j(q_i)) = 0, \quad j = 1 \dots n,\, k = 1 \dots m,\, i = 1 \dots d. \tag{12.62}$$

We proceed in the same manner as before, starting with virtual power expressed in generalized coordinates,

$$\begin{gathered} \delta P = \delta\dot{q}_i \left(Q - \frac{\partial V}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial T}{\partial q_i}\right) = 0 \\ \forall \left\{ \delta\dot{q}_i \,|\, \delta\dot{C}_k = 0 \right\}, \quad i = 1 \dots d. \end{gathered} \tag{12.63}$$

The virtual change in constraint velocity is zero, as in

$$\delta\dot{C}_k = 0 \qquad \Rightarrow \qquad C_{k,i}\delta\dot{q}_i = 0. \tag{12.64}$$

Again we include this kinematic constraint with the help of the Lagrange multipliers λ_k, as in

$$\delta\dot{q}_i \left(Q_i - \frac{\partial V}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial T}{\partial q_i}\right) = \lambda_k C_{k,i}\delta\dot{q}_i \qquad \forall\, \delta\dot{q}_i. \tag{12.65}$$

Since this must hold for arbitrary generalized virtual velocities, the result is the constrained equations of motion,

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}}_{\text{generalized inertia forces}} + C_{k,i}\lambda_k = Q_i - \frac{\partial V}{\partial q_i}. \tag{12.66}$$

Let us have a closer look at the generalized inertia forces. We expect the inertia terms to be linear in the accelerations plus some terms quadratic in the velocities, as in

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \bar{M}_{ij}(q_l)\ddot{q}_j + h_i(q_l, \dot{q}_l). \tag{12.67}$$

Here we have introduced a generalized mass matrix $\bar{M}_{ij}(q_l)$, which in general will depend on the configuration, and convective inertia terms

$h_i(q_l,\, \dot{q}_l)$ which we expect to be quadratic in the velocities (with terms like $m\omega^2 r$ or $2mv_\mathrm{r}\omega$).

To derive these terms, we start with the kinetic energy of the system expressed in the CoM coordinate velocities,

$$T = \tfrac{1}{2}\dot{x}_i M_{ij} \dot{x}_j, \qquad i,\, j = 1 \ldots n. \tag{12.68}$$

The CoM coordinate velocities expressed in terms of the generalized coordinate velocities are

$$\dot{x}_i = x_{i,k}\dot{q}_k, \quad i = 1 \ldots n,\, k = 1 \ldots d. \tag{12.69}$$

Substitution in the kinetic energy leads to

$$\begin{aligned} T &= \tfrac{1}{2} x_{i,k}\dot{q}_k M_{ij} x_{j,l}\dot{q}_l, \quad i,j = 1 \ldots n,\, k,l = 1 \ldots d, \\ &= \tfrac{1}{2}\dot{q}_k \bar{M}_{kl}\dot{q}_l, \end{aligned} \tag{12.70}$$

from which we conclude that the generalized mass matrix is given by

$$\bar{M}_{kl} = x_{i,k} M_{ij} x_{j,l}. \tag{12.71}$$

Although the CoM mass matrix M_{ij} is a constant matrix, the generalized mass matrix $\bar{M}_{kl}$ is not and depends on the configuration, because the Jacobian matrix $x_{i,k}$ depends on the configuration. The transformation of the mass matrix from the CoM coordinates to the generalized coordinates can also be illustrated from a matrix multiplication form. If we associate the Jacobian matrix $x_{i,k}$ with the matrix $\mathbf{T}$ and the CoM-based mass terms M_{ij} with a mass matrix $\mathbf{M}$, then the generalized mass matrix can be written in matrix multiplication form as

$$\bar{\mathbf{M}} = \mathbf{T}^\mathrm{T}\mathbf{M}\mathbf{T}. \tag{12.72}$$

In the next chapter, we will focus a little more on this elegant transformation.

Still, the convective inertia terms h_i remain. We start with the kinetic energy expressed in terms of the generalized mass matrix,

$$T = \tfrac{1}{2}\dot{q}_k \bar{M}_{kl}\dot{q}_l, \tag{12.73}$$

and substituting this in the expression for the generalized inertia forces (12.67) results in

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{M}_{ij}\dot{q}_j\right) - \tfrac{1}{2}\bar{M}_{jk,i}\dot{q}_j\dot{q}_k \\ &= \bar{M}_{ij}\ddot{q}_j + \bar{M}_{ij,k}\dot{q}_j\dot{q}_k - \tfrac{1}{2}\bar{M}_{jk,i}\dot{q}_j\dot{q}_k \\ &= \bar{M}_{ij}\ddot{q}_j + \left(\bar{M}_{ij,k} - \tfrac{1}{2}\bar{M}_{jk,i}\right)\dot{q}_j\dot{q}_k \\ &= \bar{M}_{ij}\ddot{q}_j + h_i(q_j,\, \dot{q}_j). \end{aligned} \tag{12.74}$$

From this, we conclude that the convective inertia terms are

$$h_i(q_j,\, \dot{q}_j) = \left(\bar{M}_{ij,k} - \tfrac{1}{2}\bar{M}_{jk,i}\right) \dot{q}_j \dot{q}_k, \tag{12.75}$$

which are clearly bilinear in $\dot{q}_j$ (with terms like $m\omega^2 r$, $2mv_{\mathrm{r}}\omega$, etc.).

We conclude this section with the constrained equations of motion derived in terms of generalized coordinates by the method of Lagrange, where we have added some additional kinematic constraints C_k as

$$\begin{pmatrix} \bar{M}_{ij} & C_{k,i} \\ C_{k,j} & 0_{kk} \end{pmatrix} \begin{pmatrix} \ddot{q}_j \\ \lambda_k \end{pmatrix} = \begin{pmatrix} Q_i - \frac{\partial V}{\partial q_i} - h_i \\ -C_{k,ij}\dot{q}_i\dot{q}_j - 2C_{k,it}\dot{q}_i - C_{k,tt} \end{pmatrix}, \qquad i,j = 1\ldots d, \qquad k = 1\ldots m. \tag{12.76}$$

Note we have included the case where additional constraints can depend on time, as in $C_k = C_k(q_j,\, t)$, and that comma subscript t refers to a partial derivative to time. Such a type of time dependent constraint can be useful for instance if a prescribed motion is a function of time.

12.5 Impact

To derive the impact equations expressed in terms of generalized coordinates, we start with Lagrange's equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i, \quad i = 1\ldots d. \tag{12.77}$$

At impact we have contact constraints expressed by

$$C_k(q_i) = 0, \quad k = 1\ldots c, \tag{12.78}$$

with c the number of contacts. Note that the case of simultaneous impacts, $c > 1$, is very unlikely, and that we usually have to deal with one impact at a time, that is, $c = 1$. Next, we include contact forces by means of the Lagrange multiplier method,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_j - C_{k,i}\lambda_k, \tag{12.79}$$

with λ_k the Lagrange multipliers associated with the constraints of the contacts, and $C_{k,i}\lambda_k$ the contact forces in terms of the generalized forces.

For an impact, we assume that the duration of contact is short, $t^- \to t^+$ and that the contact forces are high, or in other words, the period

of contact goes to zero while the forces go to infinity. However, the integral over time is finite. Taking this limit case for the integral over time results in

$$\lim_{t^-\to t^+}\int_{t^-}^{t^+}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}\right)\mathrm{d}t=\lim_{t^-\to t^+}\int_{t^-}^{t^+}\left(Q_i-C_{k,i}\lambda_k\right)\mathrm{d}t. \tag{12.80}$$

The individual contributions in this equation will be evaluated piece by piece. First, we introduce the generalized applied impulses

$$s_i=\lim_{t^-\to t^+}\int_{t^-}^{t^+}Q_i\,\mathrm{d}t, \tag{12.81}$$

which are usually zero in impacting problems. Next, we introduce the contact impulses

$$\rho_k=\lim_{t^-\to t^+}\int_{t^-}^{t^+}\lambda_k\,\mathrm{d}t. \tag{12.82}$$

Finally, the convective generalized inertia forces, $\bar{M}_{jk,i}\dot{q}_j\dot{q}_k$, and the potential forces, mg or $k\Delta x$, stay finite, and therefore these terms will go to zero in the limit case,

$$\lim_{t^-\to t^+}\int_{t^-}^{t^+}\left(-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}\right)\mathrm{d}t=0. \tag{12.83}$$

Substituing (12.81), (12.82), and (12.83) in (12.80) yields

$$\left(\frac{\partial T}{\partial\dot{q}_i}\right)^+-\left(\frac{\partial T}{\partial\dot{q}_i}\right)^-=s_i-C_{k,i}\rho_k. \tag{12.84}$$

Substitution of the kinetic energy, which is bilinear in the generalized velocities, $T=\frac{1}{2}\bar{M}_{ij}\dot{q}_i\dot{q}_j$ with the generalized mass matrix $\bar{M}_{ij}=\bar{M}_{ij}(q_i)$, results in the Lagrange impact equations,

$$\bar{M}_{ij}\dot{q}_j^+ + C_{k,i}\rho_k = s_i+\bar{M}_{ij}\dot{q}_j^-. \tag{12.85}$$

For the constitutive behavior at the contact point, we assume Newton's impact law,

$$v_{\mathrm{rel}}^+=-ev_{\mathrm{rel}}^-,\quad \dot{C}_k^+=-e\dot{C}_k^- \quad\Rightarrow\quad C_{k,i}\dot{q}_i^+=-eC_{k,i}\dot{q}_i^-. \tag{12.86}$$

The coefficient of restitution, e, is an empirical constant and depends on the nature of the surfaces in contact. The values range from $e=1$,

which is purely elastic and implies no energy losses during impact, to $e = 0$ which is purely plastic, such that all impact energy, that is the work done during the impact by the impact forces, is assumed to be lost. Adding Newton's impact law to the equations of motion results in the complete set of Lagrange impact equations,

$$\begin{pmatrix} \bar{M}_{ij} & C_{k,i} \\ C_{k,j} & 0_{kk} \end{pmatrix} \begin{pmatrix} \dot{q}_j^+ \\ \rho_k \end{pmatrix} = \begin{pmatrix} s_i + \bar{M}_{ij}\dot{q}_j^- \\ -eC_{k,j}\dot{q}_j^- \end{pmatrix}, \; i,j = 1 \dots d, \; k = 1 \dots c. \tag{12.87}$$

From these equations, we can solve the generalized velocities after impact $\dot{q}_j^+$ together with the contact impulses ρ_k, given the generalized velocities before impact $\dot{q}_j^-$, and the applied generalized impulses s_i (usually zero, because this only happens in rare examples like hitting a system with a hammer simultaneous with the impact).

12.6 Problems

Problem 12.2 Identify the generalized forces Q_s and Q_φ in the load swing model Figure 12.8, as well as their physical units. What are the physical units of the entries in the mass matrix? ▶

Problem 12.3 How can you resolve the problem of a zero determinant mass matrix, as discussed in Example 12.5? ▶

Problem 12.4 Redo the double pendulum, Problem 11.3, but now use independent generalized coordinates and Lagrange's equations to derive the equations of motion. Compare and discuss your results with the results from Problem 11.3. Note that this cannot be done for the constraint forces, since, by using independent generalized coordinates, you have eliminated these forces from the equations. ▶

Problem 12.5 Redo the double pendulum problem with the added spring, Problem 11.5, but now use independent generalized coordinates and Lagrange's equations to derive the equations of motion. Compare and discuss your results with the results from Problem 11.5. Note that this cannot be done for the constraint forces; by using independent generalized coordinates, you have eliminated these forces from the equations of motion. ▶

Problem 12.6 Redo the double pendulum problem with the added motor, Problem 11.6, but now use independent generalized coordinates and Lagrange's equations to derive the equations of motion. Compare and discuss your results with the results from Problem 11.6. Note that this cannot be

done for the constraint forces; by using independent generalized coordinates, you have eliminated these forces from the equations of motion. ▶

Problem 12.7 Redo the double pendulum impact, Problem 11.7, but now use independent generalized coordinates and Lagrange's equations to derive the equations of motion. Compare and discuss your results with the results from Problem 11.7. Note that this cannot be done for the constraint impulses, since, by using independent generalized coordinates, you have eliminated these forces from the equations of motion. ▶

Problem 12.8 Derive a general method to determine, besides the accelerations of the degrees of freedom, *one* arbitrary constraint force when using Lagrange's equations of motion. Demonstrate your method on the double pendulum, with the initial conditions from Problem 11.3d, to determine the horizontal constraint force in joint B, between the two bodies. Check your result.

13 TMT Method: Virtual Power and Generalized Coordinates

In the first two chapters, we have seen that the dynamic behavior of multibody systems can be derived from the Newton–Euler equations of motion for a rigid body together with the constraint equations describing the ideal joints between the bodies. A major disadvantage of this approach is the numerical trouble we run into if we want to integrate these DAEs numerically. If we do this in a straightforward manner, we see that bodies fly apart, or in other words, that the kinematic constraints drift from zero. In the previous chapter, we have shown that, by choosing independent generalized coordinates, we can derive Lagrange's equations of motion in terms of these generalized coordinates. This technique works fine for small systems, that is, systems having a low number of degrees of freedom. Once we run into more complex systems, the symbolic computation of the partial derivatives results in very lengthy and computationally inefficient expressions. Fortunately, there is a solution to all these problems: For this, we go back to the basic ideas, as formulated by Lagrange in his monumental work "Méchanique Analitique" (1788), and make use of the following three ingredients: virtual power, independent generalized coordinates, and inertia force contributions via d'Alembert forces. With these ingredients, we will present a method to derive the equations of motion for a multibody system that is simple, clear, and computationally efficient.

13.1 Equations of Motion

The starting point is the virtual power expression δP for the unconstrained system of rigid bodies,

$$\delta P = \delta \dot{x}_i \left(F_i - M_{ij} \ddot{x}_j \right), \qquad i,\, j = 1 \ldots n, \tag{13.1}$$

with the CoM coordinates, $x_i = (x,\, y,\, \varphi)_b$, the applied forces and moments at the CoM, $F_i = (F_x,\, F_y,\, M)_b$, and the diagonal mass matrix

$M_{ij} = \text{diag}\,(m,\, m,\, I)_b$, and $b = 1 \ldots N$ for the N rigid bodies, were usually $n = 3N$.

The system is in equilibrium if $\delta P = 0$ for all $\delta\dot{x}$ which comply with the kinematic constraints

$$C_k(x_i) = 0 \rightarrow C_{k,i}\delta\dot{x}_i = 0, \qquad k = 1 \ldots m, \quad j = 1 \ldots n. \tag{13.2}$$

We assume that we can write the CoM coordinates of the bodies as functions of a set of d generalized coordinates q_j such that all constraints are fulfilled,

$$x_i = T_i(q_j), \quad i = 1 \ldots n, \quad j = 1 \ldots d \quad (d = n - m). \tag{13.3}$$

In general, we cannot find these functions T_i in explicit form. Even for such a simple system as a four-bar mechanism this is not so easy. However, then we will use a numerical approach, as will be explained in Chapter 15 on closed-loop kinematic chains.

The velocities of the CoM coordinates are

$$\dot{x}_i = T_{i,j}\dot{q}_j, \qquad i = 1 \ldots n, \quad j = 1 \ldots d, \tag{13.4}$$

and the virtual velocities are

$$\delta\dot{x}_i = T_{i,j}\delta\dot{q}_j. \tag{13.5}$$

Since we have introduced new coordinates q_j, we also have a new set of corresponding generalized forces Q_j. Therefore, the total virtual power of the applied forces F_i, together with the d'Alembert inertia forces $-M_{ik}\ddot{x}_k$, and the generalized forces Q_j of the system is

$$\delta P^* = \delta\dot{x}_i\,(F_i - M_{ik}\ddot{x}_k) + \delta\dot{q}_j Q_j, \quad i,k = 1 \ldots n,\, j = 1 \ldots d. \tag{13.6}$$

The virtual velocities which satisfy the constraints are

$$\delta\dot{x}_i = T_{i,j}\delta\dot{q}_j, \tag{13.7}$$

and the accelerations which satisfy the constraints are

$$\ddot{x}_k = T_{k,l}\ddot{q}_l + T_{k,lm}\dot{q}_l\dot{q}_m, \qquad l,\, m = 1 \ldots d, \tag{13.8}$$

$$\ddot{x}_k = T_{k,l}\ddot{q}_l + g_k, \tag{13.9}$$

where we introduced the *convective acceleration terms* $g_k = T_{k,lm}\dot{q}_l\dot{q}_m$, which are bilinear in the velocities (with terms like $\omega^2 r$ or $2v_\mathrm{r}\omega$). Substitution of these results in (13.6) yields the virtual power expressed in generalized coordinates and their derivatives,

$$\delta P^* = T_{i,j}\delta\dot{q}_j \left(F_i - M_{ik}T_{k,l}\ddot{q}_l - M_{ik}g_k\right) + \delta\dot{q}_j Q_j. \tag{13.10}$$

The system is in (dynamic) equilibrium when $\delta P^* = 0 \;\forall\; \delta\dot{q}_j$, which results in j equations of motion,

$$T_{i,j}\left(F_i - M_{ik}T_{k,l}\ddot{q}_l - M_{ik}g_k\right) + Q_j = 0. \tag{13.11}$$

Rearranging the terms gives us the equations of motion for the system in terms of generalized coordinates,

$$T_{i,j}M_{ik}T_{k,l}\ddot{q}_l = Q_j + T_{i,j}\left(F_i - M_{ik}g_k\right), \tag{13.12}$$

or if we associate the Jacobian matrix $T_{k,l}$ with the matrix $\mathbf{T}$, then these equations of motion are, in matrix–vector form,

$$\mathbf{T}^\mathrm{T}\mathbf{M}\mathbf{T}\cdot\ddot{\boldsymbol{q}} = \boldsymbol{Q} + \mathbf{T}^\mathrm{T}\left(\boldsymbol{F} - \mathbf{M}\boldsymbol{g}\right). \tag{13.13}$$

We see that the global mass matrix from the CoM coordinates, $\mathbf{M}$, is symmetrically transformed by this pre- and post-multiplication of the Jacobian matrix, respectively $\mathbf{T}^\mathrm{T}$ and $\mathbf{T}$, hence the name *TMT-method*. The resulting mass matrix is smaller, denser and coordinate-dependent. On the right-hand side we see that the global CoM forces are transformed to the generalized forces by this transpose of the Jacobian matrix $\mathbf{T}^\mathrm{T}$. This transformation is a projection from the high-dimensional space of CoM forces, $\dim(\boldsymbol{F}) = n$, to the much lower-dimensional space of generalized forces, $\dim(\boldsymbol{Q}) = d$. We will demonstrate this in Example 13.1.

The equations of motion expressed in generalized coordinates (13.12) can even be written in a more compact form,

$$\bar{M}_{jl}\ddot{q}_l = \bar{Q}_j, \quad j,l = 1\ldots d, \tag{13.14}$$

with the reduced generalized mass-matrix,

$$\bar{M}_{jl} = T_{i,j}M_{ik}T_{k,l}, \tag{13.15}$$

or in matrix-vector format,

$$\bar{\mathbf{M}} = \mathbf{T}^\mathrm{T}\mathbf{M}\mathbf{T}, \tag{13.16}$$

and the reduced generalized force vector,

$$\bar{Q}_j = Q_j + T_{i,j}\left(F_i - M_{ik}T_{k,lm}\dot{q}_l\dot{q}_m\right) \tag{13.17}$$

or in vector form,

$$\bar{\boldsymbol{Q}} = \boldsymbol{Q} + \mathbf{T}^{\mathrm{T}}\left(\boldsymbol{F} - \mathbf{M}\boldsymbol{g}\right). \tag{13.18}$$

By choosing generalized coordinates that satisfy the kinematic constraints, we have again eliminated the constraint force, just like in Lagrange's equations of motion. In fact, Lagrange's equations of motion (12.31), with the coordinate transformation $x_i = T_i(q_j)$, should lead to the same result, see Problem 13.1. However, where Lagrange steps "up", via integration, to the level of kinetic and potential energy, T and V, and then "down" again by differentiation, to forces, the TMT-method stays on the force level, and reduces the amount of algebraic manipulation (= effort). We will now demonstrate the TMT-method on a small example.

Example 13.1 This Example is about the load swing of Example 12.4 on page 333. However, instead of a point-mass model for the container, we now use a rigid body with mass m and mass moment of inertia I at the CoM connected by a rigid rod to the hinge in A.

The CoM coordinates of the rigid body are $x_i = (x,\, y,\, \varphi)^{\mathrm{T}}$, and as generalized coordinates we use $q_j = (s,\, \alpha)^{\mathrm{T}}$, see Figure 13.1.

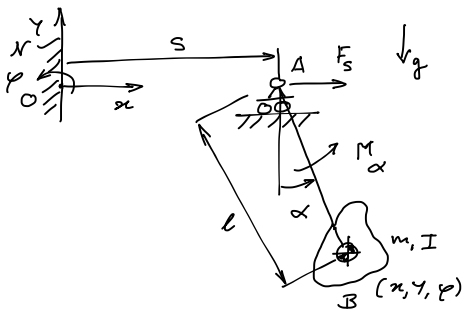


Figure 13.1: Model of load swing in an overhead container crane.

The coordinate transformations, expressing the CoM coordinates in the generalized coordinates,

$$x_i = T_i(q_j), \tag{13.19E}$$

are

$$\begin{pmatrix} x \\ y \\ \varphi \end{pmatrix} = \begin{pmatrix} s + l\sin\alpha \\ -l\cos\alpha \\ \alpha \end{pmatrix}. \tag{13.20E}$$

For the equations of motion derived by the TMT-method,

$$T_{i,j}M_{ik}T_{k,l}\ddot{q}_l = Q_j + T_{i,j}\left(F_i - M_{ik}g_k\right), \tag{13.21E}$$

we need the velocity transformation matrix $T_{i,j}$ and the convective acceleration terms, $g_k = T_{k,lm}\dot{q}_l\dot{q}_m$. By comparing

$$\dot{x}_i = T_{i,j}\dot{q}_j, \tag{13.22E}$$

with

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\varphi}\end{pmatrix} = \begin{pmatrix}\dot{s} + l\dot{\alpha}\cos\alpha\\ l\dot{\alpha}\sin\alpha\\ \dot{\alpha}\end{pmatrix} = \begin{pmatrix}1 & l\cos\alpha\\ 0 & l\sin\alpha\\ 0 & 1\end{pmatrix}\begin{pmatrix}\dot{s}\\ \dot{\alpha}\end{pmatrix}, \tag{13.23E}$$

we obtain

$$T_{i,j} = \begin{pmatrix}1 & l\cos\alpha\\ 0 & l\sin\alpha\\ 0 & 1\end{pmatrix}. \tag{13.24E}$$

We differentiate again to get g_k as in

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{\varphi}\end{pmatrix} = \begin{pmatrix}\ddot{s} + l\ddot{\alpha}\cos\alpha - l\dot{\alpha}^2\sin\alpha\\ l\ddot{\alpha}\sin\alpha + l\dot{\alpha}^2\cos\alpha\\ \ddot{\alpha}\end{pmatrix} \tag{13.25E}$$

$$= \begin{pmatrix}\ddot{s} + l\ddot{\alpha}\cos\alpha\\ l\ddot{\alpha}\sin\alpha\\ \ddot{\alpha}\end{pmatrix} + \begin{pmatrix}-l\dot{\alpha}^2\sin\alpha\\ l\dot{\alpha}^2\cos\alpha\\ 0\end{pmatrix}, \tag{13.26E}$$

which corresponds to

$$\ddot{x}_i = T_{i,j}\ddot{q}_j + T_{i,kl}\dot{q}_k\dot{q}_l. \tag{13.27E}$$

The second set of terms in (13.26E) are the convective acceleration terms

$$g_k = \begin{pmatrix}-l\dot{\alpha}^2\sin\alpha & l\dot{\alpha}^2\cos\alpha & 0\end{pmatrix}^{\mathrm{T}}. \tag{13.28E}$$

Now, with the CoM-based mass matrix $M_{ij} = \mathrm{diag}(m, m, I)$, the transformed generalized mass matrix becomes

$$T_{i,j}M_{ik}T_{k,l} = \begin{pmatrix}1 & 0 & 0\\ l\cos\alpha & l\sin\alpha & 1\end{pmatrix}\begin{pmatrix}m & 0 & 0\\ 0 & m & 0\\ 0 & 0 & I\end{pmatrix}\begin{pmatrix}1 & l\cos\alpha\\ 0 & l\sin\alpha\\ 0 & 1\end{pmatrix} \tag{13.29E}$$

$$= \begin{pmatrix}m & ml\cos\alpha\\ ml\cos\alpha & ml^2 + I\end{pmatrix}. \tag{13.30E}$$

With the CoM-applied forces $F_i = (0,\, -mg,\, 0)^{\mathrm{T}}$, the transformed forces on the right-hand side are

$$T_{i,j}\,(F_i - M_{ik} g_k) \tag{13.31E}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ l\cos\alpha & l\sin\alpha & 1 \end{pmatrix} \left(\begin{pmatrix} 0 \\ -mg \\ 0 \end{pmatrix} - \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} -l\dot{\alpha}^2 \sin\alpha \\ l\dot{\alpha}^2 \cos\alpha \\ 0 \end{pmatrix} \right), \tag{13.32E}$$

which resolves to

$$T_{i,j} = \begin{pmatrix} ml\dot{\alpha}^2 \sin\alpha \\ -mgl\sin\alpha \end{pmatrix}. \tag{13.33E}$$

Collecting and reordering the terms results in the equations of motion expressed in terms of generalized coordinates,

$$\begin{pmatrix} m & ml\cos\alpha \\ ml\cos\alpha & ml^2 + I \end{pmatrix} \begin{pmatrix} \ddot{s} \\ \ddot{\alpha} \end{pmatrix} = \begin{pmatrix} F_s + ml\dot{\alpha}^2 \sin\alpha \\ M_\alpha - mgl\sin\alpha \end{pmatrix}. \tag{13.34E}$$

Besides the additional mass moment of inertia contribution, these equations agree completely with the equations of motion as derived by means of the method of Lagrange (12.44E). Note that for a finite mass, length, and mass moment of inertia, the generalized mass matrix will never be singular,

$$\det(\bar{\mathbf{M}}) = m^2 l^2 \left(1 - \cos^2\alpha\right) + mI > 0, \tag{13.35E}$$

and the equations will always have a unique solution, as expected.

13.2 Elimination of Constraint Forces

We will now demonstrate how the constraint forces are eliminated when using generalized coordinates that fulfill all the constraints. If we have m kinematic constraints of the form $C_k(x_i) = 0$ on the CoM coordinates, then constraint equations of motion are given by

$$M_{ij}\ddot{x}_j = F_i - C_{k,i}\lambda_k, \qquad i,\, j = 1 \ldots n, \quad k = 1 \ldots m. \tag{13.36}$$

We have seen in the previous chapters that the Lagrange multipliers multiplied by the Jacobian matrix of the constraints yield forces at the constraints. Defining these constraint forces as

$$R_i = -C_{k,i}\lambda_k, \tag{13.37}$$

the equations of motions for the rigid bodies become

$$M_{ij}\ddot{x}_j = F_i + R_i, \qquad i,\, j = 1 \ldots n, \tag{13.38}$$

which are the equations of motion we obtain when we cut the bodies free from the joints, and introduce action–reaction constraint forces R_i at the cuts, similar to the pencil-and-paper method from Chapter 10. The next step is to transform these equations of motion, by means of the Jacobian matrix of the transformation function, $T_{i,j}$, to the space of generalized forces Q_j^*, as in

$$T_{i,j}\left(F_i + R_i - M_{ij}\ddot{x}_j\right) = Q_j^*. \tag{13.39}$$

We expect that the equations of motion in the generalized coordinates will have the form

$$Q_j + Q_j^* = 0, \qquad \text{or} \tag{13.40}$$

$$Q_j + T_{i,j}\left(F_i + R_i - M_{ij}\ddot{x}_j\right) = 0, \tag{13.41}$$

and that the terms with the constraint forces R_i will all disappear. In other words, we expect that

$$T_{i,j}R_i = -T_{i,j}C_{k,i}\lambda_k = 0_j, \quad i = 1\ldots n,\, j = 1\ldots d,\, k = 1\ldots m, \tag{13.42}$$

for arbitrary values of the Lagrange multipliers λ_k. Then the leading matrix should be zero, as in,

$$T_{i,j}C_{k,i} = 0_{jk}, \quad i = 1\ldots n, \quad j = 1\ldots d, \quad k = 1\ldots m. \tag{13.43}$$

This follows directly from substitution of the coordinate transformation, $x_i = T_i(q_j)$, in the kinematic constraints, $C_k(x_i) = 0$, and taking the time derivatives,

$$C_k(T_i(q_j)) = 0 \quad \Rightarrow \quad C_{k,i}T_{i,j}\dot{q}_j = 0, \tag{13.44}$$

which should hold for arbitrary generalized velocities $\dot{q}_j$. From this we conclude that all entries in the leading matrix are zero,

$$C_{k,i}T_{i,j} = 0_{kj}, \tag{13.45}$$

which proves that all constraint forces are eliminated from the equations of motion.

13.3 Passive and Active Elements

To add the contribution of passive and active elements to the equations of motion, we start again with the sum of the virtual power of the

applied forces F_i, the d'Alembert inertia forces, and the generalized forces Q_j,

$$\delta P^* = \delta\dot{x}_i \left(F_i - M_{ik}\ddot{x}_k\right) + \delta\dot{q}_j Q_j. \tag{13.46}$$

Adding the virtual power from the extra elements, $-\sigma_l \delta\dot{C}_l$, results in

$$\delta P^{**} = \delta\dot{x}_i \left(F_i - M_{ik}\ddot{x}_k\right) + \delta\dot{q}_j Q_j - \sigma_l \delta\dot{C}_l, \tag{13.47}$$

where C_l is some generalized relative coordinate to describe the state of the element (see Table 12.1 on page 338). We are used to expressing C_l in terms of the CoM coordinates x_i,

$$C_l = C_l(x_i). \tag{13.48}$$

The virtual generalized element velocities are

$$\delta\dot{C}_l = C_{l,i}\delta\dot{x}_i. \tag{13.49}$$

which we substitute in the virtual power expression (13.47), resulting in

$$\delta P^{**} = \delta\dot{x}_i \left(F_i - M_{ik}\ddot{x}_k\right) + \delta\dot{q}_j Q_j - \sigma_l C_{l,i}\delta\dot{x}_i. \tag{13.50}$$

Transforming the CoM coordinates $\delta\dot{x}_i$ to the generalized coordinates q_j, and their speeds yields

$$x_i = T_i(q_j) \qquad \Rightarrow \qquad \delta\dot{x}_i = T_{i,j}\delta\dot{q}_j, \tag{13.51}$$

expressing the CoM accelerations in terms of the generalized coordinates, their speed and accelerations,

$$\ddot{x}_k = T_{k,l}\ddot{q}_l + g_k, \tag{13.52}$$

and substitution of these into the virtual power expression results in

$$\delta P^{**} = \delta\dot{q}_j T_{i,j} \left(F_i - M_{ik}\left(T_{k,l}\ddot{q}_l + g_k\right) - \sigma_p C_{p,i}\right) + \delta\dot{q}_j Q_j. \tag{13.53}$$

The system is in equilibrium if $\delta P^{**} = 0 \;\forall\; \delta\dot{q}_j$, which results in the equations of motion,

$$T_{i,j}M_{ik}T_{k,l}\ddot{q}_l = Q_j + T_{i,j}\left(F_i - M_{ik}g_k - C_{p,i}\sigma_p\right). \tag{13.54}$$

Note that the element forces σ_p are transformed to the generalized forces Q_j according to these two Jacobian matrices,

$$-C_{p,i}T_{i,j}\sigma_p = -\frac{\partial C_p}{\partial x_i}\frac{\partial x_i}{\partial q_j}\sigma_p, \tag{13.55}$$

which according to the chain rule, is identical to differentiating C_p with respect to the generalized coordinates q_j

$$\frac{\partial C_p}{\partial x_i}\frac{\partial x_i}{\partial q_j}\sigma_p = \frac{\partial C_p}{\partial q_j}\sigma_p. \tag{13.56}$$

13.4 Adding Further Constraints

Assume that, besides the constraints that are already included through the choice of the generalized coordinates, we want to add m further kinematic constraints of the form

$$\begin{aligned} C_c = C_c(x_i) = 0 \quad \text{or} \quad C_c(T_i(q_j)) = 0, \\ c = 1 \dots m, \quad i = 1 \dots n, \quad j = 1 \dots d. \end{aligned} \tag{13.57}$$

We include these kinematic constraints here again with the help of the Lagrange multipliers λ_c. The resulting constrained equations of motion are

$$T_{i,j} M_{ik} T_{k,l} \ddot{q}_l + C_{c,j} \lambda_c = Q_j + T_{i,j} \left(F_i - M_{ik} g_k \right). \tag{13.58}$$

Differentiating the constraints with respect to time twice and adding these to the constrained equations of motion results in the complete set of constrained equations of motion, from which we are able to solve for the acceleration of the generalized coordinates $\ddot{q}_l$, together with the unknown Lagrange multipliers λ_c,

$$\begin{pmatrix} T_{i,j} M_{ik} T_{k,l} & C_{c,j} \\ C_{c,l} & 0_{cc} \end{pmatrix} \begin{pmatrix} \ddot{q}_l \\ \lambda_c \end{pmatrix} = \begin{pmatrix} Q_j + T_{i,j} \left(F_i - M_{ik} g_k \right) \\ -C_{c,jl} \dot{q}_j \dot{q}_l \end{pmatrix}, \tag{13.59}$$

or in matrix-vector notation (with the matrix $\mathbf{C} = \frac{\partial C_c}{\partial q_j}$),

$$\begin{pmatrix} \mathbf{T}^{\mathrm{T}} \mathbf{M} \mathbf{T} & \mathbf{C}^{\mathrm{T}} \\ \mathbf{C} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Q} + \mathbf{T}^{\mathrm{T}} \left(\boldsymbol{F} - \mathbf{M} \boldsymbol{g} \right) \\ -\mathbf{C}_2 \dot{\boldsymbol{q}} \dot{\boldsymbol{q}} \end{pmatrix}. \tag{13.60}$$

These constrained equations of motion will form the basis for deriving the generalized impact equations, where the additional kinematic constraints are the contact conditions.

13.5 Impact

We follow the same reasoning as in Section 11.4. Assume we have $c = 1 \dots n_{\mathrm{imp}}$ simultaneous impacts described by contact conditions,

$$C_c = C_c(x_i) = 0, \tag{13.61}$$

or with $x_i = T_i(q_j)$,

$$C_c = C_c(q_j) = 0. \tag{13.62}$$

Taking the integral of the constrained equations of motion with respect to time, and the limit case where the impact duration goes to zero, gives

$$\begin{aligned}\lim_{t^-\to t^+}\int_{t^-}^{t^+}&\left(T_{i,j}M_{ik}T_{k,l}\ddot{q}_l+C_{c,j}\lambda_c\right)\,\mathrm{d}t\\&=\int_{t^-}^{t^+}\left(Q_j+T_{i,j}\left(F_i-M_{ik}g_k\right)\right)\,\mathrm{d}t.\end{aligned}\tag{13.63}$$

We introduce the contact impulses as

$$\rho_c=\lim_{t^-\to t^+}\int_{t^-}^{t^+}\lambda_c\,\mathrm{d}t,\tag{13.64}$$

the applied generalized impulses (usually zero) as

$$s_j=\lim_{t^-\to t^+}\int_{t^-}^{t^+}Q_j\,\mathrm{d}t,\tag{13.65}$$

and the applied impulses (usually also zero) as

$$p_i=\lim_{t^-\to t^+}\int_{t^-}^{t^+}F_i\,\mathrm{d}t.\tag{13.66}$$

Taking the limit case where the period of contact goes to zero, $(t^+ - t^-)\to 0$, noting that all velocity jumps are finite, and adding Newton's impact law $\dot{C}_c^+ = -e\dot{C}_c^-$, we obtain the set of linear equations for calculating the velocities after impact together with contact impulses,

$$\begin{pmatrix}T_{i,j}M_{ik}T_{k,l} & C_{c,j}\\ C_{c,l} & 0_{cc}\end{pmatrix}\begin{pmatrix}\dot{q}_l^+\\ \rho_c\end{pmatrix}=\begin{pmatrix}T_{i,j}M_{ik}T_{k,l}\dot{q}_l^- + s_j + T_{i,j}p_i\\ -eC_{c,l}\dot{q}_l^-\end{pmatrix}.\tag{13.67}$$

These impact equations, expressed in the generalized coordinates and speeds, naturally show great resemblance to the impact equations as derived by means of Lagrange's method (12.87). However, calculating the contributions is more straightforward and costs less effort.

13.6 Problems

Problem 13.1 Show that by using the coordinate transformation $x_i = T_i(q_j)$ and the applied forces as, $F_i = -\frac{\partial V}{\partial x_i}$, that Lagrange's equations $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j$, will lead to the equations of motion (13.12).

Problem 13.2 Prove that in the load-swing Example 13.1, the centrifugal inertia forces at the CoM, $F_c = -m\dot{\alpha}^2 l(-\sin\alpha, \cos\alpha, 0)^{\mathrm{T}}$, do not contribute to the generalized force M_α.

Problem 13.3 A simplified 2-D model of a bipedal walking machine consists of two slender legs hinged at the hip, see Figure 13.2. We model the system in the motion phase where one leg (called the stance leg) is on the ground and stays on the ground, and the other leg (called the swing leg) is swaying. The ground is tilted clockwise by an angle γ with respect to the downward gravity g.

The two legs are identical, with length l and mass m (distributed evenly along the length of the leg). An extra point mass M is located at the hip.

As generalized coordinates, we choose the angle α of the stance leg with respect to the outer normal to the ground, and the angle β of the swing leg, also with respect to the outer normal to the ground, where clockwise is defined positive.

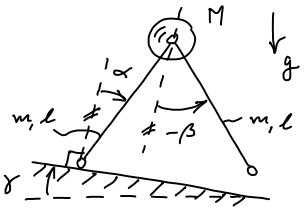


Figure 13.2: 2-D walker.

a. Derive the equations of motion in terms of the generalized coordinates $q_i = (\alpha, \beta)^{\mathrm{T}}$ by means of Lagrange's equations of motion. Write these equations in terms of a mass matrix times accelerations equals forces, as in $\bar{M}_{ij}\ddot{q}_j = \bar{Q}_i$
b. Derive the equations of motion using the principle of virtual power, the principle of d'Alembert, and the transformation of the coordinates of the CoMs of the bodies x_i in terms of the independent generalized coordinates q_j as in $x_i = T_i(q_j)$ (TMT-method).
c. Compare the results from a. and b.
d. Discuss for large systems, where large means many bodies and many degrees of freedom, the pros and cons of the two different methods.

In deriving the equations of motion you may want to use symbolic software such as MAPLE$^{\mathrm{TM}}$ or the symbolic toolbox from MATLAB$^{\mathrm{TM}}$ or MATHEMATICA$^{\mathrm{TM}}$. This can reduce the number of errors substantially. ▶

Problem 13.4 Consider a planar system which consists of two slender rigid bars and an upright wheel, see Figure 13.3. Bar 1 is connected to the

ground by a rotational joint in A. Bar 2 is connected to bar 1 in point B. A massless upright wheel with radius r is mounted on bar 2 in point C with a lateral offset d from the end of bar. The wheel rolls over a horizontal xy-plane. We assume that there is no sideslip at the wheel and that it can freely rotate about its axis. Bar 1 is a slender bar of length L_1, mass m_1, and mass moment of inertia I_1 at the CoM. Bar 2 is a slender bar of length L_2, mass m_2, and mass moment of inertia I_2 at the CoM. We assume that the CoM's of bar 1 and 2 are in the middle of the bars, that is at $L_1/2$ and $L_2/2$. The orientation of each bar is described by an angle φ_1 respectively φ_2, which is measured positive anti-clockwise relative to the x-axis. Gravity is acting in the minus z-direction.

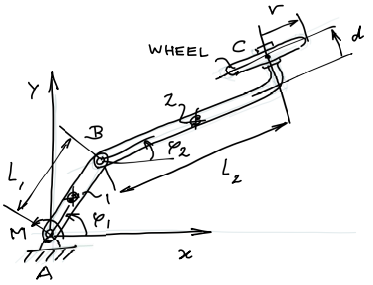


Figure 13.3: Two slender bars and a wheel.

While the system is in motion an anti-clockwise torque M is applied to bar 1. Determine at that moment in time the angular acceleration of bar 1, $\ddot{\varphi}_1$.[a]

The initial conditions for the system in motion and the value for the applied torque are: $\varphi_1 = \pi/3\,\text{rad}$, $\varphi_2 = 0\,\text{rad}$, $\dot{\varphi}_1 = 3\,\text{rad/s}$, and $M = 5.439\,63\,\text{N m}$.

For system parameters use: $L_1 = 0.3\,\text{m}$, $L_2 = 0.6\,\text{m}$, $d = 0.1\,\text{m}$, $r = 0.15\,\text{m}$, $m_1 = 2\,\text{kg}$, $m_2 = 4\,\text{kg}$, $I_1 = 0.02\,\text{kg m}^2$, $I_2 = 0.12\,\text{kg m}^2$, and $g = 9.81\,\text{N/kg}$. ▷

[a] *Hint*: This is like the double pendulum problem with an extra non-holonomic constraint: no sideslip at point C of bar 2. Determine the position of point C in terms of your two kinematic degrees of freedom $\boldsymbol{q}$, say this is the vector $\boldsymbol{r}_C(q)$. Next express the velocity of point C in terms of the degrees of freedom and their velocities, by differentiation of $\boldsymbol{r}_C$, say this is vector $\boldsymbol{v}_C = (\boldsymbol{r}_C)_{\boldsymbol{b}}\dot{\boldsymbol{q}}$. The unit vector in point C perpendicular to the wheel is $\boldsymbol{e}_p = (-\sin(\varphi_2), \cos(\varphi_2))^{\mathrm{T}}$. Then, no sideslip at C is expressed by $s1 = \boldsymbol{v}_C^{\mathrm{T}} * \boldsymbol{e}_p = 0$. Differentiate this constraint equation and add it to the equations of motion. Note that bar 2 also has an initial angular velocity and you can find that from the initial conditions and the non-holonomic constraint. Check, that for this set of parameters that initial angular velocity is $\dot{\varphi}_2 = -0.75\,\text{rad/s}$.

Problem 13.5 Consider a cart with a double pendulum attached, as shown in Figure 13.4. The cart, body 1 with mass m_1, is restricted to move over a horizontal plane. The double pendulum consist of two slender bars, body 2 and body 3 with length l_2 and l_3, and evenly along the length distributed mass m_2 and m_3 and mass moments of inertia at the CoM (at the middle of the bars) I_2 and I_3 . The bodies are attached to each other by frictionless rotational joints.

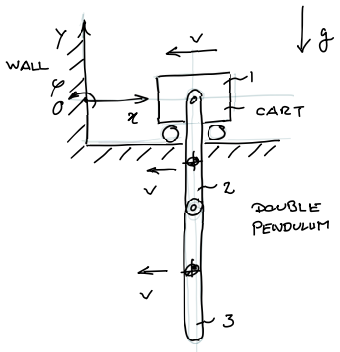


Figure 13.4: Moving cart with double pendulum system.

The system is moving with a constant horizontal speed v towards a vertical wall on the left. Both pendula are hanging vertically down and have the same horizontal speed v as the cart and zero rotational speed. Then the cart hits the wall and bounces back during a perfect elastic impact ($e = 1$). Determine the velocity of body 2 (the first pendulum attached to the cart) at its CoM in the x-direction, $\dot{x}_2$, in m/s, directly after the impact.[a] Gravity is acting vertically downwards with field strength g.

The parameter values for this system are: $l_2 = 0.4\,\text{m}$, $l_3 = 0.6\,\text{m}$, $m_1 = 10\,\text{kg}$, $m_2 = 2\,\text{kg}$, $m_3 = 9\,\text{kg}$, $I_2 = 0.03\,\text{kg}\,\text{m}^2$, $I_3 = 0.27\,\text{kg}\,\text{m}^2$, $v = 3\,\text{m/s}$ (to the left), and $g = 9.81\,\text{N/kg}$ ▷

[a] *Hint*: The system has three degrees of freedom, the linear motion of the cart and the rotations of the two pendula. For the elastic impact of the cart with the vertical wall you have to add a contact constraint to the system impulse equations such that the relative velocity after impact is minus the one before impact. If you solve these impact equations then it is expected that the velocity of the cart after an elastic impact is v again but now to the right ($+x$-direction). Next we expect that the kinetic energy before and after impact to be the same.

14 Numerical Integration of Ordinary Differential Equations

Up till now we have solved, in various ways, for the accelerations of the rigid bodies in the multibody system from the equations of motion. However, we are actually interested in the motion of the system as a function of time. Therefore, we have to solve the set of second-order differential equations of motion. In general, we will not be able to find analytic solutions for that, and therefore we have to use numerical methods to find the solutions. In this chapter, we will introduce a number of basic methods to numerically integrate the equations of motion.

The equations of motion we derived so far can be classified as ordinary differential equations, opposed to partial differential equations. They are called ordinary, because the differentiation is with respect to only one variable, in our case time t; on the other hand partial differential equations are characterized by having more independent variables or differentials, like time and place, for example as in the Navier–Stokes equations from fluid dynamics. Furthermore, we usually deal here with an initial value problem, that is an ordinary differential equation with given initial values of the state at t_0. This is opposed to, for instance, a boundary value problem, where we have conditions on both the initial values at t_0 and the final values at t_{end}.

A vast amount of literature treats the numerical integration of ordinary differential equations, of which we will treat two in particular in this chapter: The first book is "Numerical Methods for Ordinary Differential Equations" by J.C. Butcher [8], which describes the various methods in an elegant way and within a historic context. The second book is by L.F. Shampine and M.K. Gordon, entitled "Computer Solution of Ordinary Differential Equations: the Initial Value Problem" [65]; most of the methods described in this book have been implemented in various computer codes, first in the famous FORTRAN code STEP and more recently in the ODE suite [66] of MATLAB$^{\text{TM}}$.

14.1 Explicit First-Order ODE Methods

14.1.1 Basic Method: Euler

We start by introducing one of the simplest *numerical integration* methods. A natural, and perhaps somewhat naive, way to find $\boldsymbol{q}_1$ and $\dot{\boldsymbol{q}}_1$ would be *Euler's method* or the *Euler step*:

$$\boldsymbol{q}_1 = \boldsymbol{q}_0 + h\dot{\boldsymbol{q}}_0, \tag{14.1}$$

$$\dot{\boldsymbol{q}}_1 = \dot{\boldsymbol{q}}_0 + h\ddot{\boldsymbol{q}}_0. \tag{14.2}$$

Naive, because it follows almost directly from the definition of a derivative with respect to time (as linking velocity and acceleration):

$$\dot{\boldsymbol{q}}(t) = \lim_{h\to 0} \frac{\boldsymbol{q}(t+h) - \boldsymbol{q}(t)}{h} \quad \Rightarrow \quad \boldsymbol{q}(t+h) \approx \boldsymbol{q}(t) + h\dot{\boldsymbol{q}}(t). \tag{14.3}$$

In fact, such an Euler step can be justified by looking at the Taylor series expansion of a function,

$$f(x+h) = f(x) + hf'(x) + \tfrac{1}{2}h^2 f''(x) + \cdots + \frac{1}{n!}h^n f^n(x) + O(h^{n+1}), \tag{14.4}$$

and ignoring terms of order h^2 and higher. Knowing this, one could raise the question if it were not better to use also the second-order derivative, in approximation of the new coordinates, as in

$$\boldsymbol{q}_1 = \boldsymbol{q}_0 + h\dot{\boldsymbol{q}}_0 + \tfrac{1}{2}h^2\ddot{\boldsymbol{q}}_0. \tag{14.5}$$

However, what do we mean by "better", more accurate? We will get back to that question further on.

14.1.2 Standard First-Order Form

Before we move on, let us first look at the standard definition of ordinary differential equations and the system of second-order differential equations we use in mechanics.

A set of *ordinary differential equations* (ODEs) is usually, together with their initial values, expressed as

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}), \qquad \boldsymbol{y}(\boldsymbol{x}_0) = \boldsymbol{y}_0. \tag{14.6}$$

In our case the independent variable x is usually the time t and we have

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(t, \boldsymbol{y}), \qquad \boldsymbol{y}(t_0) = \boldsymbol{y}_0. \tag{14.7}$$

However, in fact we have a set of n second-order differential equations, with $2n$ initial values,

$$\bar{\mathbf{M}}\ddot{\boldsymbol{q}} = \bar{\boldsymbol{Q}}(t,\, \boldsymbol{q},\, \dot{\boldsymbol{q}}), \qquad \boldsymbol{q}(t_0) = \boldsymbol{q}_0, \qquad \dot{\boldsymbol{q}}(t_0) = \dot{\boldsymbol{q}}_0. \tag{14.8}$$

Such a system can be transformed into $2n$ first-order differential equations by substitution of $\boldsymbol{u} = \dot{\boldsymbol{q}}$ such that (14.8) becomes

$$\bar{\mathbf{M}}\dot{\boldsymbol{u}} = \bar{\boldsymbol{Q}}(t,\, \boldsymbol{q},\, \boldsymbol{u}) \tag{14.9}$$

or

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{u}, & \boldsymbol{q}(t_0) &= \boldsymbol{q}_0, \\ \dot{\boldsymbol{u}} &= \bar{\mathbf{M}}^{-1}\bar{\boldsymbol{Q}}(t,\, \boldsymbol{q},\, \boldsymbol{u}), & \boldsymbol{u}(t_0) &= \boldsymbol{u}_0 = \dot{\boldsymbol{q}}_0, \end{aligned} \tag{14.10}$$

which we can subsequently formulate into the standard first-order problem with the state vector $\boldsymbol{y} = (\boldsymbol{q}^{\mathrm{T}},\, \boldsymbol{u}^{\mathrm{T}})^{\mathrm{T}}$, as

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(t,\, \boldsymbol{y}), \qquad \boldsymbol{y}(t_0) = \boldsymbol{y}_0. \tag{14.11}$$

Example 14.1 We will demonstrate the formulation of a standard first-order ODE from a second-order system on a mass-spring-damper system in 2-D, as depicted in Figure 14.1.

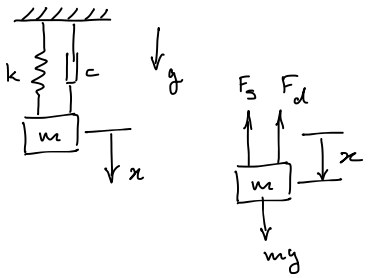


Figure 14.1: Mass-spring-damper system with FBD.

The equation of motion for this system, with linear stiffness k and linear viscous damping c, is

$$\begin{aligned} m\ddot{x} &= mg - F_{\mathrm{s}} - F_{\mathrm{d}} \\ &= mg - kx - c\dot{x}. \end{aligned} \tag{14.12E}$$

Substituting $\dot{x} = u$ results in

$$\dot{u} = g - \frac{k}{m}x - \frac{c}{m}u. \tag{14.13E}$$

Rewriting this to the standard first-order form of (14.11) leads to

$$\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ g \end{pmatrix}. \tag{14.14E}$$

Starting from an initial position x_0, and an initial velocity $\dot{x}_0$, the initial conditions on the state $\boldsymbol{y} = (x,\, u)^{\mathrm{T}}$ are

$$\boldsymbol{y}_0 = \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix}. \tag{14.15E}$$

14.1.3 Refining the Method: Heun

A somewhat more refined, or more accurate method than the Euler method, is Heun's method,

$$\boldsymbol{y}^*_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_n,\, \boldsymbol{y}_n), \tag{14.16}$$

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \tfrac{1}{2}h\left(\boldsymbol{f}(t_n,\, \boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1},\, \boldsymbol{y}^*_{n+1})\right). \tag{14.17}$$

This method first predicts the end value $\boldsymbol{y}^*_{n+1}$ by an Euler step and then uses this prediction to estimate the derivative at this end point in order to use the average of the derivative $\dot{\boldsymbol{y}}$. This way, it takes into account the changes of the derivative and results in more accurate results within one step. This method is illustrated in Figure 14.2.

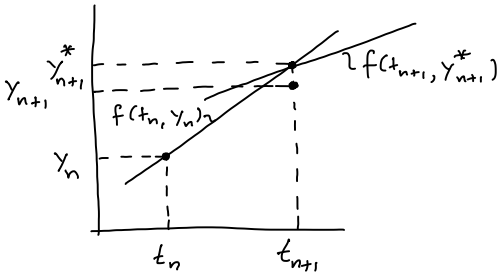


Figure 14.2: Illustration of Heun's integration method.

14.1.4 Accuracy and Error Propagation

Until now, we have seen two numerical integration methods, Euler and Heun, where we described Heun as more accurate than Euler. However, what do we mean by accuracy? Let us first examine the local *truncation*

error after one numerical integration step, which we call $\boldsymbol{\varepsilon}$. For the previously mentioned two methods these are

$$\text{Euler} \qquad \boldsymbol{\varepsilon} = \mathcal{O}(h^2), \tag{14.18}$$

$$\text{Heun} \qquad \boldsymbol{\varepsilon} = \mathcal{O}(h^3). \tag{14.19}$$

This follows directly from the truncation of the Taylor expansion of a function, as presented in (14.4). Yet, far more important than the local truncation error is the global truncation error E, which is the sum of all local truncation errors over a fixed time span. At a step size h, the total number of truncation errors is T/h, and therefore the global truncation error is

$$\boldsymbol{E} = \frac{T}{h}\boldsymbol{\varepsilon}. \tag{14.20}$$

Therefore, the global truncation errors for the two methods are

$$\text{Euler} \qquad \boldsymbol{E} = \mathcal{O}(h), \tag{14.21}$$

$$\text{Heun} \qquad \boldsymbol{E} = \mathcal{O}(h^2). \tag{14.22}$$

The local and global truncation errors as described above only provide an order of magnitude. In practice, we would like to estimate the global truncation error. For this purpose, we define the approximate result as

$$\boldsymbol{y} = \bar{\boldsymbol{y}} + \boldsymbol{C}_1 h^p, \tag{14.23}$$

with the (unknown) true answer $\bar{\boldsymbol{y}}$, the (unknown) coefficients $\boldsymbol{C}_1$, and the exponent p. This exponent depends on the method, for Euler $p = 1$ and for Heun $p = 2$.

To determine the unknowns, we integrate twice over a fixed timespan $t = [0, T]$, first with stepsize h to calculate the result $\boldsymbol{y}_h$, and next with step size $\frac{1}{2}h$ to calculate the result $\boldsymbol{y}_{h/2}$, and then we compare the results. The two results can be written as,

$$\boldsymbol{y}_h = \bar{\boldsymbol{y}} + \boldsymbol{C}_1 h^p, \tag{14.24}$$

$$\boldsymbol{y}_{h/2} = \bar{\boldsymbol{y}} + \boldsymbol{C}_1 \left(\tfrac{1}{2}h\right)^p. \tag{14.25}$$

These are two equations in terms of the two unknowns, the true value $\bar{\boldsymbol{y}}$ and the constant $\boldsymbol{C}_1$. Under the assumption that the constant $\boldsymbol{C}_1$ indeed remains constant, we can estimate the global error in the result $\boldsymbol{y}_{h/2}$ as

$$\boldsymbol{E} = \bar{\boldsymbol{y}} - \boldsymbol{y}_{h/2} = \frac{1}{2^p - 1}\left(\boldsymbol{y}_{h/2} - \boldsymbol{y}_h\right). \tag{14.26}$$

This is exactly what we do in an intuitive way: We compare the two results and estimate the difference between them to be the global error.

14.1.5 Stability of Numerical Integration Methods

So far, we have only looked at accuracy. What is far more important is the so-called *stability* of the method, which concerns the question how previously made errors propagate.

To investigate stability, we need a test equation. As such, we will use the linear differential equation,

$$\dot{\boldsymbol{y}} = \lambda \boldsymbol{y}. \tag{14.27}$$

The general solution to this equation is $\boldsymbol{y} = \hat{\boldsymbol{y}}_0 e^{\lambda t}$, which, with complex $\lambda = a + bi$, describes a complete family of solutions of the form

$$\begin{aligned} \boldsymbol{y} &= \hat{\boldsymbol{y}}_0 e^{(a+bi)t} \\ &= \hat{\boldsymbol{y}}_0 e^{at} e^{bit} \\ &= \hat{\boldsymbol{y}}_0 e^{at} \left(\cos(bt) + i \sin(bt)\right). \end{aligned} \tag{14.28}$$

The first part, e^{at}, describes growth or decay, and the second part, $(\cos(bt) + i \sin(bt))$, describes an oscillation. Indeed, these are the general solutions for a system of linear differential equations, as we will illustrate in the next example.

Example 14.2 This Example is about the mass-spring-damper system of Example 14.1 on page 363. It demonstrates the connection between the general solution of a set of linear differential equations and the test equation we will use for the numerical stability analysis.

The equation of motion for the system, with linear stiffness k and viscous damping c and an additional external force f, is

$$m\ddot{x} + c\dot{x} + kx = f. \tag{14.29E}$$

For the general homogeneous solution, we look at the homogeneous equation of motion, that is for $f = 0$,

$$m\ddot{x} + c\dot{x} + kx = 0. \tag{14.30E}$$

We introduce new constants, namely the eigenfrequency $\omega_0 = \sqrt{k/m}$ and the relative damping β, with $2\beta\omega_0 = c/m$, such that we can re-rewrite as

$$\ddot{x} + 2\beta\omega_0\dot{x} + \omega_0^2 x = 0. \tag{14.31E}$$

Next, we transform this single second-order differential equation into two

first-order equations, with $u = \dot{x}$,

$$\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\beta\omega_0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \tag{14.32E}$$

$$\dot{\boldsymbol{y}} = \mathbf{A}\boldsymbol{y}. \tag{14.33E}$$

Assuming a solution of the form $\boldsymbol{y} = \hat{\boldsymbol{y}}_0 e^{\lambda t}$ results in

$$\lambda\hat{\boldsymbol{y}}_0 e^{\lambda t} = \mathbf{A}\hat{\boldsymbol{y}}_0 e^{\lambda t} \tag{14.34E}$$

$$\Rightarrow \quad (\mathbf{A} - \lambda\mathbf{E})\,\hat{\boldsymbol{y}}_0 = \mathbf{0} \tag{14.35E}$$

$$\Rightarrow \quad \det(\mathbf{A} - \lambda\mathbf{E}) = 0. \tag{14.36E}$$

This system of equations only has a nonzero solution for $\hat{\boldsymbol{y}}_0$ if the determinant of the leading matrix, $\det(\mathbf{A} - \lambda\mathbf{E})$, is zero. Expressing this determinant in terms of a polynomial in λ results in the characteristic equation of the eigenvalue problem,

$$\det \begin{pmatrix} -\lambda & 1 \\ -\omega_0^2 & -\lambda - 2\beta\omega_0 \end{pmatrix} = \lambda^2 + 2\beta\omega_0\lambda + \omega_0^2 = 0. \tag{14.37E}$$

This polynomial equation can be solved for the unknown λ:

$$\lambda = -\beta\omega_0 \pm \sqrt{1-\beta^2}\omega_0 i. \tag{14.38E}$$

The real part $\mathrm{Re}(\lambda) = -\beta\omega_0$ reflects the damping, and the imaginary part $\mathrm{Im}(\lambda) = \sqrt{1-\beta^2}\omega_0$ the frequency of the oscillation.

We continue with the test equation $\dot{\boldsymbol{y}} = \lambda\boldsymbol{y}$. In this equation, we substitute the solution as the sum of the true value and the error, as in $\boldsymbol{y} = \bar{\boldsymbol{y}} + \boldsymbol{\varepsilon}$, with $\bar{\boldsymbol{y}}$ the true solution and $\boldsymbol{\varepsilon}$ the truncation error of the method:

$$\dot{\bar{\boldsymbol{y}}} + \dot{\boldsymbol{\varepsilon}} = \lambda\left(\bar{\boldsymbol{y}} + \boldsymbol{\varepsilon}\right). \tag{14.39}$$

Subtracting the true solution $\dot{\bar{\boldsymbol{y}}} = \lambda\bar{\boldsymbol{y}}$ from this equation leaves the error propagation equation,

$$\dot{\boldsymbol{\varepsilon}} = \lambda\boldsymbol{\varepsilon}. \tag{14.40}$$

By comparison, we conclude that the error propagation behaves just as the test equation. Therefore, we can conclude for the error propagation that

$\mathrm{Re}(\lambda) > 0$	stable only on a fixed interval,
$\mathrm{Re}(\lambda) < 0$	unconditionally stable.

Now let us apply this to the Euler method,

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n), \tag{14.41}$$

with the test equation

$$\boldsymbol{f}(t_n, \boldsymbol{y}_n) = \lambda \boldsymbol{y}_n. \tag{14.42}$$

Substitution of the test equation in the Euler step results in

$$\begin{aligned} \boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + h\lambda \boldsymbol{y}_n \\ &= (1 + h\lambda)\, \boldsymbol{y}_n \\ &= C(h\lambda)\boldsymbol{y}_n, \end{aligned} \tag{14.43}$$

where we have introduced the *amplification factor* $C(h\lambda)$, which for the Euler method is $C(h\lambda) = (1 + h\lambda)$.

For an error not to get amplified in the subsequent step, we need the amplification factor to be $C(h\lambda) < 1$. Since λ can be complex, $h\lambda = a + bi$, we have to look at $|C(h\lambda)| < 1$. Therefore the stability condition for an Euler step is,

$$|1 + h\lambda| < 1, \tag{14.44}$$

which defines the shaded area in the complex $h\lambda$ plane, see Figure 14.3. Thus, we have to choose a numerical integration stepsize h such that $h\lambda$ is within the shaded area.

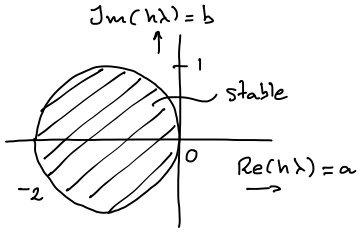


Figure 14.3: Stability area for the Euler method in the complex $h\lambda$ plane.

For a pure oscillatory system, like a simple undamped pendulum, the eigenvalue is purely imaginary and the Euler method will result in unstable solutions. In other words, the errors will be amplified and the solutions will diverge (see Figure 14.4).

Problem 14.1 Apply Euler's method to the differential equation of a simple undamped pendulum and see that even for small time steps h, the solution will diverge. Use for physical parameters a pendulum length of $l = 0.25\,\mathrm{m}$, in a gravity field with strength $g = 9.81\,\mathrm{N/kg}$, and take as initial

conditions a small angle of $\varphi_0 = 10^\circ$ from the vertical and zero speed.

For Heun's method, (14.16)(14.17), the stability conditions can be derived in the same way from the test equation:

$$\boldsymbol{y}^*_{n+1} = \boldsymbol{y}_n + h\lambda \boldsymbol{y}_n, \tag{14.45}$$

$$\begin{aligned} \boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \tfrac{1}{2}h\left(\lambda \boldsymbol{y}_n + \lambda\left(\boldsymbol{y}_n + h\lambda \boldsymbol{y}_n\right)\right) \\ &= \boldsymbol{y}_n + \tfrac{1}{2}h\lambda \boldsymbol{y}_n + \tfrac{1}{2}h\lambda \boldsymbol{y}_n + \tfrac{1}{2}h^2\lambda^2 \boldsymbol{y}_n \\ &= \left(1 + h\lambda + \tfrac{1}{2}(h\lambda)^2\right)\boldsymbol{y}_n \\ &= C(h\lambda)\boldsymbol{y}_n. \end{aligned} \tag{14.46}$$

We conclude that the amplification factor for Heun's method is $C(h\lambda) = \left(1 + h\lambda + \frac{1}{2}(h\lambda)^2\right)$, and that the stability condition for Heun's method becomes

$$\left|1 + (h\lambda) + \tfrac{1}{2}(h\lambda)^2\right| < 1, \tag{14.47}$$

which defines the shaded region in the complex $h\lambda$-plane, as displayed in Figure 14.5. Although the stability area for Heun's method is somewhat larger than for Euler's method, we see that, here also, purely oscillatory systems will result in unstable numerical solutions.

14.1.6 Runge–Kutta 4

A very popular and widespread numerical integration scheme is the so-called *Runge–Kutta 4*, or RK4, method. The history behind the method is quite interesting. Carl D. T. Runge [53] and Martin W. Kutta [52] were both German mathematicians. The method, which is a generalization of Euler's method by allowing a number of evaluations of the derivative within a step, was initially developed by Runge (1895), with further contributions by Heun (1900), and Kutta (1901), and first published in 1900 in Kutta's doctoral thesis entitled "Beiträge zur näherungsweisen Integration totaler Differentialgleichungen". Figure 14.6 shows their

Figure 14.4: The motion of an undamped pendulum solved by numerical integration using Euler's method.

portraits. Details on the history of the Runge-Kutta methods can be found in a 1996 paper by Butcher [7].

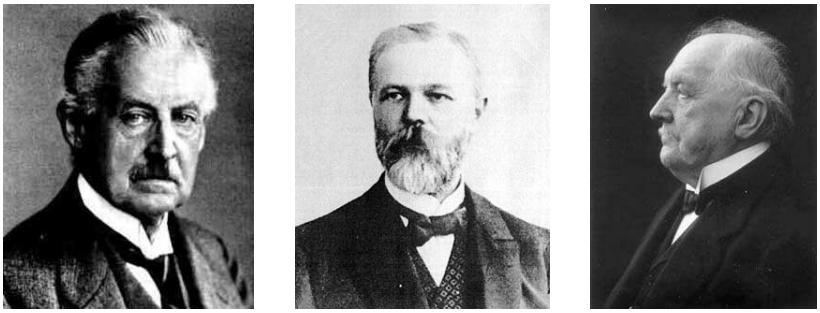


Figure 14.6: Carl David Tolme Runge (1856–1927), Karl Heun (1859–1929), and Martin Wilhelm Kutta (1867–1944), now famous for their RK4 numerical integration method.

The steps for the Runge–Kutta 4 method are

$$\boldsymbol{k}_1 = \boldsymbol{f}(t_n,\, \boldsymbol{y}_n), \tag{14.48}$$

$$\boldsymbol{k}_2 = \boldsymbol{f}(t_n + \tfrac{1}{2}h,\, \boldsymbol{y}_n + \tfrac{1}{2}h\boldsymbol{k}_1), \tag{14.49}$$

$$\boldsymbol{k}_3 = \boldsymbol{f}(t_n + \tfrac{1}{2}h,\, \boldsymbol{y}_n + \tfrac{1}{2}h\boldsymbol{k}_2), \tag{14.50}$$

$$\boldsymbol{k}_4 = \boldsymbol{f}(t_n + h,\, \boldsymbol{y}_n + h\boldsymbol{k}_3), \tag{14.51}$$

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \tfrac{1}{6}h\left(\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4\right). \tag{14.52}$$

The local truncation error of the RK4 method is of the order 5, $\boldsymbol{\varepsilon} = \mathcal{O}(h^5)$, and the amplification factor $C(h\lambda)$ is

$$C(h\lambda) = 1 + h\lambda + \tfrac{1}{2}\left(h\lambda\right)^2 + \tfrac{1}{6}\left(h\lambda\right)^3 + \tfrac{1}{24}\left(h\lambda\right)^4. \tag{14.53}$$

When plotting the stability region in the complex $h\lambda$-plane, as done in Figure 14.7, we notice that part of the imaginary axis is included in

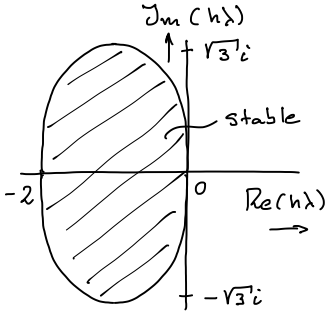


Figure 14.5: Stability area for Heun's method in the complex $h\lambda$ plane .

the stability region. We therefore conclude that the RK4 can be used for stable numerical integration of pure oscillatory systems.

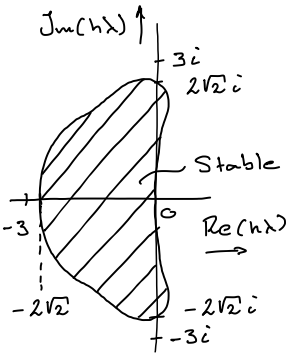


Figure 14.7: Runge-Kutta 4 stability area in the complex plane.

For a purely oscillatory system with eigenvalues $\lambda = \pm\omega_0 i$, the condition we have to set in the RK4 stepsize h is then

$$|h\lambda| < 2\sqrt{2} \qquad \Rightarrow \qquad h < \frac{2\sqrt{2}}{\omega_0}. \tag{14.54}$$

If we think in terms of period instead of frequency, with the period of vibration being $T_0 = \frac{2\pi}{\omega_0}$, the RK4 stepsize restriction is

$$h < \frac{2\sqrt{2}}{2\pi} T_0 \qquad \Rightarrow \qquad h < 0.45 T_0. \tag{14.55}$$

This means that already with three steps in one period, we can integrate a purely oscillatory system stably with the RK4 method. Of course, the result will not be very accurate, and for more accurate results we will have to refine the step size.

14.2 Method for Second-Order ODEs

The second-order differential equations we have derived until now usually have the following structure when reduced to a double set of first-order differential equations:

$$\dot{\boldsymbol{q}} = \boldsymbol{z}, \tag{14.56}$$

$$\dot{\boldsymbol{z}} = \boldsymbol{f}(t, \boldsymbol{q}, \boldsymbol{z}). \tag{14.57}$$

Here we see that the first set of differential equations, $\dot{\boldsymbol{q}} = \boldsymbol{z}$, are quite simple. The Euler 2 method exploits this and reads

$$\boldsymbol{k}_1 = \boldsymbol{f}(t_n + \tfrac{1}{2}h,\, \boldsymbol{q}_n + \tfrac{1}{2}h\boldsymbol{z}_n,\, \boldsymbol{z}_n), \tag{14.58}$$

$$\boldsymbol{q}_{n+1} = \boldsymbol{q}_n + h\boldsymbol{z}_n + \tfrac{1}{2}h^2\boldsymbol{k}_1, \tag{14.59}$$

$$\boldsymbol{z}_{n+1} = \boldsymbol{z}_n + h\boldsymbol{k}_1. \tag{14.60}$$

Note that the accelerations are evaluated at the midpoint $t_n + \frac{1}{2}h$. The estimated midpoint position is $\boldsymbol{q}_n + \frac{1}{2}h\boldsymbol{z}_n$, and the estimated midpoint velocity is just the velocity at the start of the step, $\boldsymbol{z}_n$, since we have no more information.

The local truncation error $\boldsymbol{\varepsilon}$ is of the order $\mathcal{O}(h^2)$, but if the accelerations $\boldsymbol{f}(t, \boldsymbol{q}, \boldsymbol{z})$ only weakly depend on velocities $\boldsymbol{z}$, then the order is one higher, namely $\mathcal{O}(h^3)$.

To investigate the stability of the Euler 2 method, we use a simple second-order differential equation as a test equation, namely that of a linear damped oscillatory system,

$$\ddot{\boldsymbol{q}} + 2\beta\omega_0\dot{\boldsymbol{q}} + \omega_0^2\boldsymbol{q} = \boldsymbol{0}. \tag{14.61}$$

With $\dot{\boldsymbol{q}} = \boldsymbol{z}$ we obtain

$$\begin{pmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{z}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\beta\omega_0 \end{pmatrix} \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{z} \end{pmatrix}. \tag{14.62}$$

Comparing this to the general test equation $\dot{\boldsymbol{y}} = \lambda\boldsymbol{y}$, then the eigenvalues λ are given by

$$\lambda_{1,2} = -\beta\omega_0 \pm \sqrt{1-\beta^2}\omega_0 i. \tag{14.63}$$

We can derive the amplification factor $C(h\lambda)$, which in this case will be an amplification matrix, by applying the Euler 2 method to the test equation. Next, we plot the stability region $|C(h\lambda)| < 1$ in the complex $h\lambda$ plane, where the result is shown in Figure 14.8. Clearly, the Euler 2 method is stable for pure oscillatory systems, as long as the step size is $h < 2/\mathrm{Im}(\lambda)$, which for slightly damped system is approximately $h < T_0/\pi$.

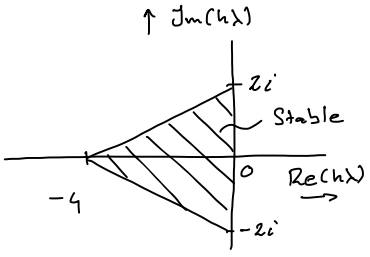


Figure 14.8: Euler 2 stability area in the complex plane.

The Euler 2 method is sometimes referred to as an "arcade" scheme, suitable for the entertainment industry, because it is moderately accurate but fast, and it only uses one function call per step. Note that this function call is at the midpoint $t_n + \frac{1}{2}h$, so after one step, we know the position $\boldsymbol{q}$ and the velocity $\boldsymbol{z}$ at the discrete points, but the acceleration $\ddot{\boldsymbol{q}}$ and all forces etc. are only known at the midpoints.

14.3 Error Estimates

14.3.1 Global Error Estimates

To get more accurate results, a first guess could be to just decrease the step size h. However, this only works until a certain step size. Until now, we only looked at local truncation errors of the type $\varepsilon = \mathcal{O}(h^{p+1})$, or, after $n = T/h$ steps, the global truncation error of $E = \mathcal{O}(h^p)$. However, since we work with finite word length in the computer to represent our numbers, we also have round-off errors which are of the type $\varepsilon = C_2$, which is some constant depending on the representation method of real numbers in a specific computer, and which is usually a very small number. For double precision in MATLAB$^{\text{TM}}$, C_2 is approximately 10^{-16}. However, because we take $n = T/h$ steps, the accumulated or global round-off error is

$$E = \frac{T}{h} C_2 = \mathcal{O}(h^{-1}). \tag{14.64}$$

In refining our step size, there will be a moment were this round-off error dominates over the truncation error. So, the total global error is something like the sum of the two,

$$E = C_1 h^p + \frac{C_2}{h}. \tag{14.65}$$

For the Euler method, this global error estimate is

$$E = C_1 h + \frac{C_2}{h}, \tag{14.66}$$

and for Heun's method the global error estimates is

$$E = C_1 h^2 + \frac{C_2}{h}. \tag{14.67}$$

As an example, we have plotted this global error estimate for Heun's method in the left of Figure 14.9. Because of the power laws, it is easier to plot these error estimates on a log-log scale, because then the curved lines become constant slopes, as shown in the right of Figure 14.9.

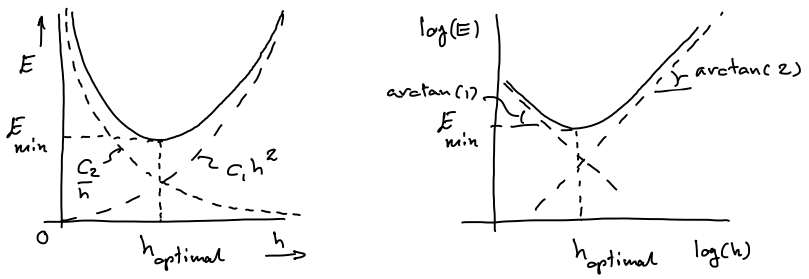


Figure 14.9: Global error estimate E as a function of the step size h for Heun's method, left on a linear scale, right on a log-log scale.

Because of the two power laws, there is clearly an optimal step size for which the global error is minimal. This minimum value depends on the problem at hand and the method used. The only way to decrease the minimal error is to use a higher-order method.

14.3.2 Error Estimates in Practice

In practice, we have to find a way to estimate the error in the solution. The numerical solution has the form,

$$\boldsymbol{y} = \bar{\boldsymbol{y}} + \boldsymbol{C}_1 h^p + \boldsymbol{C}_2 \frac{1}{h}, \tag{14.68}$$

with the true value $\bar{\boldsymbol{y}}$ (which we do not know), and the two approximately constant vectors $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$, which of course heavily depend on the problem at hand. Now we expect to be either in the region of the first step-size term or in the region of the second step-size term, that is to say,

$$\text{dominant truncation} \qquad \boldsymbol{y} \approx \bar{\boldsymbol{y}} + \boldsymbol{C}_1 h^p, \tag{14.69}$$

$$\text{dominant round-off} \qquad \boldsymbol{y} \approx \bar{\boldsymbol{y}} + \boldsymbol{C}_2 \frac{1}{h}. \tag{14.70}$$

To estimate the global error we proceed in the following manner: We take a number of successive steps with step sizes $h = T/2^n$ (with time period T and usually starting with $n = 6$ or $n = 8$) and calculate the absolute values of the difference between the two solutions as

$$\boldsymbol{D}_n = \left| \boldsymbol{y}_n - \boldsymbol{y}_{n-1} \right|, \tag{14.71}$$

where $\boldsymbol{y}_n$ is the result at $t = T$ for the numerical integration from $t = 0$ until $t = T$ with a step size of $h = T/2^n$. A plot of these differences on a log-log scale is shown in Figure 14.10.

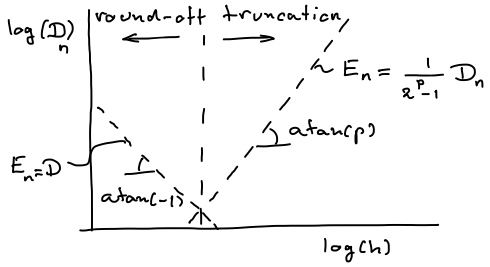


Figure 14.10: Absolute differences D_n between two successive solutions with step size $h = T/2^{n-1}$ and $h = T/2^n$ as a function of the step size h, on a log-log scale.

From the figure, we can first check the order of the applied method by looking at the slope of the line in the truncation region. Next, we can set a desired global error level and estimate for which step size we will be able to reach that (or will never reach it!).

From the figure, we can also see that what we intuitively would guess as the global error, the difference between two solutions, is actually not so bad, besides the constant factor $1/(2^p - 1)$. So, comparing the two solutions and assuming that the digits which do not change from one solution to the other are the accurate result, is not so far from the truth.

◯ **Example 14.3** Mario Gomes and Andy Ruina calculated error estimates for their ape brachiating analysis [26]. Ape brachiating is the way gibbons can move around in the forest, by swinging from, and flying between branches. This is an interesting motion, because it can be almost energy conservative, due to the hand contact having approximately no impact losses. An example of this ape brachiating can be seen in a video [76].

Gomes and Ruina developed a number of mechanical models with increasing complexity to simulate brachiating. In order to find cyclic motions, they had to solve a boundary value problem, and for that they needed accurate numerical results.

To study the accuracy of the various numerical methods applied, they made the following global error estimate, see Figure 14.11. It is obvious that the Euler method is rather inaccurate, and that higher-order methods are able to achieve a higher accuracy with a much larger step size. Note the slope of the various lines and that they show a good agreement with the theoretical order of the method. We also see that the maximum accuracy possible to achieve in this problem is of the order 10^{-13}.

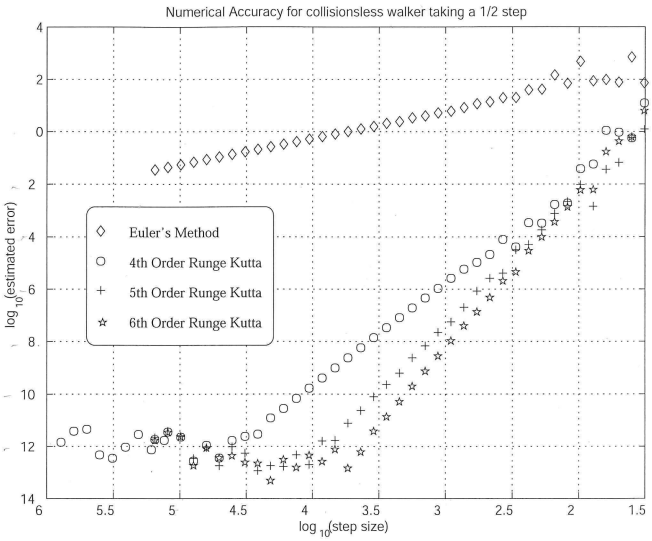


Figure 14.11: Global error estimates E in the cyclic solutions for the ape brachiating problem (= upside down collisionsless walker) for various numerical integration methods, as functions of the step size h on a log-log scale, by Mario Gomes and Andy Ruina (figure courtesy of Andy Ruina).

14.4 Implicit ODE Methods

Up to this point, we discussed explicit numerical integration methods. An example of this is the *forward Euler* method (14.41),

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n), \tag{14.72}$$

where the derivative is evaluated at the start of the step. In contrast, there are also methods which use the derivative at the *end* of the step.

As an example for such an *implicit numerical integration* method, we show here the *backward Euler* method,

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}). \tag{14.73}$$

This is clearly an implicit equation, because in order to determine $\boldsymbol{y}_{n+1}$, we need to evaluate the derivative at t_{n+1} which needs $\boldsymbol{y}_{n+1}$, which we do not know, so in general we will have to solve a set of nonlinear equations to determine $\boldsymbol{y}_{n+1}$.

Implicit numerical integration methods have the advantage that they exhibit a much larger stability region. In other words, with implicit methods, we are able to integrate the system which a much larger step size and still get stable results. This is particularly important for systems with large and small eigenvalues or high- and low-frequency components. We are usually only interested in the slowly varying solution, but the high-frequency components limit the maximal step size considerably. In this case we sometimes call the differential equations *stiff*.

We will demonstrate this stability behavior on the backward Euler method. The local truncation error of the backward Euler method is $\varepsilon = \mathcal{O}(h^2)$, which is the same as for the forward Euler. For the stability, we use again the same test equation as before,

$$\dot{\boldsymbol{y}} = \lambda \boldsymbol{y}. \tag{14.74}$$

Substitution in the backward Euler method results in

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\lambda \boldsymbol{y}_{n+1}. \tag{14.75}$$

So, for the error propagation we have the same, namely

$$\begin{aligned} \boldsymbol{\varepsilon}_{n+1} &= \boldsymbol{\varepsilon}_n + h\lambda \boldsymbol{\varepsilon}_{n+1} \\ &= \frac{1}{1-h\lambda} \boldsymbol{\varepsilon}_n. \end{aligned} \tag{14.76}$$

We conclude that the amplification factor is

$$C(h\lambda) = \frac{1}{1-h\lambda}. \tag{14.77}$$

Stability is expressed by $|C(h\lambda)|$, and with λ being complex, we obtain a stability region as in shown in Figure 14.12. This region is very large, and it includes purely oscillatory and even unstable systems ($\mathrm{Re}(\lambda) > 0$).

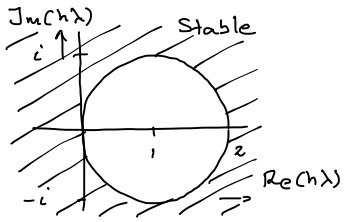


Figure 14.12: Backward Euler stability region.

To illustrate why an implicit method gives so much better results than an explicit method for systems having high-frequency components, we look at the exemplary system in Figure 14.13.

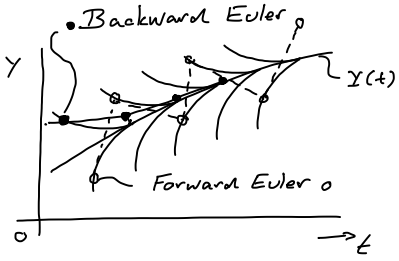


Figure 14.13: Comparison of the solutions of a differential equation with fast and slowly-varying components, found with the forward and backward Euler methods, from [21].

If we start the system with any initial value, it will quickly convergence to the slowly-varying solution, denoted by $Y(t)$. Then, in an explicit method, we evaluate the derivative on the high-frequency transient part of the solution and usually overshoot our estimate, and therefore generate unstable solutions. However, in an implicit method, we determine the derivative at the solution, which is close to the derivative on the slowly-varying solution, and the solutions will quickly converge to the slowly-varying solution.

Problem 14.2 Apply the implicit backward Euler numerical integration method (14.73) to the simple undamped pendulum from Problem 14.1, and show that the results are stable for arbitrary step sizes.

14.5 Butcher Arrays

Most of the numerical integration methods have been implemented in MATLAB$^{\text{TM}}$, for example in `ode45` or `ode23`, see also [66]. To understand which methods these implementations use, we first have to introduce a very clear and handy notation for numerical integration methods, introduced by J.C. Butcher, the so-called *Butcher arrays* [8].

The Butcher array is a general notation for an s-stage Runge-Kutta method for solving the ordinary differential equations $\dot{\boldsymbol{y}} = \boldsymbol{f}(t, \boldsymbol{y})$, where a numerical integration step has the form

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h \sum_{i=1}^{s} b_i \boldsymbol{k}_i, \tag{14.78}$$

with the derivatives defined as

$$\boldsymbol{k}_i = \boldsymbol{f}(t_n + c_i h,\, \boldsymbol{y}_n + h \sum_{j=1}^{s} a_{ij} \boldsymbol{k}_j). \tag{14.79}$$

The method is characterized by the number of stages, s, and the coefficients a_{ij}, b_i, and c_i. These coefficients are usually written in the following scheme called the Butcher array,

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array} \quad \text{or} \quad \begin{array}{c|c} \boldsymbol{c} & \mathbf{A} \\ \hline & \boldsymbol{b}^{\mathrm{T}} \end{array} \quad . \tag{14.80}$$

Example 14.4 For the well-known Runge–Kutta 4 method,

$$\boldsymbol{k}_1 = \boldsymbol{f}(t_n,\, \boldsymbol{y}_n), \tag{14.81E}$$
$$\boldsymbol{k}_2 = \boldsymbol{f}(t_n + \tfrac{1}{2}h,\, \boldsymbol{y}_n + \tfrac{1}{2}h\boldsymbol{k}_1), \tag{14.82E}$$
$$\boldsymbol{k}_3 = \boldsymbol{f}(t_n + \tfrac{1}{2}h,\, \boldsymbol{y}_n + \tfrac{1}{2}h\boldsymbol{k}_2), \tag{14.83E}$$
$$\boldsymbol{k}_4 = \boldsymbol{f}(t_n + h,\, \boldsymbol{y}_n + h\boldsymbol{k}_3), \tag{14.84E}$$
$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \tfrac{1}{6}h(\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4), \tag{14.85E}$$

the corresponding Butcher array takes on the form

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array} \quad . \tag{14.86E}$$

Several characteristics of a specific method can be identified from the Butcher array:

$a_{ij} = 0,$	$j \geq i$:	explicit method,
$a_{ij} = 0,$	$j > i$:	diagonally implicit,
	else :	implicit *sensu stricto*.

For convergence, the c_i coefficients have to fulfill $c_i = \sum_{j=1}^{s} a_{ij}$, and the b_i coefficients have to fulfill $\sum_{i=1}^{s} b_i = 1$.

Methods with a single stage, $s = 1$, are the Euler methods.

Example 14.5 The Butcher array for the forward Euler is

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array} \quad , \tag{14.87E}$$

whereas, for the backward Euler, the Butcher array is given by

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array} \quad . \tag{14.88E}$$

An explicit method with two stages, $s = 2$, is the well-known Heun method,

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \quad , \tag{14.89}$$

which is in fact a special case of a family of explicit 2-stage methods

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2\alpha} & \frac{1}{2\alpha} & 0 \\ \hline & 1-\alpha & \alpha \end{array} \quad , \tag{14.90}$$

which all have the property of a local truncation error of $\boldsymbol{\varepsilon} = \mathcal{O}(h^3)$.

Before we move on to the MATLAB$^{\text{TM}}$ implementations, we will briefly discuss the basis of these RK s-stage methods. The ODE $\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y})$ has a solution $\boldsymbol{y}(t)$, for which we can write down a Taylor expansion, as in

$$\boldsymbol{y}(t+h) = \boldsymbol{y}(t) + h\dot{\boldsymbol{y}} + \tfrac{1}{2}h^2\ddot{\boldsymbol{y}} + \cdots + \tfrac{1}{p!}h^p\boldsymbol{y}^{(p)} + \mathcal{O}(h^{p+1}), \tag{14.91}$$

and from which we can estimate the order of the truncation error of the method.

For example, in an explicit Runge–Kutta two-stage method, we hope for $\boldsymbol{\varepsilon} = \mathcal{O}(h^3)$, where the general form of such a method is

$$\boldsymbol{k}_1 = \boldsymbol{f}\left(\boldsymbol{y}_n\right), \tag{14.92}$$

$$\boldsymbol{k}_2 = \boldsymbol{f}\left(\boldsymbol{y}_n + ah\boldsymbol{k}_1\right), \tag{14.93}$$

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\left(c_1\boldsymbol{k}_1 + c_2\boldsymbol{k}_2\right). \tag{14.94}$$

The corresponding Butcher array is

$$\begin{array}{c|cc} 0 & 0 & 0 \\ b_2 & a_{21} & 0 \\ \hline & c_1 & c_2 \end{array} \quad \Rightarrow \quad \begin{array}{c|cc} 0 & 0 & 0 \\ a & a & 0 \\ \hline & c_1 & c_2 \end{array} \quad . \tag{14.95}$$

The coefficients a, c_1, and c_2 are yet unknown. We obtain these by starting off with the Taylor expansion

$$\boldsymbol{k}_2 = \boldsymbol{f}(\boldsymbol{y}_n) + ah\boldsymbol{k}_1\dot{\boldsymbol{f}}(\boldsymbol{y}_n) + \tfrac{1}{2}\left(ah\boldsymbol{k}_1\right)^2\ddot{\boldsymbol{f}}(\boldsymbol{y}_n) + \ldots, \tag{14.96}$$

which can be substituted to obtain the Taylor expansion of the solution

$$\begin{aligned} \boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + hc_1\boldsymbol{f}(\boldsymbol{y}_n) + hc_2\boldsymbol{f}(\boldsymbol{y}_n) + hc_2ah\boldsymbol{k}_1\dot{\boldsymbol{f}}(\boldsymbol{y}_n) + \mathcal{O}(h^3) \\ &= \boldsymbol{y}_n + h\left(c_1 + c_2\right)\boldsymbol{f}(\boldsymbol{y}_n) + h^2c_2a\boldsymbol{k}_1\dot{\boldsymbol{f}}(\boldsymbol{y}_n) + \mathcal{O}(h^3) \qquad (14.97) \\ &= \boldsymbol{y}_n + h\dot{\boldsymbol{y}}_n + \tfrac{1}{2}h^2\ddot{\boldsymbol{y}}_n + \mathcal{O}(h^3). \qquad (14.98) \end{aligned}$$

When we realize that

$$\ddot{\boldsymbol{y}} = \dot{\boldsymbol{f}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}\frac{\partial \boldsymbol{y}}{\partial t} = \boldsymbol{f}'\dot{\boldsymbol{y}} = \boldsymbol{f}'\boldsymbol{f} \tag{14.99}$$

we can compare (14.97) and (14.98) and see that

$$h\left(c_1 + c_2\right) = h \qquad \Rightarrow \qquad c_1 + c_2 = 1, \tag{14.100}$$

$$h^2c_2a = \tfrac{1}{2}h^2 \qquad \Rightarrow \qquad c_2a = \tfrac{1}{2}. \tag{14.101}$$

We now have two equations with three unknowns, which means that we have one free variable, for example the α from the 2-stage method shown in (14.90).

14.6 MATLAB Implementation of RK Methods

Now, we try and connect this theory to MATLAB$^{\text{TM}}$'s `ode23`. This is an implementation of an explicit Runge–Kutta three-stage method, which in general has the following non-zero Butcher array coefficients,

$$b_1 + b_2 + b_3 = 1, \tag{14.102}$$

$$b_2c_2 + b_3c_3 = \tfrac{1}{2}, \tag{14.103}$$

$$b_2c_2^2 + b_3c_3^2 = \tfrac{1}{3}, \tag{14.104}$$

$$b_3a_{32}a_{21} = \tfrac{1}{6}. \tag{14.105}$$

There are six coefficients and four equations, so we have two free parameters. Methods according to this scheme are third order, with truncation error of the order $\varepsilon = \mathcal{O}(h^4)$. An example of such an RK3 method is the following scheme,

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & 0 & \frac{2}{3} & 0 \\ \hline & \frac{1}{4} & 0 & \frac{3}{4} \end{array} . \tag{14.106}$$

However, the MATLAB$^{\text{TM}}$ `ode23` implementation uses the following Butcher array:

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{3}{4} & 0 & \frac{3}{4} & 0 \\ \hline & \frac{2}{9} & \frac{1}{3} & \frac{4}{9} \end{array} . \tag{14.107}$$

Written down as a regular numerical integration scheme, the method of MATLAB$^{\text{TM}}$'s `ode23` is given by

$$\boldsymbol{k}_1 = \boldsymbol{f}(t_n, \boldsymbol{y}_n), \tag{14.108}$$
$$\boldsymbol{k}_2 = \boldsymbol{f}(t_n + \tfrac{1}{2}h, \boldsymbol{y}_n + \tfrac{1}{2}h\boldsymbol{k}_1), \tag{14.109}$$
$$\boldsymbol{k}_3 = \boldsymbol{f}(t_n + \tfrac{3}{4}h, \boldsymbol{y}_n + \tfrac{3}{4}h\boldsymbol{k}_2), \tag{14.110}$$
$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \tfrac{1}{9}h(2\boldsymbol{k}_1 + 3\boldsymbol{k}_2 + 4\boldsymbol{k}_3). \tag{14.111}$$

Now we will examine the MATLAB$^{\text{TM}}$(R2015b) code of `ode23.m` and look at the following relevant parts:

```
1  function varargout = ode23(ode,tspan,y0,options,varargin)
2  %ODE23  Solve non-stiff differential equations, low order method.
3  %   [TOUT,YOUT] = ODE23(ODEFUN,TSPAN,Y0) with TSPAN = [T0 TFINAL] integrates

                                    ...

200  % Initialize method parameters.
201  pow = 1/3;
202  A = [1/2, 3/4, 1];
203  B = [
204      1/2          0               2/9
205      0            3/4             1/3
206      0            0               4/9
207      0            0               0
208      ];
209  E = [-5/72; 1/12; 1/9; -1/8];
210  f = zeros(neq,4,dataType);
211  hmin = 16*eps(t);
```

...

```
252    % LOOP FOR ADVANCING ONE STEP.
253    nofailed = true;                      % no failed attempts
254    while true
255      hB = h * B;
256      f(:,2) = feval(odeFcn,t+h*A(1),y+f*hB(:,1),odeArgs{:});
257      f(:,3) = feval(odeFcn,t+h*A(2),y+f*hB(:,2),odeArgs{:});
258      tnew = t + h*A(3);
259      if done
260        tnew = tfinal;   % Hit end point exactly.
261      end
262      h = tnew - t;      % Purify h.
263
264      ynew = y + f*hB(:,3);
265      f(:,4) = feval(odeFcn,tnew,ynew,odeArgs{:});
266      nfevals = nfevals + 3;
267
268      % Estimate the error.
269      NNrejectStep = false;
270      if normcontrol
271        normynew = norm(ynew);
272        errwt = max(max(normy,normynew),threshold);
273        err = absh * (norm(f * E) / errwt);
274        if nonNegative && (err <= rtol) && any(ynew(idxNonNegative)<0)
275          errNN = norm( max(0,-ynew(idxNonNegative)) ) / errwt ;
276          if errNN > rtol
277            err = errNN;
278            NNrejectStep = true;
279          end
280        end
281      else
282        err = absh * norm((f * E) ./ max(max(abs(y),abs(ynew)),threshold),inf);
283        if nonNegative && (err <= rtol) && any(ynew(idxNonNegative)<0)
284          errNN = norm( max(0,-ynew(idxNonNegative)) ./ thresholdNonNegative , inf);
285          if errNN > rtol
286            err = errNN;
287            NNrejectStep = true;
288          end
289        end
290      end
291
292      % Accept the solution only if the weighted error is no more than the
293      % tolerance rtol.  Estimate an h that will yield an error of rtol on
294      % the next step or the next try at taking this step, as the case may be,
295      % and use 0.8 of this value to avoid failures.
296      if err > rtol                         % Failed step
```

...

```
449  % Advance the integration one step.
450    t = tnew;
451    y = ynew;
452    if normcontrol
453      normy = normynew;
454    end
455    if NNreset_f4
456      % Used f4 for unperturbed solution to interpolate.
457      % Now reset f4 to move along constraint.
458      f(:,4) = feval(odeFcn,tnew,ynew,odeArgs{:});
459      nfevals = nfevals + 1;
460    end
461    f(:,1) = f(:,4);  % Already have f(tnew,ynew)
```

We recognize the following Butcher arrays in the code:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ & & & & \\ \boldsymbol{A}^{\mathrm{T}} & & \mathbf{B}^{\mathrm{T}} & & \\ & & & & \\ \hline & & \boldsymbol{E}^{\mathrm{T}} & & \end{array} = \begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{3}{4} & 0 & \frac{3}{4} & 0 & 0 \\ 1 & \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & 0 \\ \hline & -\frac{5}{72} & \frac{1}{12} & \frac{1}{9} & -\frac{1}{8} \end{array} . \tag{14.112}$$

The last column of $\mathbf{B}$ is used to calculate $\boldsymbol{y}_{n+1}$, as evident from code-line 264. Subsequently, in line 265, the estimated $\boldsymbol{y}_{n+1}$ is used to calculate a new approximation of the derivative $\boldsymbol{k}_4 = \mathbf{f}(t_{n+1}, \boldsymbol{y}_{n+1})$ at the end of the step. These derivatives are then used to estimate the local truncation error,

$$\boldsymbol{\varepsilon} = h\left(-\tfrac{5}{72}\boldsymbol{k}_1 + \tfrac{1}{12}\boldsymbol{k}_2 + \tfrac{1}{9}\boldsymbol{k}_3 - \tfrac{1}{8}\boldsymbol{k}_4\right), \tag{14.113}$$

which the code realizes by multiplication of the matrix of derivatives $\mathbf{f}$ with the vector E, as in

$$\boldsymbol{\varepsilon} = |h|\mathrm{norm}(\mathbf{f}\boldsymbol{E}). \tag{14.114}$$

Note that the absolute value of h is used, because the integration can go back in time, and we use the norm, because that term is multidimensional. If the step is accepted, then the values of the first column of $\mathbf{f}$, so $\boldsymbol{k}_1$, are assigned the same values as in the fourth column, $\boldsymbol{k}_4$. This is done in line 461. So, this appears a four-stage method, although in fact it is a three-stage method.

Finally, for dense output or event detection interpolation, we want to interpolate accurately between $\boldsymbol{y}_n$ and $\boldsymbol{y}_{n+1}$, without any extra function evaluations of the derivatives $\mathbf{f}$. To this end, `ode23` uses the interpolation function

```
1  function [yinterp,ypinterp] = ntrp23(tinterp,t,y,~,~,h,f,idxNonNegative)
2  %NTRP23  Interpolation helper function for ODE23.

                                    ...

22 BI = [
23      1       -4/3      5/9
24      0         1      -2/3
25      0        4/3     -8/9
26      0        -1        1
27      ];
28
29 s = (tinterp - t)/h;
30 yinterp = y(:,ones(size(tinterp))) + f*(h*BI)*cumprod([s;s;s]);
```

Apparently the interpolation has the form

$$\begin{aligned}\boldsymbol{y}_{n+\alpha} = \boldsymbol{y}_n &+ h\boldsymbol{k}_1\left(\alpha - \tfrac{4}{3}\alpha^2 + \tfrac{5}{9}\alpha^3\right)\\ &+ h\boldsymbol{k}_2\left(\alpha^2 - \tfrac{2}{3}\alpha^3\right)\\ &+ h\boldsymbol{k}_3\left(\tfrac{4}{3}\alpha^2 - \tfrac{8}{9}\alpha^3\right)\\ &+ h\boldsymbol{k}_4\left(-\alpha^2 + \alpha^3\right) \qquad \left(+\mathcal{O}(h^4)?\right),\end{aligned} \tag{14.115}$$

where the interpolation argument is $\alpha = (t - t_n)/(t_{n+1} - t_n)$.

We conclude that the MATLAB$^{\text{TM}}$ `ode23` function is an implementation of an explicit RK3 method with local truncation error control to determine an acceptable step size, together with an accurate interpolation method for intermediate values, as needed for, among other things, event detection.

Problem 14.3 Inspect MATLAB$^{\text{TM}}$'s `ode45`, and uncover the ingredients as done above for `ode23`. The relevant MATLAB$^{\text{TM}}$(R2015b) code for `ode45` is:

```
  1  function varargout = ode45(ode,tspan,y0,options,varargin)
  2  %ODE45  Solve non-stiff differential equations, medium order method.
  3  %   [TOUT,YOUT] = ODE45(ODEFUN,TSPAN,Y0) with TSPAN = [T0 TFINAL] integrates

                                        ...

201  % Initialize method parameters.
202  pow = 1/5;
203  A = [1/5, 3/10, 4/5, 8/9, 1, 1];
204  B = [
205      1/5          3/40     44/45    19372/6561       9017/3168        35/384
206      0            9/40     -56/15   -25360/2187      -355/33          0
207      0            0        32/9     64448/6561       46732/5247       500/1113
208      0            0        0        -212/729         49/176           125/192
209      0            0        0        0                -5103/18656      -2187/6784
210      0            0        0        0                0                11/84
211      0            0        0        0                0                0
212      ];
213  E = [71/57600; 0; -71/16695; 71/1920; -17253/339200; 22/525; -1/40];
214  f = zeros(neq,7,dataType);
215  hmin = 16*eps(t);

                                        ...

256    % LOOP FOR ADVANCING ONE STEP.
257    nofailed = true;                      % no failed attempts
258    while true
259      hA = h * A;
260      hB = h * B;
261      f(:,2) = feval(odeFcn,t+hA(1),y+f*hB(:,1),odeArgs{:});
262      f(:,3) = feval(odeFcn,t+hA(2),y+f*hB(:,2),odeArgs{:});
263      f(:,4) = feval(odeFcn,t+hA(3),y+f*hB(:,3),odeArgs{:});
264      f(:,5) = feval(odeFcn,t+hA(4),y+f*hB(:,4),odeArgs{:});
265      f(:,6) = feval(odeFcn,t+hA(5),y+f*hB(:,5),odeArgs{:});
266
267      tnew = t + hA(6);
268      if done
269        tnew = tfinal;   % Hit end point exactly.
270      end
271      h = tnew - t;      % Purify h.
272
273      ynew = y + f*hB(:,6);
274      f(:,7) = feval(odeFcn,tnew,ynew,odeArgs{:});
275      nfevals = nfevals + 6;
```

```
276
277      % Estimate the error.
278      NNrejectStep = false;
279      if normcontrol
280        normynew = norm(ynew);
281        errwt = max(max(normy,normynew),threshold);
282        err = absh * (norm(f * E) / errwt);
283        if nonNegative && (err <= rtol) && any(ynew(idxNonNegative)<0)
284          errNN = norm( max(0,-ynew(idxNonNegative)) ) / errwt ;
285          if errNN > rtol
286            err = errNN;
287            NNrejectStep = true;
288          end
289        end
290      else
291        err = absh * norm((f * E) ./ max(max(abs(y),abs(ynew)),threshold),inf);
292        if nonNegative && (err <= rtol) && any(ynew(idxNonNegative)<0)
293          errNN = norm( max(0,-ynew(idxNonNegative)) ./ thresholdNonNegative , inf)
                 ;
294          if errNN > rtol
295            err = errNN;
296            NNrejectStep = true;
297          end
298        end
299      end
300
301      % Accept the solution only if the weighted error is no more than the
302      % tolerance rtol.  Estimate an h that will yield an error of rtol on
303      % the next step or the next try at taking this step, as the case may be,
304      % and use 0.8 of this value to avoid failures.
305      if err > rtol                         % Failed step
```

Moreover, the function `ode45` uses the following auxiliary function:

```
 1  function [yinterp,ypinterp] = ntrp45(tinterp,t,y,~,~,h,f,idxNonNegative)
 2  %NTRP45  Interpolation helper function for ODE45.

                                        ...

22  BI = [
23       1        -183/64       37/12        -145/128
24       0           0            0             0
25       0        1500/371     -1000/159     1000/371
26       0        -125/32        125/12      -375/64
27       0        9477/3392    -729/106     25515/6784
28       0         -11/7         11/3         -55/28
29       0          3/2          -4            5/2
30       ];
31
32  s = (tinterp - t)/h;
33  yinterp = y(:,ones(size(tinterp))) + f*(h*BI)*cumprod([s;s;s;s]);
```

14.7 Problems

Problem 14.4 Make a global error estimate plot for the angle and angular speed at $T = 1\,\mathrm{s}$ for the simple pendulum from Problem 14.2 (simple pendulum, backward Euler method), by using a number of decreasing step sizes, and plot the results on a log-log scale.

Problem 14.5 Determine the motion of the double pendulum from Problem 10.2 by numerical integration of the equations of motion expressed in terms of independent generalized coordinates.

The initial conditions are defined by both bars up in the $+y$-direction, at

zero speed. We assume a gravitational field operating in the $+x$-direction with a field strength of $g = 9.81\,\mathrm{N/kg}$.

Determine the angle, in radians, of both bars with respect to the horizontal axis after 3.0 seconds with a maximal absolute error of 10^{-6} rad.

a. Determine the respective maximal constant step sizes for the following numerical integration methods:

   1) Euler.
   2) Heun.
   3) A Runge–Kutta third order.
   4) Classical Runge–Kutta fourth order.

   Use an error estimation method based on the method-inherent truncation error and the round-off error due to the finite precision, as explained in this chapter. For each angle, plot in one figure the $\log_{10}$ (estimated error) versus the $\log_{10}$ (step size) for all four methods. In some cases, it may not be feasible to determine such a maximum step size. Either the method used is unstable, or the step size becomes so small that it is impractical to reach the end point at $t = 3.0\,\mathrm{s}$. If so, do not spend too much time on that, but state your case clearly.

b. Finally use the three ODE solvers `ode23`, `ode45`, and `ode113` from MATLAB$^{\mathrm{TM}}$. Set the error tolerance `RelTol` and `AbsTol` such that you obtain the same final accuracy (global error) as above, and integrate the equations of motion for 3.0 seconds.

   1) Compare the angles of both bars at $t = 3.0\,\mathrm{s}$ with the results from above.
   2) Determine the average step size and the total number of function evaluations (calls to the differential equation $\boldsymbol{f}(t, \boldsymbol{y})$) as used in the three methods. Do these agree with your previous results?

Tabulate all your results with enough digits to show convergence (for instance 15) and discuss them. ►

15 Closed-Loop Kinematic Chains

We have seen, in the previous chapters, that using a minimal set of generalized coordinates to describe the kinematics has the advantage of a small set of equations to solve, and no drift in the constraints when generating a time series solution by numerical integration. However, this is complicated for systems that include *closed-loop kinematic chains*, that is, kinematic chains where each body is connected to at least two other bodies. For these systems, it is generally not possible to find the coordinate transformation which expresses the CoM coordinates of the rigid bodies as functions of the generalized coordinates explicitly in closed form. In this chapter, two methods to solve that problem are presented: Cutting the loops in combination with the coordinate projection method to suppress drift, and the coordinate partitioning method.

15.1 Coordinate Projection Method

For systems with closed-loop kinematic chains, finding closed-form expressions for the CoM coordinates of the rigid bodies in terms of the minimal set of generalized coordinates, $x_i = T_i(q_j)$, is generally not easy, and even for moderate-size systems impossible. However, finding this transformation for an open-loop kinematic chain is easy. The method we present here is first to make a number of cuts that open the closed kinematic loops. Next, we introduce as many generalized coordinates as there are cuts. Then, it is fairly easy to write the coordinate transformation in explicit form. By introducing the constraints to close the loops, we have created the familiar problem of drift in the constraints and in the coordinates during numerical integration. The *coordinate projection method* removes these drifts after each integration step.

15.1.1 Cutting the Kinematic Loops

In the following, we illustrate the first part of this method, cutting the loops, on the example of a four-bar linkage.

Example 15.1 The four-bar linkage as depicted in Figure 15.1 is a prominent example of a closed-loop kinematic chain.

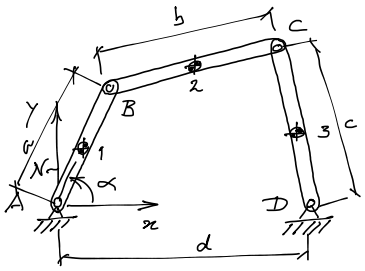


Figure 15.1: Four-bar linkage, an example of a closed-loop kinematic chain.

The system has one degree of freedom (check this), and we choose as generalized coordinate the angle α which bar 1 encloses with the horizontal axis, measured counter-clockwise. Even for such a simple system, writing down the expressions for the CoM coordinates of the three rigid bodies in terms of the generalized coordinate α is not trivial. We start from body 1,

$$x_1 = \tfrac{1}{2} a \cos\alpha, \tag{15.1E}$$

$$y_1 = \tfrac{1}{2} a \sin\alpha, \tag{15.2E}$$

$$\varphi_1 = \alpha, \tag{15.3E}$$

$$x_2 = a \cos\alpha + \dots . \tag{15.4E}$$

From here on it becomes hard, due to the closed-loop nature of the system. Example 15.2 illustrates the complex explicit expressions for the relation between two angles in the four-bar linkage.

Contrary to the ingenious math work in Example 15.2, our strategy will be: divide and conquer. By making appropriate cuts in the system, we can create a system with open-loop chains, at the cost of more generalized coordinates q_j, plus some constraints. For an open-loop chain, it is easy to write down $x_i = T_i(q_j)$, but the larger set of q_j is now dependent. This dependency is expressed by the joints we had to cut to open the loops. Therefore, we incorporate the constraints imposed by the cut joints as constraint equations. With these constraints, we can write down the DAEs and solve for the accelerations of the generalized coordinates q_j together with the Lagrange multipliers λ_c, which represent the joint forces at the cut joints.

We cut the system at joint D, as shown in Figure 15.2, and create an open-chain triple pendulum. This system has three degrees of freedom, and we choose the following generalized coordinates: $q_k = (\alpha, \beta, \gamma)$.

The coordinates of the CoM of the three rigid bodies in terms of these generalized coordinates, $x_i = T_i(q_j)$, can now easily be found as

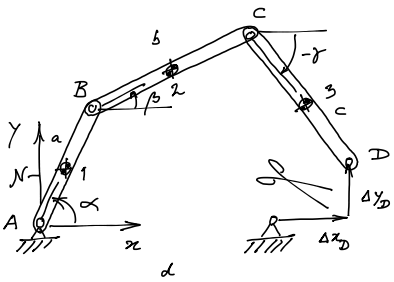


Figure 15.2: The four-bar linkage cut at the joint in D, to create an open-loop kinematic chain.

$$x_1 = \tfrac{1}{2}a\cos\alpha, \tag{15.5E}$$
$$y_1 = \tfrac{1}{2}a\sin\alpha, \tag{15.6E}$$
$$\varphi_1 = \alpha, \tag{15.7E}$$
$$x_2 = a\cos\alpha + \tfrac{1}{2}b\cos\beta, \tag{15.8E}$$
$$y_2 = a\sin\alpha + \tfrac{1}{2}b\sin\beta, \tag{15.9E}$$
$$\varphi_2 = \beta, \tag{15.10E}$$
$$x_3 = a\cos\alpha + b\cos\beta + \tfrac{1}{2}c\cos\gamma, \tag{15.11E}$$
$$y_3 = a\sin\alpha + b\sin\beta + \tfrac{1}{2}c\sin\gamma, \tag{15.12E}$$
$$\varphi_3 = \gamma. \tag{15.13E}$$

The two constraints needed to revert to the four-bar linkage, with the closed-loop kinematic chain, are the joint constraints in joint D,

$$C_1 = \Delta x_D = a\cos\alpha + b\cos\beta + c\cos\gamma - d = 0, \tag{15.14E}$$
$$C_2 = \Delta y_D = a\sin\alpha + b\sin\beta + c\sin\gamma = 0. \tag{15.15E}$$

Example 15.2 G. J. Talbourdet's 1941 article entitled "Part I - Analysis of Single Four-Bar Linkage" [72] contains a clever derivation of the relation between two angles in a four-bar linkage. Talbourdet worked in the Research Division of United Shoe Machinery Corporation where such a mechanism is used in every shoe-making machine. In fact, such a mechanism is also used in many other machines. The article nicely illustrates the complexity of the expressions for a (simple) closed-loop system. Its specific derivation is relevant when the four-bar linkage is used to control the angle of a driven crank by driving another crank as shown in Figure 15.3. In the remainder of this example, the equations in Talbourdet's article are reproduced here verbatim (exactly the same as in the paper):

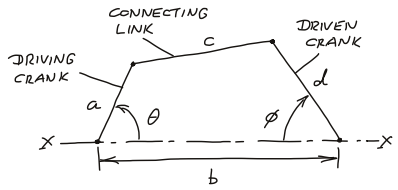


Figure 15.3: A four-bar linkage with in- and output cranks and angles (re-drawn).

θ: Angular displacement of driving crank.
ϕ: Angular displacement of driven crank.
$\frac{\mathrm{d}\theta}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$: Angular velocity and acceleration respectively of the driving crank.
$\frac{\mathrm{d}\phi}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}$: Angular velocity and acceleration respectively of the driven crank.

Calculate the values of the following factors from the geometry of the linkage:

$$K = a^2 + b^2 - c^2 + d^2, \tag{15.16E}$$

$$A = a\sin\theta, \tag{15.17E}$$

$$B = a^2 + b^2 - 2ab\cos\theta, \tag{15.18E}$$

$$D = K - 2ab\cos\theta, \tag{15.19E}$$

$$S = \sqrt{4d^2B - D^2}. \tag{15.20E}$$

Then,

$$\phi = \arctan\left(\frac{A}{b - a\cos\theta}\right) + \arccos\left(\frac{D}{2d\sqrt{B}}\right) \tag{15.21E}$$

and

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\left[\frac{a}{B}\left(b\cos\theta - a\right) - \frac{Ab}{S}\left(2 - \frac{D}{B}\right)\right] \tag{15.22E}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} =& \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\left[\frac{a}{B}\left(b\cos\theta - a\right) - \frac{Ab}{S}\left(2 - \frac{D}{B}\right)\right] \\ &+ \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\left[\frac{2A^2b^2}{BS}\left(1 - \frac{D}{B}\right)\right. \\ &\quad + \left(2 - \frac{D}{B}\right)\left(\frac{2A^2b^2\left(2d^2 - D\right)}{S^3} - \frac{ab\cos\theta}{S}\right) \\ &\quad \left. - \frac{Ab}{B}\left(1 + \frac{2a\left(b\cos\theta - a\right)}{B}\right)\right].\end{aligned} \tag{15.23E}$$

When the motion of the driving crank is uniform, $\mathrm{d}\theta/\mathrm{d}t = \omega$, and $\mathrm{d}^2\theta/\mathrm{d}t^2 = 0$, in which case we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega \left[\frac{a}{b} \left(b\cos\theta - a \right) - \frac{Ab}{S} \left(2 - \frac{D}{B} \right) \right] \tag{15.24E}$$

and

$$\begin{aligned} \frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = \omega^2 \Bigg[& \frac{2A^2b^2}{BS} \left(1 - \frac{D}{B} \right) \\ & + \left(2 - \frac{D}{B} \right) \left(\frac{2A^2b^2 \left(2d^2 - D \right)}{S^3} - \frac{ab\cos\theta}{S} \right) \\ & - \frac{Ab}{B} \left(1 + \frac{2a\left(b\cos\theta - a \right)}{B} \right) \Bigg]. \end{aligned} \tag{15.25E}$$

Problem 15.1 There is a typo in (15.24E) in the original manuscript, which we have repeated here. Use units to check consistency and find the typo. ▶

In general, after cutting the loops of a system, we are able to formulate the constrained equations of motion (DAEs) as

$$\begin{pmatrix} T_{i,l} M_{ij} T_{j,k} & C_{c,l} \\ C_{c,k} & 0_{cc} \end{pmatrix} \begin{pmatrix} \ddot{q}_k \\ \lambda_c \end{pmatrix} = \begin{pmatrix} Q_l + T_{i,l} \left(F_i - M_{ij} g_j \right) \\ -C_{c,kl} \dot{q}_k \dot{q}_l \end{pmatrix}. \tag{15.26}$$

From these equations, we can solve for the unknown accelerations, which then can be used to generate the motion of the system as a function of time by numerical integration.

Four-bar linkages are among the most widespread mechanisms in engineering applications. A possible reason is that such systems are relatively simple and able to generate non-linear input–ouput relations. In the 1940's, Hrones and Nelson created an atlas of four-bar linkage motions [31], which could be used by designers for mechanism synthesis (that is finding the mechanism that generates a specific motion). This 700-pages folio book from 1951 shows 500,000 solutions of the coupler curve for a general four-bar linkage. These motions were constructed with the help of a mechanism. Today, engineering students can generate these motions with a relatively simple computer program.

15.1.2 Projecting the Coordinates

From the constrained equations of motion (15.26), we can solve for the accelerations $\ddot{\boldsymbol{q}}$ and the Lagrange multipliers λ_c, and then numerically

integrate the state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ in one step from t_n to t_{n+1}, as in

$$\begin{pmatrix} \bar{\boldsymbol{q}} \\ \dot{\bar{\boldsymbol{q}}} \end{pmatrix}_{n+1} = \begin{pmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{pmatrix}_{n} + \int_{t_n}^{t_{n+1}} \begin{pmatrix} \dot{\boldsymbol{q}} \\ \ddot{\boldsymbol{q}} \end{pmatrix}_{n} \mathrm{d}t, \tag{15.27}$$

in order to obtain the motion in time. However, even when the state variables $\boldsymbol{q}_n$ and $\dot{\boldsymbol{q}}_n$ at step n fulfill the constraints, the approximate values, $\bar{\boldsymbol{q}}_{n+1}$ and $\dot{\bar{\boldsymbol{q}}}_{n+1}$, at step $n+1$, will in general *not* fulfill the constraints, and things will fly apart (as in the double pendulum Example 10.6, where the system flies apart, as shown in Figure 10.4). A way to visualize this is by drawing the constraint surface $\boldsymbol{C}(\boldsymbol{q}) = \boldsymbol{0}$. Starting from a point $\boldsymbol{q}_n$, which is on the constraint surface, we move to a point $\boldsymbol{q}_{n+1}$, which is not on the surface, as shown in Figure 15.4.

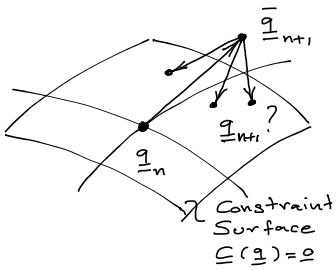


Figure 15.4: Visualization of a constraint surface $\boldsymbol{C}(\boldsymbol{q}) = \boldsymbol{0}$, and the result from a numerical integration step with a DAE approach, resulting in an approximate value after one step $\bar{\boldsymbol{q}}_{n+1}$, which is not on the constraint surface.

From the infinitely many ways of getting back onto the surface, we propose to use the one that has the minimal length, or formally, where the norm $\left\|\bar{\boldsymbol{q}}_{n+1} - \boldsymbol{q}_{n+1}\right\|_2$ is minimal, for all $\boldsymbol{q}_{n+1}$ that fulfill the constraints $\boldsymbol{C}(\boldsymbol{q}_{n+1}) = \boldsymbol{0}$. Such a problem is known in mathematics as a *non-linear constrained least-square problem*,

$$\left\|\bar{\boldsymbol{q}}_{n+1} - \boldsymbol{q}_{n+1}\right\|_2 = \min_{\boldsymbol{q}_{n+1}}, \quad \forall \quad \{\boldsymbol{q}_{n+1} | \boldsymbol{C}(\boldsymbol{q}_{n+1}) = \boldsymbol{0}\}. \tag{15.28}$$

One way of solving this is by using the so-called Gauss-Newton method.

15.1.3 Gauss-Newton Method

The Gauss-Newton method is an iterative method for solving a non-linear constrained least square problem (15.28). It starts by linearizing the system around $\bar{\boldsymbol{q}}_{n+1}$ by assuming small changes $\Delta\boldsymbol{q}_{n+1}$ relative to this point,

$$\boldsymbol{q}_{n+1} = \bar{\boldsymbol{q}}_{n+1} + \Delta\boldsymbol{q}_{n+1}. \tag{15.29}$$

Substitution in (15.28) results into

$$\left\|\Delta \boldsymbol{q}_{n+1}\right\|_2 = \min_{\boldsymbol{q}_{n+1}}, \quad \text{and} \quad \boldsymbol{C}(\bar{\boldsymbol{q}}_{n+1}) + \boldsymbol{C}_{,\boldsymbol{q}}\left(\bar{\boldsymbol{q}}_{n+1}\right)\Delta \boldsymbol{q}_{n+1} = \boldsymbol{0}. \tag{15.30}$$

This is a linear constrained optimization problem which can be solved by the Lagrange multiplier technique, where we introduce the Lagrange multipliers $\boldsymbol{\mu}$, as many as there are constraints. This results in the following (quite familiar) system of linear equations:

$$\begin{pmatrix} \mathbf{E} & \boldsymbol{C}_{,\boldsymbol{q}}^{\mathrm{T}} \\ \boldsymbol{C}_{,\boldsymbol{q}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \Delta \boldsymbol{q}_{n+1} \\ \boldsymbol{\mu} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ -\boldsymbol{C}\left(\bar{\boldsymbol{q}}_{n+1}\right) \end{pmatrix}, \tag{15.31}$$

or with the shorthand notations,

$$\boldsymbol{\Delta} = \Delta \boldsymbol{q}_{n+1}, \qquad \mathbf{C} = \boldsymbol{C}_{,\boldsymbol{q}}, \qquad \text{and} \qquad \boldsymbol{e} = -\boldsymbol{C}\left(\bar{\boldsymbol{q}}_{n+1}\right), \tag{15.32}$$

in a more compact form,

$$\begin{pmatrix} \mathbf{E} & \mathbf{C}^{\mathrm{T}} \\ \mathbf{C} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\Delta} \\ \boldsymbol{\mu} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{e} \end{pmatrix}. \tag{15.33}$$

Solving for $\boldsymbol{\Delta}$ and $\boldsymbol{\mu}$ is straightforward: First, we substitute the first row $\boldsymbol{\Delta} = -\mathbf{C}^{\mathrm{T}}\boldsymbol{\mu}$ in the second row $\mathbf{C}\boldsymbol{\Delta} = \boldsymbol{e}$, to get

$$-\mathbf{C}\mathbf{C}^{\mathrm{T}}\boldsymbol{\mu} = \boldsymbol{e}. \tag{15.34}$$

Then, although the matrix $\mathbf{C}$ is non-square and large, $m \times n$ with $n > m$, the matrix product $\mathbf{C}\mathbf{C}^{\mathrm{T}}$ is square with the smaller dimension $m \times m$. Generally, if the linearized constraints are independent, we can solve for $\boldsymbol{\mu}$ and Δ,

$$\boldsymbol{\mu} = -\left(\mathbf{C}\mathbf{C}^{\mathrm{T}}\right)^{-1}\boldsymbol{e}, \tag{15.35}$$

$$\boldsymbol{\Delta} = \mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\mathbf{C}^{\mathrm{T}}\right)^{-1}\boldsymbol{e}. \tag{15.36}$$

More in general, for such an undetermined linear system of equations $\mathbf{C}\boldsymbol{\Delta} = \boldsymbol{e}$ (fewer equations than unknown), the matrix

$$\mathbf{C}^{+} = \mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\mathbf{C}^{\mathrm{T}}\right)^{-1} \tag{15.37}$$

is called the *Moore–Penrose pseudo-inverse*, and it gives us the least square solution of the problem, as in

$$\Delta = \mathbf{C}^{+}\boldsymbol{e}. \tag{15.38}$$

This Moore–Penrose pseudo-inverse is implemented in MATLAB$^{\mathrm{TM}}$ as `pinv`. However, it is numerically preferable to implement $\mathbf{C}^{+}\boldsymbol{e}$ as `C\e`.

Example 15.3 To illustrate the Gauss-Newton method, let us look at a simple linear example. Assume we have three coordinates (q_1, q_2, q_3), and two linear constraints,

$$q_2 - q_1 = 0, \tag{15.39E}$$
$$q_2 - q_3 = 0, \tag{15.40E}$$

and we have an approximate solution $(q_1, q_2, q_3) = (0.9, 1, 1)$, which clearly does not fulfill the constraints. Then the constraint Jacobian matrix is

$$\mathbf{C} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \tag{15.41E}$$

and the approximate position,

$$\bar{\boldsymbol{q}} = \begin{pmatrix} 0.9 & 1 & 1 \end{pmatrix}^{\mathrm{T}} \tag{15.42E}$$

and consequently, the "errors"

$$\mathbf{C}\bar{\boldsymbol{q}} = \begin{pmatrix} 0.1 & 0 \end{pmatrix}^{\mathrm{T}}. \tag{15.43E}$$

Introducing the increments $\Delta\boldsymbol{q}$ and setting up the equations results in

$$\boldsymbol{q} = \bar{\boldsymbol{q}} + \Delta\boldsymbol{q} \tag{15.44E}$$
$$\Rightarrow \quad \mathbf{C}\bar{\boldsymbol{q}} + \mathbf{C}\Delta\boldsymbol{q} = \mathbf{0} \tag{15.45E}$$
$$\Rightarrow \quad \mathbf{C}\Delta\boldsymbol{q} = -\mathbf{C}\bar{\boldsymbol{q}} \tag{15.46E}$$
$$\Rightarrow \quad \Delta\boldsymbol{q} = \mathbf{C}^{+}\left(-\mathbf{C}\bar{\boldsymbol{q}}\right). \tag{15.47E}$$

With

$$\mathbf{C}\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{15.48E}$$

and the inverse

$$\left(\mathbf{C}\mathbf{C}^{\mathrm{T}}\right)^{-1} = \tfrac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \tag{15.49E}$$

we obtain the pseudo-inverse

$$\mathbf{C}^{+} = \mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\mathbf{C}^{\mathrm{T}}\right)^{-1} = \tfrac{1}{3}\begin{pmatrix} -2 & -1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix}, \tag{15.50E}$$

and the least square solution is

$$\Delta \boldsymbol{q} = \mathbf{C}^{+}\left(-\mathbf{C}\bar{\boldsymbol{q}}\right) = \tfrac{1}{3}\begin{pmatrix} -2 & -1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -0.1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{15} \\ -\frac{1}{30} \\ -\frac{1}{30} \end{pmatrix}, \tag{15.51E}$$

$$\|\Delta \boldsymbol{q}\| = \tfrac{1}{30}\sqrt{6}. \tag{15.52E}$$

The coordinates, which are now on the constraint surface and a minimal distance away from the approximate solution, are

$$\boldsymbol{q} = \begin{pmatrix} 0.9 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{15} \\ -\frac{1}{30} \\ -\frac{1}{30} \end{pmatrix} = \tfrac{1}{30}\begin{pmatrix} 29 \\ 29 \\ 29 \end{pmatrix}. \tag{15.53E}$$

We can check that this solution is indeed at the shortest distance from the approximate solution (0.9, 1, 1). One can easily come up with other solutions being also on the surface, such as (1, 1, 1), but these are always further away.

The above example only considered linear constraints, allowing for a one-step solution. In case of non-linear constraints, we have to linearize the constraint equations and repeat the process until the constraint errors are within bounds.

Finally, we present an algorithmic form of the Gauss-Newton iteration method for our type of problems. First, we project the generalized coordinates back onto the constraint surface,

```
iterat    = 0
tol       = 1e-12
q_n1      = qb_n1
maxiterat = 10
C = C(q_n1)
repeat
  Dq_n1  = -C,q^T((C,q*C,q^T)^(-1))*C
  q_n1   = q_n1 + Dq_n1
  C      = C(q_n1)
  iterat = iterat+1
until max(abs(C))<tol  or  iterat>maxiterat
```

Next, we determine the speeds which fulfill the constraints. These are linear equations, so we have a linear least squares problem, which we can solve in *one* step,

```
Cd     = C,q*qd_n1
Dqd_n1 = -C,q^T((C,q*C,q^T)^(-1))*Cd
```

```
set  qd_n1 = qd_n1 + Dqd_n1
```

The coordinate projection method is discussed in detail in the 1998 book "Numerical Methods in Multibody Dynamics" by Edda Eich-Soellner and Claus Führer [14]. Among other things, they prove that the error made by the coordinate projection is of the same order as the error made during the numerical integration step. Therefore, the error estimates for the numeric method at hand are also valid when correcting results via coordinate projection after each numerical integration step.

15.1.4 Coordinate Projection and Large Systems of DAEs

The method of coordinate projection can also be applied to the larger set of constrained equations of motion (11.11), namely the DAEs as derived in Chapter 11,

$$\begin{pmatrix} M_{ij} & C_{k,i} \\ C_{k,j} & 0_{kk} \end{pmatrix} \begin{pmatrix} \ddot{x}_j \\ \lambda_k \end{pmatrix} = \begin{pmatrix} F_i \\ -C_{k,ij}\dot{x}_i\dot{x}_j \end{pmatrix}. \tag{15.54}$$

After one numerical integration step, we have the approximate CoM coordinates of the rigid bodies, $x_j = \bar{x}_j$, which have to fulfill the usually non-linear constraints $C_k(x_j) = 0$. For the velocities, we have $\dot{x}_i = \dot{\bar{x}}_i$, which have to fulfill the linear constraints $C_{k,i}(x_j)\dot{x}_i = 0$. Both can easily be solved with the Gauss-Newton method from above. In fact, this way of establishing the equations of motion and solving the drift in the constraints is rather straightforward, and has found its way into various multibody dynamics computer codes.

15.2 Coordinate Partitioning Method

For systems with closed-loop kinematic chains, we are in general not able to determine closed-form expressions for the CoM coordinates of the rigid bodies, x_i, in terms of the generalized coordinates q_j, that is, the mapping $x_i = T_i(q_j)$. Instead of cutting the closed kinematic loops and introducing additional generalized coordinates together with constraints, an alternative is to use an incremental approach to determine this coordinate mapping. In this incremental approach, we are able to find the velocity and acceleration mappings, and from those we are then able to find the CoM coordinates.

First, we determine the number of degrees of freedom, say d. Next, we introduce generalized coordinates, q_j, as many as there are degrees

of freedom. Then, we we formulate d constraint equations which make the connection between the newly introduced generalized coordinates, q_j, and the CoM coordinates, x_i, of the rigid bodies. Usually this is very easy, since we are free in our choice of generalized coordinates. This extra set of d constraint equations are added to the existing set of m constraint equations. Then, the complete set of constraint equations is

$$C_k(x_i,\, q_j) = 0, \qquad k = 1 \dots m + d,\; i = 1 \dots n,\; j = 1 \dots d, \tag{15.55}$$

with n the total number of CoM coordinates of the rigid bodies, m the original number of constraints, and d the number of degrees of freedom. If all constraints are independent, then $d = n - m$. Although we are not able to find expressions for the coordinates $x_i = T_i(q_j)$, we can find the expressions for the velocities from (15.55). Differentiation with respect to time results in

$$C_{k,i}\dot{x}_i + C_{k,j}\dot{q}_j = 0. \tag{15.56}$$

The number of equations $m + d$ is equal to the number of CoM coordinates n, and the matrix $C_{k,i}$ is square, and if all constraints are independent, then we can solve for the CoM velocities in terms of the generalized velocities, as in

$$\dot{x}_i = -(C_{k,i})^{-1} C_{k,j}\dot{q}_j. \tag{15.57}$$

Even without the explicit expressions for the mapping $x_i = T_i(q_j)$, we have found the tangent mapping, or Jacobian matrix of the coordinate transformation $T_{i,j}$, from the constraints as

$$T_{i,j} = -(C_{k,i})^{-1} C_{k,j}. \tag{15.58}$$

This Jacobian can now be used to transform the equations of motion from the CoM coordinates to the generalized coordinates, according to (13.12) from Chapter 13, which we repeat here,

$$T_{i,j} M_{ik} T_{k,l} \ddot{q}_l = Q_j + T_{i,j} \left(F_i - M_{ik} g_k \right).$$

What remains are the expressions for the convective terms $g_k = T_{i,jl}\dot{q}_j\dot{q}_l$. These can be obtained in the same manner as above: First, we differentiate the velocities (15.56) to get the accelerations,

$$C_{k,i}\ddot{x}_i + C_{k,j}\ddot{q}_j + C_{k,rs}\dot{x}_r\dot{x}_s + 2C_{k,ru}\dot{x}_r\dot{q}_u + C_{k,uv}\dot{q}_u\dot{q}_v = 0. \tag{15.59}$$

Then, we solve for the accelerations of the CoM coordinates as

$$\ddot{x}_i = -(C_{k,i})^{-1} C_{k,j} \ddot{q}_j - (C_{k,i})^{-1} \left[C_{k,rs} \dot{x}_r \dot{x}_s + 2 C_{k,ru} \dot{x}_r \dot{q}_u + C_{k,uv} \dot{q}_u \dot{q}_v \right]. \tag{15.60}$$

Here, we recognize in the first part the Jacobian matrix (15.58), so we rewrite (15.60) as

$$\ddot{x}_i = T_{i,j} \ddot{q}_j + g_k, \tag{15.61}$$

while the second part are the convective terms,

$$g_k = -(C_{k,i})^{-1} \left[C_{k,rs} \dot{x}_r \dot{x}_s + 2 C_{k,ru} \dot{x}_r \dot{q}_u + C_{k,uv} \dot{q}_u \dot{q}_v \right]. \tag{15.62}$$

From the TMT transformed equations of motion (13.12), we can solve for the accelerations of the generalized coordinates $\ddot{q}_j$ and perform one numerical integration step from t_n to t_{n+1} to obtain the generalized coordinates and speeds in the next moment in time,

$$\begin{pmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{pmatrix}_{n+1} = \begin{pmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{pmatrix}_{n} + \int_{t_n}^{t_{n+1}} \begin{pmatrix} \dot{\boldsymbol{q}} \\ \ddot{\boldsymbol{q}} \end{pmatrix}_{n} \mathrm{d}t. \tag{15.63}$$

Opposed to generalized coordinates obtained in the cut-loop method from Section 15.1.1, the ones found here are drift-free.

The last step is to generate the new CoM coordinates. We estimate the new CoM positions and velocities at time step $n+1$, and any intermediate step, using the velocity and acceleration expressions (15.57) and (15.61), resulting in

$$\dot{\boldsymbol{x}}_n = \mathbf{T} \dot{\boldsymbol{q}}_n, \tag{15.64}$$

$$\ddot{\boldsymbol{x}}_n = \mathbf{T} \ddot{\boldsymbol{q}}_n + \boldsymbol{g}(\boldsymbol{q}_n, \dot{\boldsymbol{q}}_n). \tag{15.65}$$

Then the estimated CoM coordinates, $\bar{\boldsymbol{x}}_{n+1}$, at time step $n+1$ become

$$\bar{\boldsymbol{x}}_{n+1} = \boldsymbol{x}_n + h \dot{\boldsymbol{x}}_n + \tfrac{1}{2} h^2 \ddot{\boldsymbol{x}}_n. \tag{15.66}$$

These estimated coordinates can be corrected in precisely the same manner as used in the coordinate projection method from Section 15.1, with the augmented constraints (15.55) serving as the non-linear constraints. An algorithmic presentation of the Gauss-Newton iteration method for this non-linear constrained least squares problem is, as follows:

```
iterat    = 0
tol       = 1e-12
x_n1      = xb_n1
```

```
maxiterat = 10
C = C(x_n1,q_n1)
while (max(abs(C))>tol  and  iterat<maxiterat)
  Dx_n1  = -(C,x^(-1))*C
  x_n1   = x_n1 + Dx_n1
  C = C(x_n1,q_n1)
  iterat = iterat+1
end
```

Example 15.4 The four-bar slider mechanism, shown in Figure 15.5, has three rigid bodies and therefore nine CoM coordinates. The joints in A, B, C, and D result in seven joint constraint equations, which leaves nine minus seven is two degrees of freedom. For the two generalized coordinates q_j describing the degrees of freedom, we choose the relative rotation α between bar 1 and bar 2, and the horizontal distance s from joint A to slider joint D.

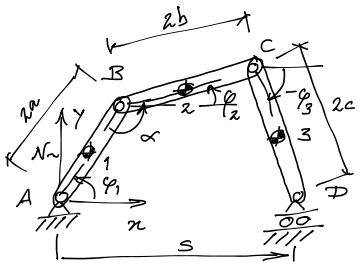


Figure 15.5: Four-bar slider mechanism.

The seven original joint constraint expressions, C_k, expressed in the CoM coordinates $x_i = (x_1,\, y_1,\, \varphi_1,\, x_2,\, y_2,\, \varphi_2,\, x_3,\, y_3,\, \varphi_3)^{\mathrm{T}}$ and some constants are

$$C_1 = x_1 - a\cos\varphi_1, \tag{15.67E}$$
$$C_2 = y_1 - a\sin\varphi_1, \tag{15.68E}$$
$$C_3 = (x_1 + a\cos\varphi_1) - (x_2 - b\cos\varphi_2)\,, \tag{15.69E}$$
$$C_4 = (y_1 + a\sin\varphi_1) - (y_2 - b\sin\varphi_2)\,, \tag{15.70E}$$
$$C_5 = (x_2 + b\cos\varphi_2) - (x_3 - c\cos\varphi_3)\,, \tag{15.71E}$$
$$C_6 = (y_2 + b\sin\varphi_2) - (y_3 - c\sin\varphi_3)\,, \tag{15.72E}$$
$$C_7 = y_3 + c\sin\varphi_3. \tag{15.73E}$$

The two additional constraints, which express the relations between the two generalized coordinates $q_j = (\alpha,\, s)^{\mathrm{T}}$, and the CoM coordinates x_i, are

$$C_8 = \alpha - (\varphi_2 - \varphi_1 + \pi), \tag{15.74E}$$
$$C_9 = s - (x_3 + c\cos\varphi_3)\,. \tag{15.75E}$$

Note that in this example, they are linear in the generalized coordinates, which is not necessary; they can be any algebraic expression.

Next, for an initial configuration, we calculate the mapping $T_{i,j}$, the CoM velocities $\dot{x}_i$, and the convective terms g_k. The initial configuration is shown in Figure 15.6.

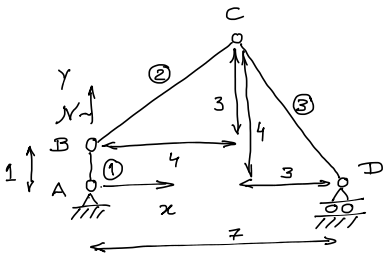


Figure 15.6: Four-bar slider mechanism, initial configuration.

The corresponding coordinates are (omitting units):

$$x_i = (0,\, 0.5,\, 1.5708,\, 2,\, 2.5,\, 0.6435,\, 5.5,\, 2,\, -0.9273)^{\mathrm{T}}, \tag{15.76E}$$

$$q_j = (2.2143,\, 7)^{\mathrm{T}}, \tag{15.77E}$$

and the length parameters $2a = 1$, $2b = 5$, and $2c = 5$. In this configuration, the transformation matrix $\mathbf{T}$, the Jacobian $T_{i,j}$ of the coordinate mapping T_i, can be calculated (15.58) as,

$$T_{i,j} = \begin{pmatrix} 0.4464, & 0.0536 \\ 0.0000, & 0.0000 \\ -0.8929, & -0.1071 \\ 0.7321, & 0.2679 \\ 0.2143, & -0.2143 \\ 0.1071, & -0.1071 \\ 0.2857, & 0.7143 \\ 0.2143, & -0.2143 \\ -0.1429, & 0.1429 \end{pmatrix}. \tag{15.78E}$$

If we assume an initial velocity of the degrees of freedom $\dot{q}_j = (1,\, -1)^{\mathrm{T}}$, then the initial velocities on the CoM coordinates can be calculated according to (15.57) as

$$\dot{x}_i = (0.3929,\, 0.0000,\, -0.7857,\, 0.4643,\, 0.4286,\, 0.2143,\, -0.4286,\, 0.4286,\, -0.2857)^{\mathrm{T}}, \tag{15.79E}$$

and with these velocities, the convective terms (15.62) are

$$g_k = (-0.0077,\, -0.3087,\, 0.0153,\, -0.1301,\, -0.6556,\, 0.0153,\, -0.1224,\, -0.3469,\, 0.1224)^{\mathrm{T}}. \tag{15.80E}$$

With these ingredients, the TMT transformed equations of motion (13.12) can be formulated and solved for the accelerations of the degrees of freedom $\ddot{q}_j$.

As a last step, we calculate the estimated CoM coordinates $\bar{x}_i$ after a time step of $h = 0.1$. Assume for now for simplicity that the generalized accelerations $\ddot{q}_j$ are all zero. Then the accelerations of the CoM coordinates $\ddot{x}_i$ are the convective accelerations g_k from (15.80E). The estimated CoM coordinates at $t + h$ according to (15.66) are then

$$\bar{x}_i = (0.0392,\, 0.4985,\, 1.4923,\, 2.0458,\, 2.5396,\, 0.6650,\, 5.4565,\, 2.0411,\, -0.9553)^{\mathrm{T}}, \tag{15.81E}$$

and the new values of the generalized coordinates are

$$q_j = (2.3143,\, 6.9)^{\mathrm{T}}. \tag{15.82E}$$

If we substitute these values in the constraint equations (15.67E) to (15.75E), we see that the constraint violations are all less than 10^{-4}. We conclude that the approximated position is already very close to the constraint surface. Applying only a few Gauss-Newton iterations will lead the coordinates back onto the surface within the desired error bound. As an exercise, check these results.

15.3 Problems

Problem 15.2 Consider the quick-return mechanism from Figure 15.7. The crank 2 drives rocker 4 via slider 3, and finally connecting bar 5 moves slider 6. The center of mass of link i is denoted by G_i.

The specification of the mechanism is as follows:
$O_2A = 0.2\,\mathrm{m}$, $O_4B = 0.7\,\mathrm{m}$, $BC = 0.6\,\mathrm{m}$, $O_4O_2 = 0.3\,\mathrm{m}$, $O_4G_4 = 0.4\,\mathrm{m}$, $BG_5 = 0.3\,\mathrm{m}$, $y_C = 0.9\,\mathrm{m}$, $m_3 = 0.5\,\mathrm{kg}$, $m_4 = 6\,\mathrm{kg}$, $m_5 = 4\,\mathrm{kg}$, $m_6 = 2\,\mathrm{kg}$, $I_4 = 10\,\mathrm{kg\,m^2}$, $I_5 = 6\,\mathrm{kg\,m^2}$, $F = 1\,\mathrm{kN}$, $T = 0$.

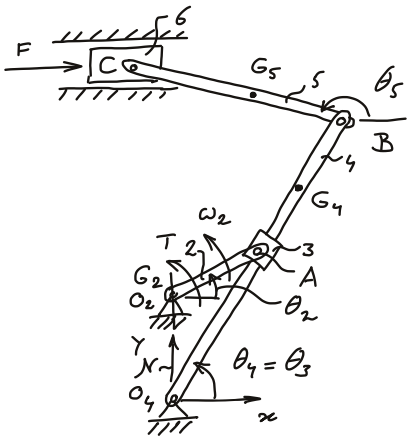


Figure 15.7: Quick-return mechanism.

The reduced mass moment of inertia at the balanced crank ($G_2 = O_2$) is $I_2 = 100\,\mathrm{kg\,m^2}$. The initial angular velocity of the crank is $\omega_2 = 75\,\mathrm{rpm}$, counter clockwise, at $\theta_2 = 0^\circ$. We assume no friction and zero gravity.

Determine the motion of the mechanism by numerical integration of the equations of motion. Derive these equations in terms of generalized coordinates and use the loop cutting method for this system with closed kinematic chain. Use the Coordinate Projection Method to enforce the constraints on the solution. Try not to derive the equations of motion in an explicit form, but evaluate your equations in a step-by-step manner.

Address the following questions:

a. Describe your algorithm in words and equations.

Show for two revolutions of the crank as a function of time:

b. - The angular speed of crank 2, rocker 4 and connecting bar 5.
c. - The sliding speed of slider 3 with respect to rocker 4.
   - The horizontal position, speed and acceleration of slider 6.
d. - The normal force exerted by the slider 3 on the rocker 4.
   - The normal force exerted by slider 6 on the ground.

Finally,

e. The mechanism's motion is clearly periodic, determine the period T in seconds $\pm 0.25\%$, and describe the method you used to find this period.

►

16 Rolling Contact, Non-Holonomic Constraints

In broadening our horizons we make use of different means of transport, evolving from tricycles, skates, scooters, bicycles, motor scooters and cars to trains, ships and planes. The motion of these types of systems can be investigated in an approximate way by a mechanical model having non-holonomic constraints. These constraints are formulated on the velocity level, and they express the conditions of vanishing slip (relative sliding velocity) at the contact points. A mechanical system with at least one non-holonomic constraint is called non-holonomic. The dynamics of mechanical systems with ideal holonomic constraints was almost completed by the publication of Lagrange's monumental Méchanique Analitique [38], while Heinrich Hertz [30] was the first to describe and name systems with non-holonomic constraints. Although the *principle of minimal action* fails for these systems, the principle of virtual power and the principle of d'Alembert can be applied. In their excellent book [48], Neĭmark and Fufaev treat the kinematics and dynamics of non-holonomic mechanical systems in great detail.

In this chapter, we will derive the equations of motion for planar mechanical system that consist of rigid bodies with constraints, in particular constraints on velocity, or non-holonomic constraints. We will also demonstrate how to obtain the motion in time by numerical integration, and how to avoid drift in the solutions by applying a coordinate projection method or Gauss–Newton process on the constraints.

16.1 Rolling Without Slip in "Planar" Systems

We will investigate the rolling constraints of a wheel that moves over the xy-plane, and where the plane of the wheel will always remain perpendicular to the the xy-plane; the straight upright configuration, see Figure 16.1. Such a wheel fits well into a 2-D MBS analysis.

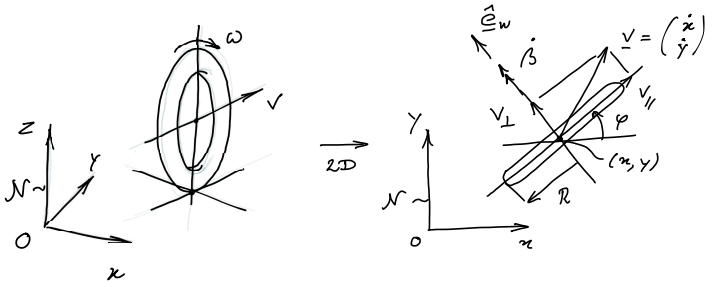


Figure 16.1: Straight upright (non-leaning) rolling wheel, in a perspective and a top-view projection.

16.1.1 Velocity Constraints for the Upright Wheel

The position and orientation of the wheel can be described by the CoM coordinates $(x,\, y)$ and the angle φ, the orientation of the wheel, or more specifically the rotation of the wheel about the z-axis. However, for a rolling wheel, also a second angle is relevant, namely the rotation about the wheel axis, which we will denote by the angle β. Thus, the coordinates which describe the position and orientation of the upright wheel are

$$x_i = (x,\, y,\, \varphi,\, \beta)^{\mathrm{T}}\,. \tag{16.1}$$

As a rigid body, this wheel can be positioned anywhere on the xy-plane at any orientation angle φ and at any angle β, so it has four degrees of freedom. However, for a purely rolling wheel, the velocity vector of the CoM of the wheel, $\boldsymbol{v} = (\dot{x},\, \dot{y})$, should always be in the plane of the wheel. In other words, there should be no sideslip, the orthogonal velocity component $v_\perp = 0$. A second condition for pure rolling is that there must be no slip in the longitudinal direction of the wheel. In other words, the forward velocity component $v_{\shortparallel}$ is a function of the rotational speed $\dot{\beta}$ and the radius R, via $v_{\shortparallel} = \dot{\beta} R$. These two conditions are non-integrable constraints on the level of the velocities, for which we use the terminology *non-holonomic constraints*, as introduced by Heinrich Hertz [30]. These non-holonomic constraints are

$$v_\perp = 0 \quad \Rightarrow \quad v_\perp = \boldsymbol{v} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} = -\dot{x}\sin\varphi + \dot{y}\cos\varphi, \tag{16.2}$$

$$v_{\shortparallel} = \dot{\beta} R \quad \Rightarrow \quad v_{\shortparallel} = \boldsymbol{v} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \dot{x}\cos\varphi + \dot{y}\sin\varphi = \dot{\beta} R, \tag{16.3}$$

where we have used the wheel axis direction vector $\hat{\boldsymbol{e}}_{\mathrm{w}} = (-\sin\varphi, \cos\varphi)$ and its perpendicular direction $(\cos\varphi, \sin\varphi)$, which is the driving direction of the wheel. Although we have four degrees of freedom in the configuration space, due to the two non-holonomic constraints, we only have two degrees of freedom in the velocity space. The number of velocity degrees of freedom is sometimes called the *mobility*; this system has a mobility of two.

16.1.2 A Skate with No Sideslip

A convenient way of thinking about pure rolling is to first write down an equation for the slip velocity at the contact. Then the constraint equation is just zero slip velocity. This gives us the opportunity either to introduce this as a constraint or to allow slip. If that slip generates forces, like from friction, we can add a constitutive equation for the behavior. However, such a constitutive equation can also come from a tire model, where lateral or sideslip generates a lateral force, the so-called cornering force of a tire.

Sideslip in accordance with (16.2) can be defined as

$$S_1 = -\dot{x}\sin\varphi + \dot{y}\cos\varphi. \tag{16.4}$$

For now, we focus on sideslip only and forget about any slip in the longitudinal direction. Then the wheel is like a skate on ice, see Figure 16.2. The coordinates for such a skate as a rigid body are $x_i = (x, y, \varphi)^{\mathrm{T}}$.

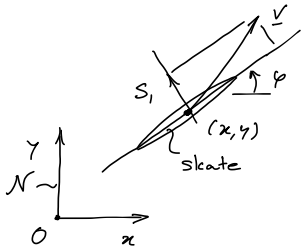


Figure 16.2: Skate with possible sideslip S_1.

The no-sideslip constraint can be expressed as zero sideslip,

$$S_1 = S_{1i}\dot{x}_i = 0 \tag{16.5}$$

$$= \begin{pmatrix} -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\varphi} \end{pmatrix} = 0. \tag{16.6}$$

To obtain the constraint on the accelerations $\ddot{x}_i$, we differentiate with respect to time,

$$\dot{S}_1 = S_{1i}\ddot{x}_i + S_{1i,j}\dot{x}_i\dot{x}_j = 0 \tag{16.7}$$

$$= \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\varphi} \end{pmatrix} + \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\varphi} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 0 & 0 & -\cos\varphi \\ 0 & 0 & -\sin\varphi \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\varphi} \end{pmatrix} = 0. \tag{16.8}$$

We see that the convective acceleration terms, $h_1 = S_{1i,j}\dot{x}_i\dot{x}_j$ in $\dot{S}_1 = S_{1i}\ddot{x}_i + h_1$, are

$$h_1 = -\dot{x}\dot{\varphi}\cos\varphi - \dot{y}\dot{\varphi}\sin\varphi. \tag{16.9}$$

Interestingly, the matrix $S_{1i,j}$ is not symmetric in i and j. This is in fact a hallmark of a non-holonomic constraint. In mathematical terms, it means that there exists no primitive of S_1, or, in other words, that S_1 is not a derivative of any function in terms of the coordinates.

To prove that S_1 cannot be integrated, we assume a function for $\int S_1 \,\mathrm{d}t = f(x_i)$ then

$$S_1 = f_{,i}\dot{x}_i, \tag{16.10}$$

$$\dot{S}_1 = f_{,i}\ddot{x}_i + f_{,ij}\dot{x}_i\dot{x}_j, \tag{16.11}$$

with

$$f_{,ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} = f_{,ji}. \tag{16.12}$$

Ergo, $f_{,ij}$ is symmetric, but clearly $S_{1i,j}$ is not. Therefore, S_1 has no primitive function (is not integrable), and is a non-holonomic constraint.

16.1.3 Pure Rolling in Forward Direction

Now, we investigate pure rolling in the forward or longitudinal direction. The constraint enforcing that is $v_{\shortparallel} = \dot{\beta}R$. Again, we rewrite this constraint as a zero-slip condition, by introducing the longitudinal slip

$$S_2 = \dot{x}\cos\varphi + \dot{y}\sin\varphi - \dot{\beta}R. \tag{16.13}$$

Pure rolling in the longitudinal direction is then expressed by $S_2 = 0$. As we are now dealing with a wheel, there are four coordinates for this rigid body, $x_i = (x,\, y,\, \varphi,\, \beta)^{\mathrm{T}}$.

The longitudinal slip constraint in a matrix–vector form is

$$S_2 = S_{2i}\dot{x}_i = 0 \tag{16.14}$$

$$= \begin{pmatrix}\cos\varphi & \sin\varphi & 0 & -R\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\varphi}\\ \dot{\beta}\end{pmatrix} = 0. \tag{16.15}$$

To obtain the constraint on the accelerations $\ddot{x}_i$, we differentiate again with respect to time,

$$\dot{S}_2 = S_{2i}\ddot{x}_i + S_{2i,j}\dot{x}_i\dot{x}_j = 0 \tag{16.16}$$

$$= \begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\\ -R\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{\varphi}\\ \ddot{\beta}\end{pmatrix} + \begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\varphi}\\ \dot{\beta}\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}0 & 0 & -\sin\varphi & 0\\ 0 & 0 & \cos\varphi & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\varphi}\\ \dot{\beta}\end{pmatrix} = 0. \tag{16.17}$$

The convective acceleration terms, $h_2 = S_{2i,j}\dot{x}_i\dot{x}_j$ in $\dot{S}_2 = S_{2i}\ddot{x}_i + h_2$, are

$$h_2 = -\dot{x}\dot{\varphi}\sin\varphi + \dot{y}\dot{\varphi}\cos\varphi. \tag{16.18}$$

Again we observe that zero longitudinal slip, $S_2 = 0$, is clearly a non-holonomic constraint for which no primitive (function in terms of the coordinates) exists. The condition for zero longitudinal slip can only be expressed in terms of the velocities (and the coordinates).

16.1.4 Upright Wheel Mass Matrix

To complete the treatment of the upright wheel, we have to supply the expressions for the inertia forces, $\boldsymbol{F}_{\mathrm{in}} = -\mathbf{M}\ddot{\boldsymbol{x}}$. In other words, we have to define the mass matrix $\mathbf{M}$ for such a rigid body.

The Newton–Euler equations of motion for such a wheel, as depicted in Figure 16.3, are

$$\left.\begin{aligned} m\ddot{x} &= \textstyle\sum F_x\\ m\ddot{y} &= \textstyle\sum F_y\\ I\ddot{\varphi} &= \textstyle\sum M_\varphi\\ J\ddot{\beta} &= \textstyle\sum M_\beta\end{aligned}\right\} \quad \Rightarrow \quad \begin{pmatrix}m & 0 & 0 & 0\\ 0 & m & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & J\end{pmatrix}\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{\varphi}\\ \ddot{\beta}\end{pmatrix} = \begin{pmatrix}\sum F_x\\ \sum F_y\\ \sum M_\varphi\\ \sum M_\beta\end{pmatrix}, \tag{16.19}$$

with the total mass m, the mass moment of inertia I at the CoM about the z-axis, and the mass moment of inertia J about the axis of the β-rotation of the wheel. For a thin homogeneous solid disk with mass m and radius R, we have $J = \frac{1}{2}mR^2$ and $I = \frac{1}{4}mR^2$.

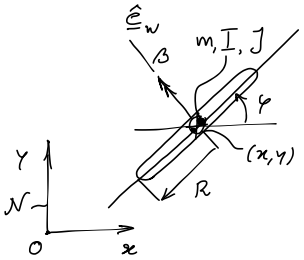


Figure 16.3: 2-D wheel, top view with parameters and coordinates.

16.2 Dynamics of Non-Holonomic Systems

We will derive the constrained equations of motion for a system with holonomic and non-holonomic constraints. The derivation is analogous to the previous derivations of DAEs with regular or holonomic constraints. We start by deriving the equations of motion for a system with only non-holonomic constraints and then extend that result to that for a system with both holonomic and non-holonomic constraints.

16.2.1 Systems with Non-Holonomic Constraints Only

Here we derive the constrained equations of motion for a system with only non-holonomic constraints, analogous to the previous derivations of DAEs with regular or holonomic constraints.

We start with the virtual power of the applied and d'Alembert inertia forces for the unconstrained system,

$$\delta P = (F_i - M_{ij}\ddot{x}_j)\,\delta\dot{x}_i. \tag{16.20}$$

Next, we add the condition that the virtual velocities should fulfill a total of h non-holonomic constraints,

$$S_k = S_{ki}\dot{x}_i = 0, \qquad k = 1 \ldots h. \tag{16.21}$$

Instead of real velocities $\dot{x}_i$, we use virtual velocities $\delta\dot{x}_i$:

$$\delta S_k = S_{ki}\delta\dot{x}_i = 0. \tag{16.22}$$

We include these constraints by means of the Lagrange multipliers λ_k in the virtual power expression,

$$\begin{aligned}\delta P^* &= (F_i - M_{ij}\ddot{x}_j)\delta\dot{x}_i - \lambda_k \delta S_k = 0 \\ &\qquad\qquad \forall\,\{\delta S_k \,|\, \delta S_k = S_{ki}\delta\dot{x}_i\} \\ &= (F_i - M_{ij}\ddot{x}_j)\delta\dot{x}_i - \lambda_k S_{ki}\delta\dot{x}_i = 0 \qquad \forall\,\delta\dot{x}_i. \end{aligned} \tag{16.23}$$

The term $\lambda_k \delta S_k$ has the dimension of power, and since δS_k is a slip or velocity, clearly λ_k is the constraint force.

For arbitrary $\delta\dot{x}_i$ we obtain i equations of motion,

$$F_i - M_{ij}\ddot{x}_j - \lambda_k S_{ki} = 0, \tag{16.24}$$

together with k constraints,

$$S_{ki}\dot{x}_i = 0. \tag{16.25}$$

Differentiating the constraints with respect to time lowers them to the level of acceleration constraints, such that they can be added to the equations of motion,

$$M_{ij}\ddot{x}_j + S_{ki}\lambda_k = F_i, \tag{16.26}$$

$$S_{kj}\ddot{x}_j = -S_{kj,l}\dot{x}_j\dot{x}_l. \tag{16.27}$$

These DAEs for a non-holonomic constrained system can be written in mixed index matrix–vector form as

$$\begin{pmatrix} M_{ij} & S_{ki} \\ S_{kj} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \ddot{x}_j \\ \lambda_k \end{pmatrix} = \begin{pmatrix} F_i \\ -S_{kj,l}\dot{x}_j\dot{x}_l \end{pmatrix}. \tag{16.28}$$

Given the positions, x_i, and velocities[1], $\dot{x}_i$, at time t, one can solve for the accelerations $\ddot{x}_i$ (and the constraint forces λ_k) at time t.

16.2.2 Systems with Mixed Constraints

The derivation of the equations of motion for a system with both holonomic and non-holonomic constraints is now easy, resulting in:

$$\begin{pmatrix} M_{ij} & C_{k,i} & S_{mi} \\ C_{k,j} & 0_{kk} & 0_{km} \\ S_{mj} & 0_{mi} & 0_{mm} \end{pmatrix} \begin{pmatrix} \ddot{x}_j \\ \lambda_k \\ \lambda_m \end{pmatrix} = \begin{pmatrix} F_i \\ -C_{k,jl}\dot{x}_j\dot{x}_l \\ -S_{mj,l}\dot{x}_j\dot{x}_l \end{pmatrix}, \tag{16.29}$$

together with the holonomic constraints $C_k(x_i) = 0$, and the non-holonomic constraints $S_{mj}(x_i)\dot{x}_j = 0$.

[1] Remember that the velocities are not independent: they must fulfill the non-holonomic constraints, $S_{ki}\dot{x}_i = 0$.

16.3 Degrees of Freedom of Non-Holonomic Systems

For non-holonomic systems, the term "degree of freedom" becomes confusing. For a mechanical systems with n CoM coordinates, m holonomic constraints, and h non-holonomic constraints, the number of free or unconstrained coordinates is equal to $n-m$, and the number of free or unconstrained velocities is $n-h-m$, because all holonomic constraints will impose the same number of constraints on the velocities. Clearly, we have more freedom in coordinates than in velocities. The degrees of freedom in the coordinates we will call the *kinematic degrees of freedom*, whereas the degrees of freedom in the velocities will be called the *velocity degrees of freedom*, or in short, the *mobility* number.

Example 16.1 A classic example of a non-holonomic system is the *Chaplygin sleigh*. This example was introduced by Sergey Alexeyevich Chaplygin (1869-1942), teacher of Leonid Ivanovitch Sedov (1907-1999). The Chaplygin sleigh is a rigid body which moves on a 2-D plane and has *one* non-holonomic constraint: a skate rigidly attached to the body (Figure 16.4).

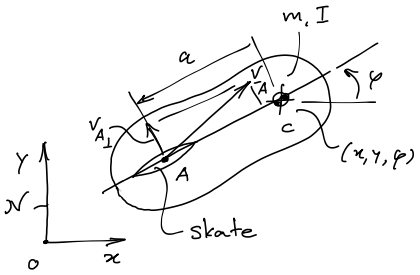


Figure 16.4: Chaplygin sleigh.

The skate in A can have no sideways velocity, and the non-holonomic constraint therefore is $v_{A\perp}=0$.

The velocity in A can be expressed in terms of the velocities at the center of mass as

$$\boldsymbol{v}_A = \boldsymbol{v}_C + \boldsymbol{\omega}\times\boldsymbol{r}_{A/C} \tag{16.30E}$$

$$= \begin{pmatrix}\dot{x}\\ \dot{y}\\ 0\end{pmatrix} + \begin{pmatrix}0\\ 0\\ \dot{\varphi}\end{pmatrix}\times\begin{pmatrix}-a\cos\varphi\\ -a\sin\varphi\\ 0\end{pmatrix} \tag{16.31E}$$

$$= \begin{pmatrix}\dot{x}+a\dot{\varphi}\sin\varphi\\ \dot{y}-a\dot{\varphi}\cos\varphi\\ 0\end{pmatrix}. \tag{16.32E}$$

The sideward velocity in A is

$$v_{A\perp} = \boldsymbol{v}_A \cdot \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \tag{16.33E}$$

$$= -\dot{x}\sin\varphi + \dot{y}\cos\varphi - a\dot{\varphi}. \tag{16.34E}$$

The zero-sideslip constraint and its derivative are therefore

$$S_1 = -\dot{x}\sin\varphi + \dot{y}\cos\varphi - a\dot{\varphi} = 0, \tag{16.35E}$$

$$\dot{S}_1 = -\ddot{x}\sin\varphi + \ddot{y}\cos\varphi - a\ddot{\varphi} + (-\dot{x}\dot{\varphi}\cos\varphi - \dot{y}\dot{\varphi}\sin\varphi) = 0, \tag{16.36E}$$

which, according to (16.28), results in the following constrained equations of motion for the Chaplygin sleigh,

$$\begin{pmatrix} m & 0 & 0 & -\sin\varphi \\ 0 & m & 0 & \cos\varphi \\ 0 & 0 & I & -a \\ -\sin\varphi & \cos\varphi & -a & 0 \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\varphi} \\ \lambda \end{pmatrix} = \begin{pmatrix} \sum F_x \\ \sum F_y \\ \sum M_C \\ \dot{x}\dot{\varphi}\cos\varphi + \dot{y}\dot{\varphi}\sin\varphi \end{pmatrix}, \tag{16.37E}$$

together with the velocity constraint,

$$-\dot{x}\sin\varphi + \dot{y}\cos\varphi - a\dot{\varphi} = 0. \tag{16.38E}$$

The Lagrange multiplier λ in this case is the lateral constraint force at the skate in A. With this one non-holonomic constraint, the Chaplygin sleigh has three kinematic degrees of freedom and two velocity degrees of freedom, or a mobility of two.

16.4 Integration of Non-Holonomic Systems

To obtain the motion in time of a non-holonomic constrained system, we use numerical integration. The constraints will introduce drift in the solution, which we will solve by a Gauss–Newton minimization process at each function evaluation of the constrained equations of motion. This is analogous to the coordinate projection method used for holonomic systems.

After one numerical integration step, we obtain a predicted solution for the coordinates $\bar{x}_i$ and the speeds $\dot{\bar{x}}_i$,

$$\bar{x}_i(t + \mathrm{d}t) = x_i(t) + \int \dot{x}_i(t)\,\mathrm{d}t, \tag{16.39}$$

$$\dot{\bar{x}}_i(t + \mathrm{d}t) = \dot{x}_i(t) + \int \ddot{x}_i(t)\,\mathrm{d}t. \tag{16.40}$$

If there are *no* holonomic constraints, then the predicted coordinates do not violate any constraints, but the predicted velocities $\dot{\tilde{x}}_i$ will in general not be on the constraint surface. In other words,

$$C_{ki}\dot{\tilde{x}}_i \neq 0. \tag{16.41}$$

This can easily be solved by a one-step Gauss–Newton process. It needs only one step, because we correct the speeds $\dot{x}_i$, and this minimization problem is *linear* in the speeds. The one-step Gauss–Newton method applied to the velocities $\dot{\tilde{x}}_i$ is

$$C_{ki}\Delta\dot{x}_i = -C_{kj}\dot{\tilde{x}}_j, \tag{16.42}$$

$$\Delta\dot{x}_i = -C_{li}\left(C_{kn}C_{ln}\right)^{-1}C_{kj}\dot{\tilde{x}}_j, \tag{16.43}$$

$$\dot{x}_i = \dot{\tilde{x}}_i + \Delta\dot{x}_i. \tag{16.44}$$

16.5 Problems

Problem 16.1

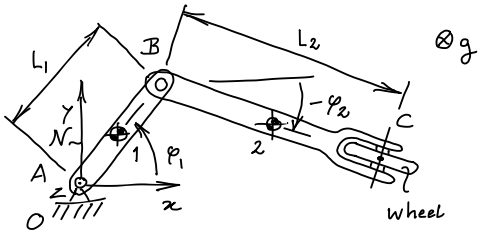


Figure 16.5: Model of a planimeter.

A model of a planimeter consists of two slender rigid bars and a wheel, see Figure 16.5. Bar 1 is connected to the ground by a rotational joint in A. Bar 2 is connected to bar 1 by a revolute joint in point B. A massless wheel is mounted on bar 2 in point C, and rolls over a horizontal xy-plane. We assume that there is no sideslip at the wheel and that it can freely rotate about its axis. Bar 1 is a very slender bar of length L_1 and with an evenly distributed mass m_1. Bar 2 is modeled as a slender bar of length L_2 and with evenly distributed mass m_2. We assume that the CoM of bar 1 and 2 are in the middle of the bars. The orientation of each bar is described by an angle φ_1 respectively φ_2, which are measured anti-clockwise relative to the x-axis. While the system is in motion an anti-clockwise torque M is applied to bar 1. Calculate at that moment in time the angular acceleration of bar 1, $\ddot{\varphi}_1$.[a]

The initial conditions for the system in motion and the value for the applied torque are: $\varphi_1 = \pi/4\,\text{rad}$, $\varphi_2 = -\pi/6\,\text{rad}$, $\dot{\varphi}_1 = 3\,\text{rad/s}$, and $M =$

12 N m.

For system parameters use: $L_1 = 0.6\,\text{m}$, $L_2 = 1.2\,\text{m}$, $m_1 = 3\,\text{kg}$, $m_2 = 6\,\text{kg}$, and gravity acts in the minus z-direction with $g = 9.81\,\text{N/kg}$. ▷

[a] *Hint*: This is like the double pendulum problem but now with an extra non-holonomic constraint: no sideslip at point C of bar 2. Determine the position of point C in terms of your two degrees of freedom $\boldsymbol{q}$, say this is the vector $\boldsymbol{r}_C(\boldsymbol{q})$. Next express the velocity of point C in terms of the degrees of freedom and their velocities, by differentiation of $\boldsymbol{r}_C$, say this is vector $\boldsymbol{v}_C = (\boldsymbol{r}_C)_{,\boldsymbol{q}}\dot{\boldsymbol{q}}$. The unit vector perpendicular to bar 2 is $\boldsymbol{e}_p = (-\sin(\varphi_2), \cos(\varphi_2))^{\mathrm{T}}$. Then, no sideslip at C is expressed by $s_1 = \boldsymbol{v}_C^{\mathrm{T}}\boldsymbol{e}_p = 0$. Differentiate this constraint equation and add it to the equations of motion. Note that bar 2 also has an initial angular velocity and you can find that from the initial conditions and the non-holonomic constraint. Check, that here $\dot{\varphi}_2 = -0.388\,229\,\text{rad/s}$.

Problem 16.2

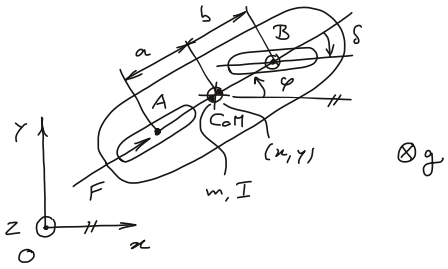


Figure 16.6: A "bicycle model" of a car.

A so-called "bicycle model" of a car, which moves over the xy-plane, consists of one rigid body with two wheels. The rear wheel is mounted in the body in A and the front wheel in B. The front wheel can be steered by an angle δ (clockwise), relative to the line AB. The CoM of the body is on the line segment AB a distance a from the rear wheel and a distance b from the front wheel. The rigid body has a mass m and a mass moment of inertia I with respect to the CoM. The wheels are massless. We assume perfect rolling (no slip) of the wheels. We only consider the zero lateral slip and do not consider the forward rolling condition. This means that each wheel can be modeled as one non-holonomic skate constraint.

The system is initially at rest with the CoM at the origin, where the vehicle is pointing in the x-direction: $(x, y, \varphi) = (0, 0, 0)$ and $(\dot{x}, \dot{y}, \dot{\varphi}) = (0, 0, 0)$. The front wheel is set at a constant steering angle $\delta = \pi/6\,\text{rad}$ (clockwise). Then a force with magnitude F is applied in A acting in the direction of the line AB. Calculate the acceleration of the CoM of the rigid body in the x-direction, $\ddot{x}$, just when the force is applied[a].

For parameter values use: $a = 0.2\,\text{m}$, $b = 0.3\,\text{m}$, $m = 3\,\text{kg}$, $I = 0.07\,\text{kg}\,\text{m}^2$, $F = 7\,\text{N}$, and gravity acts in the minus z-direction with $g = 9.81\,\text{N/kg}$.

▷

The system can be modeled by one rigid body with two non-holonomic constraints and the constrained equations of motion can be formulated in DAE form. The constraint equation for the wheel in A reads: $s_A = -\dot{x}_A \sin(\varphi) + \dot{y}_A \cos(\varphi) = 0$. Set up the DAE and check these two special cases. Case 1: $\delta = 0$ then $\ddot{x} = F/m$ and all other accelerations and Lagrange multipliers are zero. And for a non-zero steer angle, we expect $\ddot{x} < F/m$. Case 2: $\delta = \pi/2\,\text{rad}$ (90 degree), all accelerations are zero and the Lagrange multiplier in A is zero and the Lagrange multiplier in B should represent the reaction force F, keeping the system at rest.

[a] *Hint*:

Problem 16.3 The EzyRoller is advertised as the ultimate riding machine for kids. The machine is propelled by an oscillating motion of the steering assembly, which is operated by the feet. The video [74] demonstrates proper operation.

A 2-D mechanical model for this machine is shown in Figure 16.7, where gravity acts in the minus z-direction. The model consists of two rigid bodies connected by a revolute joint in B. Each body has a rolling contact: for body 1, the wheel in A, and for body 2, the wheel in C. The center of mass location is denoted by the numbers 1 and 2. With the help of this model, we wish to investigate the operation of this machine in a number of steps.

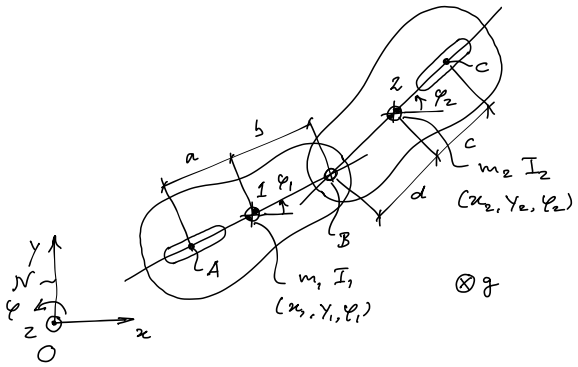


Figure 16.7: EzyRoller 2-D model.

The system consists of two rigid bodies and therefore has six coordinates. The system has two holonomic constraints, the revolute joint in B, and two non-holonomic constraints, the rolling contact conditions in A and C. Therefore, there are two constraints on the six coordinates, and four constraints on the six velocities. So, the number of degrees of freedom in the configuration space is four, but there are only two degrees of freedom in the velocity space.

You have to solve this problem by setting up the constrained equations of motion in DAE form and stabilizing the constraints by means of the coordinate projection method. Note that there are now more constraints on the velocities than on the coordinates.

Address the following questions:

a. Derive the equations of motion for this system in DAE form.
b. Formulate the coordinate projection method for the constraints on the coordinates and for the constraints on the velocities.
c. Implement the DAE together with the constraint stabilization in a MATLAB$^{\text{TM}}$ program. Do not try to form the equations of motion in an explicit form, but evaluate your equations in a step-by-step manner.
d. Determine the motion of the unpowered system by numerical integration of the equations of motion (use a fixed-step RK4 method). For dimensions of the EzyRoller, take: $a = 0.5\,\text{m}$, $b = 0.5\,\text{m}$, $c = 0.125\,\text{m}$, $d = 0.125\,\text{m}$, $m_1 = 1\,\text{kg}$, $I_1 = 0.1\,\text{kg}\,\text{m}^2$, $m_2 = 0$, $I_2 = 0$. For initial conditions, take some non-zero values for the coordinates and the speeds, and show that the system is moving properly.
e. Add an action–reaction torque in the revolute joint, such that it resembles an oscillating torque as applied by the rider on the steering assembly. For the torque, take the function $M(t) = M_0 \cos(\omega t)$, with $M_0 = 0.1\,\text{N}\,\text{m}$ and $\omega = \pi\,\text{rad/s}$. Start from rest with body 1 and body 2 aligned along the x-axis, $\varphi_1 = 0$, $\varphi_2 = \pi\,\text{rad}$ (point C between A and B). Determine the motion of the system by numerical integration of the constraint equations of motion for 100 seconds.
f. Plot the paths of points A and C.
g. Plot the linear and angular velocity of the CoMs of body 1.
h. Plot the applied torque $M(t)$ and the relative angular velocity $\omega = \dot{\varphi}_2 - \dot{\varphi}_1$, as a function of time in one figure and discuss the results. Pay extra attention to what happens in the first few seconds and when the transient has died out.
i. Determine the kinetic energy of the system and plot it as a function of time.
j. Determine the work done by the torque as a function of time and plot this as a function of time in the same figure as the kinetic energy of the system. What do you notice? Can you explain this?

►

17 From Two- to Three-Dimensional Space

As a gentle introduction to the dynamics of three-dimensional multibody systems, which consist of rigid bodies with constraints, we will start by revisiting the Newton–Euler equations of motion for a rigid body in a two- and three-dimensional world.

17.1 Newton–Euler in Two-Dimensional Space

Figure 17.1 shows a rigid body in a two-dimensional space.

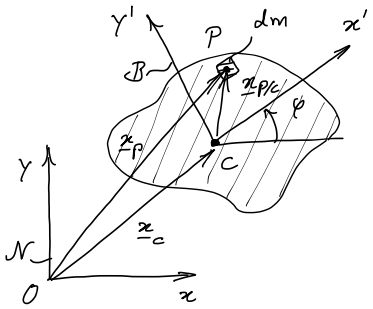


Figure 17.1: A rigid body in 2-D.

The Newton–Euler equations of motion for such a 2-D rigid body are

$$\sum F_{x_i} = m\ddot{x}_C, \tag{17.1}$$

$$\sum F_{y_i} = m\ddot{y}_C, \tag{17.2}$$

$$\sum M_C = I_C\ddot{\varphi}, \tag{17.3}$$

with the CoM position (x_C, y_C) and orientation φ of the rigid body with respect to an inertial triad $\mathcal{N}$, the mass m and mass moment of inertia at the CoM,

$$I_C = \int_m \left(x'^2 + y'^2\right) \mathrm{d}m. \tag{17.4}$$

Here, the integral over the mass is described in terms of a body-fixed triad $\mathcal{B}$, with coordinates $x'y'$ and with the origin in C. The resulting mass moment of inertia is invariant to rotation about the z-axis, so any coordinate system centered in C would give the same result.

The first two equations are the Newton part, which describe the motion of the CoM of the rigid body as if it were a point mass. The third equation is the Euler part, which describes the change in orientation due to the applied torques about the z-axis. These applied torques are usually the result of the applied forces times the distance of their lines of application to the CoM.

Before revising the three-dimensional mechanics of rigid bodies, we will trace back the roots of Euler's equation of motion and the concept of mass moment of inertia in the plane.

A way of understanding which role the mass moment of inertia plays in the equations is the following: We look at an infinitesimal mass element $\mathrm{d}m_i$ at position (x_i, y_i), as shown in Figure 17.2.

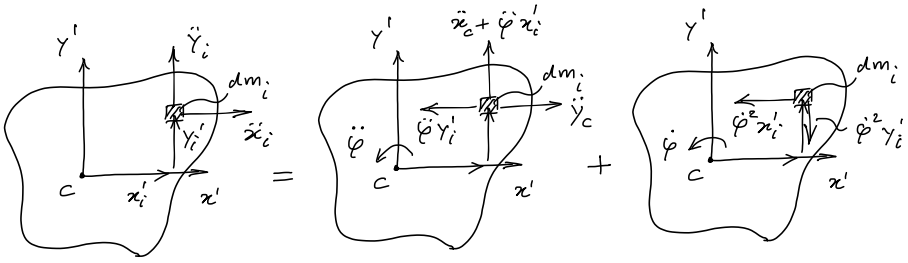


Figure 17.2: Accelerations of an infinitesimal mass element $\mathrm{d}m_i$ of a rigid body.

Because this is a rigid body, we can determine the position of the mass element, as well as its velocity and acceleration, as functions of the position of the CoM (x_C, y_C), the orientation of the body φ, and their time derivatives. The accelerations of the mass element, expressed in the inertial triad $\mathcal{N}$, are

$$\ddot{x}_i = \ddot{x}_C - y_i'\ddot{\varphi} - x_i'\dot{\varphi}^2, \tag{17.5}$$

$$\ddot{y}_i = \ddot{y}_C + x_i'\ddot{\varphi} - y_i'\dot{\varphi}^2. \tag{17.6}$$

Then Newton's equations of motion become

$$\sum F_x = \int \ddot{x}_i \,\mathrm{d}m_i = \int_V \left(\ddot{x}_C - y_i'\ddot{\varphi} - x_i'\dot{\varphi}^2\right) \,\mathrm{d}m_i, \tag{17.7}$$

$$\sum F_y = \int \ddot{y}_i \,\mathrm{d}m_i = \int_V \left(\ddot{y}_C - x_i'\ddot{\varphi} - y_i'\dot{\varphi}^2\right) \,\mathrm{d}m_i. \tag{17.8}$$

Since C is the center of mass, we know that static moments about the x' and y' axes are zero, that is $\int y'_i \,\mathrm{d}m = 0$ and $\int x'_i \,\mathrm{d}m = 0$, which is in fact the definition of the center of mass. Therefore, Newton's equations of motion for a rigid body are

$$\sum F_x = m\ddot{x}_C \quad \text{and} \quad \sum F_y = m\ddot{y}_C. \tag{17.9}$$

Next, we equate the sum of the applied torques to the inertial torques which arise from the torques from the inertia forces $\mathrm{d}F_x = \ddot{x}_i \,\mathrm{d}m_i$ and $\mathrm{d}F_y = \ddot{y}_i \,\mathrm{d}m_i$ at a distance $(x'_i,\, y'_i)$ from the CoM,

$$\sum M_C = \int -y'_i \mathrm{d}F_x + x'_i \mathrm{d}F_y \tag{17.10}$$

$$= \int -\left(\ddot{x}_C - y'_i\ddot{\varphi} - x'_i\dot{\varphi}^2\right) y'_i + \left(\ddot{y}_C + x'_i\ddot{\varphi} - y'_i\dot{\varphi}^2\right) x'_i \,\mathrm{d}m_i \tag{17.11}$$

$$= \int -\ddot{x}_C y'_i + \ddot{y}_C x'_i + \left(y_i'^2 + x_i'^2\right)\ddot{\varphi} + \left(x'_i y'_i - y'_i x'_i\right)\dot{\varphi}^2 \,\mathrm{d}m_i \tag{17.12}$$

$$= \int \left(x_i'^2 + y_i'^2\right) \mathrm{d}m_i \,\ddot{\varphi} = I_C\ddot{\varphi}, \tag{17.13}$$

This is Euler's equation of motion for a rigid body in 2-D, where the moment of inertia I_C (17.4) appears as an invariant quantity of the rigid body.

17.2 Newton–Euler in Three Dimensions

Transforming the Newton part from two to three dimensions is trivial, because for this purpose, the rigid body can be treated like a point mass, such that we simply add Newton's equation of motion for the third direction,

$$\left.\begin{array}{ll} \sum F_{x_i} & = m\ddot{x}_C \\ \sum F_{y_i} & = m\ddot{y}_C \\ \sum F_{z_i} & = m\ddot{z}_C \end{array}\right\} \quad \Rightarrow \quad \sum \boldsymbol{F}_i = m\ddot{\boldsymbol{x}}_C. \tag{17.14}$$

For Euler's equations of motion, which describe the rotational motions of the rigid body under the action of applied forces and moments, we have to consider the rigid body in three dimensions, as shown in Figure 17.3. We can proceed along similar lines as in the

two-dimensional case, but since we now have three angular speeds and three angular accelerations, the equations become somewhat more complex.

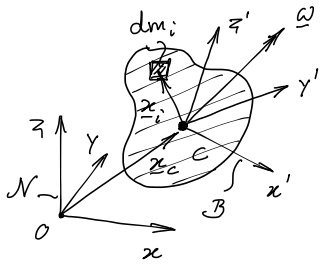


Figure 17.3: A rigid body in 3-D.

We start with the kinematics in three dimensions. The speed and acceleration of an infinitesimal mass element $\mathrm{d}m_i$ of the rigid body at the position $\boldsymbol{x}_i'$ are

$$\dot{\boldsymbol{x}}_i = \dot{\boldsymbol{x}}_C + \boldsymbol{\omega} \times \boldsymbol{x}_i', \tag{17.15}$$

$$\ddot{\boldsymbol{x}}_i = \ddot{\boldsymbol{x}}_C + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \boldsymbol{x}_i'\right) + \dot{\boldsymbol{\omega}} \times \boldsymbol{x}_i', \tag{17.16}$$

with the CoM coordinates $\boldsymbol{x}_C$, the angular speeds $\boldsymbol{\omega}$, and angular accelerations $\dot{\boldsymbol{\omega}}$.

Problem 17.1 Show that Newton's equations of motion for a rigid body (17.14) follow from integration of the infinitesimal mass element $\mathrm{d}m_i$ having a linear acceleration according to (17.16).

The torques of the applied forces and moments at the center of mass are equal to torques from the inertia forces $\mathrm{d}\boldsymbol{F}_i = \ddot{\boldsymbol{x}}_i \mathrm{d}m_i$ at a distance $\boldsymbol{x}_i$ from the center of mass, as in

$$\sum \boldsymbol{M}_C = \int \boldsymbol{x}_i' \times \mathrm{d}\boldsymbol{F}_i = \int \boldsymbol{x}_i' \times \ddot{\boldsymbol{x}}_i \, \mathrm{d}m_i. \tag{17.17}$$

Substitution of the accelerations from (17.16) and expanding terms, such that all static moments at the CoM cancel out, results finally in Euler's equations of motion for a rigid body,

$$\sum \boldsymbol{M}_C = \mathbf{I}_C \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_C \boldsymbol{\omega}), \tag{17.18}$$

with the angular velocity vector $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)^{\mathrm{T}}$ and the corresponding angular accelerations $\dot{\boldsymbol{\omega}} = (\dot{\omega}_x, \dot{\omega}_y, \dot{\omega}_z)^{\mathrm{T}}$, and the inertia tensor

$$\mathbf{I}_C = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix}. \tag{17.19}$$

The individual entries in the inertia tensor, where the negative of the off-diagonal terms are often called products of inertia, are now defined as

$$I_{xx} = \int \left(y^2 + z^2\right) \, \mathrm{d}m, \tag{17.20}$$

$$I_{yy} = \int \left(x^2 + z^2\right) \, \mathrm{d}m, \tag{17.21}$$

$$I_{zz} = \int \left(x^2 + y^2\right) \, \mathrm{d}m, \tag{17.22}$$

$$I_{xy} = I_{yx} = \int xy \, \mathrm{d}m, \tag{17.23}$$

$$I_{xz} = I_{zx} = \int xz \, \mathrm{d}m, \tag{17.24}$$

$$I_{yz} = I_{zy} = \int yz \, \mathrm{d}m. \tag{17.25}$$

Clearly, the mass moments of inertia matrix is symmetric.

We can transform this matrix to a diagonal matrix by means of the eigenvectors. The eigenvectors are the principal axes along which we have the eigenvalues as principal values of mass moments of inertia. The transformed inertia tensor then takes on the form

$$\bar{\mathbf{I}}_C = \begin{pmatrix} \bar{I}_{xx} & 0 & 0 \\ 0 & \bar{I}_{yy} & 0 \\ 0 & 0 & \bar{I}_{zz} \end{pmatrix}. \tag{17.26}$$

The principal axes and the principal values $\bar{I}_{xx}$, $\bar{I}_{yy}$, and $\bar{I}_{zz}$ describe the mass moment of inertia properties of the rigid body completely.

Problem 17.2 Derive the Euler equations of motion for a 3-D rigid body (17.18) from the expressions (17.17). It may be useful to use the tilde matrix notation for the cross product, as described in Appendix A.3, and/or to use symbolic computing (MAPLE$^{\text{TM}}$ or MATLAB$^{\text{TM}}$ symbolic toolbox).

Let us now have a closer look at the Euler equations (17.18). If the body rotates, then $\mathbf{I}_C$ is generally *not* constant. Therefore, it is often better to express Euler's equations of motion in a body-fixed triad $\mathcal{B}$:

$$\sum {}^{\mathcal{B}}\boldsymbol{M}_C = {}^{\mathcal{B}}\mathbf{I}_C \, {}^{\mathcal{B}}\dot{\boldsymbol{\omega}} + {}^{\mathcal{B}}\boldsymbol{\omega} \times \left({}^{\mathcal{B}}\mathbf{I}_C \, {}^{\mathcal{B}}\boldsymbol{\omega}\right), \tag{17.27}$$

with ${}^{\mathcal{B}}\boldsymbol{\omega}$ the angular velocity in the (local) body-fixed triad, ${}^{\mathcal{B}}\boldsymbol{M}_C$ the moment in the local body-fixed triad, and ${}^{\mathcal{B}}\mathbf{I}_C$ a new constant matrix.

It is usually convenient to align this triad with the principal axes of the body such that we get a diagonal inertia tensor,

$$ {}^{\mathcal{B}}\mathbf{I}_C = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}. \tag{17.28} $$

Summarizing, we derived the Newton–Euler equations of motion for a rigid body as shown in Figure 17.3, where the Euler part is expressed in a body-fixed triad $\mathcal{B}$, which is assumed to be aligned with the principal axes of the body:

$$ \sum F_x = m\ddot{x}_C, \tag{17.29} $$

$$ \sum F_y = m\ddot{y}_C, \tag{17.30} $$

$$ \sum F_z = m\ddot{z}_C, \tag{17.31} $$

$$ \sum M_x = I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\,\omega_y\omega_z, \tag{17.32} $$

$$ \sum M_y = I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\,\omega_z\omega_x, \tag{17.33} $$

$$ \sum M_z = I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\,\omega_x\omega_y. \tag{17.34} $$

The 3-D Euler equations look non-intuitive and complex, but 3-D motion of rigid bodies can indeed be non-intuitive and complex.

Example 17.1 Intriguing questions are: If you throw a rotating 3-D rigid body in space, will it keep rotating about the initial axis or will it start wobbling? Think of a satellite, a tennis racket, or a remote control thrown up into the air.

More formally, assume a three-dimensional rigid body with an initial angular velocity about one of the principal axes, say the first, $\boldsymbol{\omega} = (\omega,\, 0,\, 0)^{\mathrm{T}}$. Will the body continue moving this way after some small perturbation, or will it start rotating along one of the other axes and wobble? In other words, is the rotation about this axis stable?

We will analyze everything in a body-fixed triad $\mathcal{B}$ with principal axes at the CoM. We assume an initial angular velocity ${}^{\mathcal{B}}\boldsymbol{\omega}_0 = (\omega,\, 0,\, 0)^{\mathrm{T}}$. The inertia matrix is ${}^{\mathcal{B}}\mathbf{I}_C = \mathrm{diag}(A,\, B,\, C)$ and there are no applied torques, $\sum \boldsymbol{M} = \mathbf{0}$. The Euler equation of motion, expressed in the body-fixed triad are

$$ \sum {}^{\mathcal{B}}\boldsymbol{M}_C = {}^{\mathcal{B}}\mathbf{I}_C\, {}^{\mathcal{B}}\dot{\boldsymbol{\omega}} + {}^{\mathcal{B}}\boldsymbol{\omega} \times \left({}^{\mathcal{B}}\mathbf{I}_C\, {}^{\mathcal{B}}\boldsymbol{\omega} \right). \tag{17.35E} $$

We assume small changes from the steady motion, $\sum \boldsymbol{M} = \boldsymbol{M}_0 + \Delta\boldsymbol{M}$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \Delta\boldsymbol{\omega} \Rightarrow \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_0 + \Delta\dot{\boldsymbol{\omega}}$. Substitution in the Euler equations of

motion results in

$$\begin{aligned} \boldsymbol{M}_0 + \Delta\boldsymbol{M} &= \mathbf{I}_C\left(\dot{\boldsymbol{\omega}}_0 + \Delta\dot{\boldsymbol{\omega}}\right) &&+ (\boldsymbol{\omega}_0 + \Delta\boldsymbol{\omega}) \times \left(\mathbf{I}_C\left(\boldsymbol{\omega}_0 + \Delta\boldsymbol{\omega}\right)\right) && (17.36\text{E}) \\ &= \mathbf{I}_C\dot{\boldsymbol{\omega}}_0 + \mathbf{I}_C\Delta\dot{\boldsymbol{\omega}} &&+ \boldsymbol{\omega}_0 \times \mathbf{I}_C\boldsymbol{\omega}_0 + \boldsymbol{\omega}_0 \times \mathbf{I}_C\Delta\boldsymbol{\omega} \\ & && + \Delta\boldsymbol{\omega} \times \mathbf{I}_C\boldsymbol{\omega}_0 + \cancelto{\mathcal{O}(\Delta^2)}{\Delta\boldsymbol{\omega} \times \mathbf{I}_C\Delta\boldsymbol{\omega}}. && (17.37\text{E}) \end{aligned}$$

The applied moments are zero and the cross product of this angular velocity vector (with only one nonzero element) with an inertia-scaled version of itself is also zero. Therefore, the zero-order equations are

$$\cancelto{0}{\boldsymbol{M}_0} = \mathbf{I}_C\dot{\boldsymbol{\omega}}_0 + \cancelto{0}{\boldsymbol{\omega}_0 \times \mathbf{I}_C\boldsymbol{\omega}_0} \quad \Rightarrow \quad \dot{\boldsymbol{\omega}}_0 = \mathbf{0} \quad (\boldsymbol{\omega}_0 = \text{constant}), \tag{17.38E}$$

which are what we expected.

The first-order perturbation equations are

$$\Delta\boldsymbol{M} = \mathbf{I}_C\Delta\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}_0 \times \mathbf{I}_C\Delta\boldsymbol{\omega} + \Delta\boldsymbol{\omega} \times \mathbf{I}_C\boldsymbol{\omega}_0. \tag{17.39E}$$

The applied torques (which are zero) do not change, and therefore $\Delta\boldsymbol{M} = \mathbf{0}$. Next, we write for the change in the angular speed $\Delta\boldsymbol{\omega} = (\alpha,\, \beta,\, \gamma)$, and with this notation we expand the terms, resulting in

$$\mathbf{0} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} + \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} A\alpha \\ B\beta \\ C\gamma \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \times \begin{pmatrix} A\omega \\ 0 \\ 0 \end{pmatrix} \tag{17.40E}$$

$$= \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (A-C)\,\omega \\ 0 & (B-A)\,\omega & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}. \tag{17.41E}$$

This is a set of first-order differential equations for the changes in the angular speeds $\Delta\boldsymbol{\omega}$.

Assuming exponential solutions of the form $\Delta\boldsymbol{\omega} = \hat{\boldsymbol{a}}e^{\lambda t}$ leads to the following eigenvalue problem,

$$\begin{pmatrix} A\lambda & 0 & 0 \\ 0 & B\lambda & (A-C)\,\omega \\ 0 & (B-A)\,\omega & C\lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{\lambda t} = \mathbf{0}. \tag{17.42E}$$

For a nontrivial solution ($\hat{\boldsymbol{a}} \neq \mathbf{0}$), the determinant of the leading matrix should be zero, which results in the following characteristic equation:

$$A\lambda\left(BC\lambda^2 - (A-C)\,(B-A)\,\omega^2\right) = 0, \tag{17.43E}$$

from which we can solve the eigenvalues as

$$\Rightarrow \qquad \lambda_1 = 0 \qquad \text{and} \qquad \lambda_{2,3} = \pm\sqrt{\frac{(A-C)\,(B-A)}{BC}}\omega. \tag{17.44E}$$

If one of the eigenvalues is positive, $\lambda_2 > 0$, then the system is unstable, which is the case when,

$$A > C \qquad \text{and} \qquad B > A \qquad \Rightarrow \qquad B > A > C, \tag{17.45E}$$

$$\text{or} \qquad C > A \qquad \text{and} \qquad A > B \qquad \Rightarrow \qquad C > A > B. \tag{17.46E}$$

We conclude that rotation about the axis which has the *intermediate* mass moment of inertia is *unstable*. In other words, stable torque-free rotation of a rigid body in space is only possible around an axis which has either the smallest or the largest mass moment of inertia.

As an example of this, we look at a remote control (or a smartphone) of mass m and dimensions as drawn in Figure 17.4.

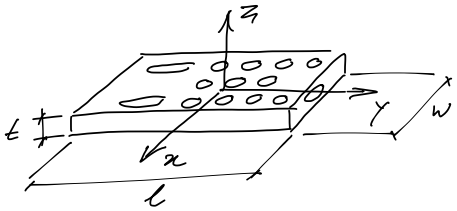


Figure 17.4: Remote control with its three principal axes.

The mass moments of inertia around the three principal axes are

$$I_{xx} = m\left(l^2 + t^2\right), \tag{17.47E}$$

$$I_{yy} = m\left(w^2 + t^2\right), \tag{17.48E}$$

$$I_{zz} = m\left(l^2 + w^2\right). \tag{17.49E}$$

With the length ratios as depicted in the figure, we see that we have $I_{zz} > I_{xx} > I_{yy}$. Thus, rotation around the remote control's y- and z-axis is stable. However, rotation about the x-axis is unstable (do try this at home by flipping up the remote in the air with a rotation about the x-axis and observe it wobbling, irrespective of how hard you try to initiate a pure rotation about the x-axis).

A similar analysis can be conducted for other objects with easily distinguishable mass moments of inertia around their principal axes.

17.3 Problems

Problem 17.3 Investigate stability of a standard tennis racket, such as the one shown in Figure 17.5. Show that torque-free rotation about the x-axis is unstable. Start by stating a relationship between the mass moments of inertia.

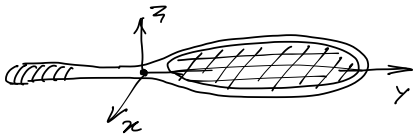


Figure 17.5: Tennis racket with its three principal axes.

Problem 17.4 In the Salyut-7 space station, circling around our globe, one of the astronauts has been working with a T-shaped tool. While he uses the tool to unscrew a bolt, he observes a very strange behavior, after having released the rotating tool. While rotating seemingly stably around the axis of operation, the tool suddenly flips 180°, and after a few seconds it flips back again. This flipping is repeated in a periodic manner. In this problem, we will investigate this behavior.

A model of the T-shaped tool, as shown in Figure 17.6, consists of two slender rigid bodies, where one end of body 2 is rigidly attached, at a right angle, to the center of body 1. At the CoM of the tool, a distance d from the rigid connection, a body-fixed coordinate system xyz with the origin in the CoM(cm) and directions of triad $\mathcal{B}$ is attached.

The length of body 1 is $l_1 = 0.08\,\text{m}$, and the length of body 2 is $l_2 = 0.04\,\text{m}$. The mass per unit length for both bodies is $\rho = 0.15\,\text{kg/m}$.

In calculating the mass moments of inertia, the diameter of the cross section may be neglected with respect to the length of the bodies. There is no gravity nor any other applied force or moment acting on the tool while it is in free flight.

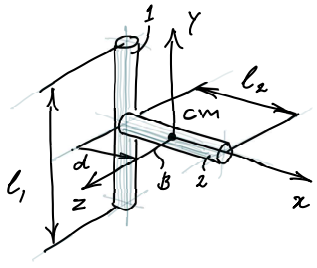


Figure 17.6: T-shaped tool from Salyut-7.

a. Determine the distance d of the center of mass with respect to the rigid connection.
b. Determine the values of the mass moments of inertia about the principal axis, I_{xx}, I_{yy}, and I_{zz}, expressed in the body-fixed triad $\mathcal{B}$.
c. Derive the equations of motion for the angular velocities ${}^{\mathcal{B}}\boldsymbol{\omega}$ of the system expressed in the body-fixed triad $\mathcal{B}$.
d. Determine the angular velocities of the tool system by numerical integration of the equations of motion for a period of 30 seconds. The initial angular velocity is such that the tool is mainly rotating about the x-axis with 60 rpm and that there is very small rotational speed about the z-axis, such that at $t = 0$ we start with ${}^{\mathcal{B}}\boldsymbol{\omega} = 2\pi(1, 0, 10^{-6})$ rad/s. Plot the three components $\omega_x, \omega_y, \omega_z$ of the angular velocity vector ${}^{\mathcal{B}}\boldsymbol{\omega}$, together with the magnitude of the angular velocity vector, $|{}^{\mathcal{B}}\boldsymbol{\omega}|$, as expressed in the body-fixed triad $\mathcal{B}$, functions of time in one figure. Discuss the results.
e. Plot the three components H_x, H_y, H_z of the angular momentum vector, ${}^{\mathcal{B}}\boldsymbol{H}$, together with the magnitude of the angular momentum vector, $|{}^{\mathcal{B}}\boldsymbol{H}|$, as expressed in the body-fixed triad $\mathcal{B}$, as functions of time in one figure. Discuss the results.
f. Determine the kinetic energy of the system and plot it as a function of time, and discuss the result.
g. Plot the absolute values of the angular velocity component about the body-fixed z-axis, $|\omega_z|$, as a function of time in a semi-logarithmic plot, where you use a logarithmic scale for the omega-axis and a linear scale for the time axis. Use the plot to determine the coefficients of the exponential growth and decay.
h. Use the stability analysis from Example 17.1 to determine the eigenvalues of the small motions of the tool spinning about its principal axis, and compare these to the values as found above.

▶

18 Rotation Matrix, Euler Angles, Quaternions

The extension from two-dimensional to three-dimensional analysis of multibody systems involves a detailed description of the kinematics of a rigid body in space. In two dimensions, two coordinates can uniquely describe the position of the CoM, and one angle the orientation of a rigid body. In space we need three coordinates for the position of the CoM, and three angles or parameters to describe the orientation of the rigid body. The extension of the position from two to three is straight-forward, but the extension from one rotation to three rotations is complex. In this chapter, we will therefore focus on the description of the orientation, or attitude, of a rigid body in space. We will introduce the concept of a rotation matrix, and various ways to parameterize this rotation matrix, particularly Euler angles, the axis-angle description, and Euler parameters. Since Euler parameters are unit quaternions, some attention is given to the quaternion analysis and algebra.

18.1 Rotation Matrix

The motion of a rigid body in space can be described by the displacement of the body's center of mass, combined with the change in orientation of the rigid body with respect to some reference system, see Figure 18.1. The position of a point P in the rigid body can then be decomposed as the sum of the CoM position and the position of P relative to the CoM, as in

$$\boldsymbol{r}_P = \boldsymbol{r}_C + \boldsymbol{r}_{P/C}. \tag{18.1}$$

Thus, by considering the motion of the CoM and the change in orientation of the body, we are able to track any point on the rigid body.

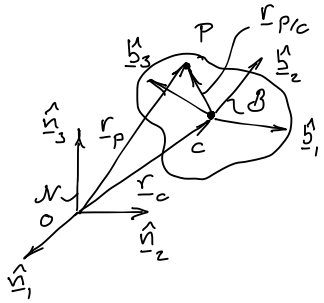


Figure 18.1: A rigid body rotating and translating.

Describing the effect of the change in orientation on the location of P is nontrivial. Therefore, we will focus here on this effect only, by setting $\boldsymbol{r}_C = \boldsymbol{0}$. The position of point P is then simply denoted by the vector $\boldsymbol{p}$, as shown in Figure 18.2.

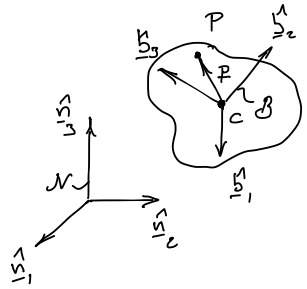


Figure 18.2: Position of P within a rigid body.

We introduce two triads, where the first is the inertial triad $\mathcal{N}$ with the associated unit base vectors $(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3)$. The second triad is a body-fixed triad $\mathcal{B}$ with the associated unit base vectors $(\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3)$.

The position vector of point P of the body can now be expressed in terms of the inertial triad base vectors as

$$\boldsymbol{p} = p_{n1}\hat{\boldsymbol{n}}_1 + p_{n2}\hat{\boldsymbol{n}}_2 + p_{n3}\hat{\boldsymbol{n}}_3. \tag{18.2}$$

The three scalars p_{n1}, p_{n2} and p_{n3} are the *components* or *projections* of the vector $\boldsymbol{p}$ with respect to the triad $\mathcal{N}$. Note that here, the components are also *coordinates* that describe the location of point P with respect to point C in directions of the $\mathcal{N}$-triad.

We usually write these components in a column vector, where we use a superscript to express that we are considering an algebraic component

vector of vector $\boldsymbol{p}$ expressed in triad $\mathcal{N}$, as in

$$^{\mathcal{N}}\boldsymbol{p} = \begin{pmatrix} p_{n1} \\ p_{n2} \\ p_{n3} \end{pmatrix}. \tag{18.3}$$

These components can be found by taking the dot product of the vector $\boldsymbol{p}$ with the three base vectors $\hat{\boldsymbol{n}}_i$, as in

$$p_{ni} = \boldsymbol{p} \cdot \hat{\boldsymbol{n}}_i, \qquad i = 1 \ldots 3. \tag{18.4}$$

This is possible because the unit base vectors are orthogonal, that is,

$$\hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{n}}_i = \delta_{ij}, \qquad i, j = 1 \ldots 3, \tag{18.5}$$

where we have used the so-called *Kronecker delta*, which has the properties that $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

We can also write the position vector of point P with respect to the body-fixed triad $\mathcal{B}$, as in

$$\boldsymbol{p} = p_{b1}\hat{\boldsymbol{b}}_1 + p_{b2}\hat{\boldsymbol{b}}_2 + p_{b3}\hat{\boldsymbol{b}}_3, \tag{18.6}$$

with the components p_{bi} with respect the body-fixed triad $\mathcal{B}$. In analogy to the inertial triad, we can determine these components by the taking the dot product of the vector $\boldsymbol{p}$, but now with the body-fixed triad base vectors $\hat{\boldsymbol{b}}_i$,

$$p_{bi} = \boldsymbol{p} \cdot \hat{\boldsymbol{b}}_i, \quad i = 1 \ldots 3. \tag{18.7}$$

It is important to note that for a rigid body, these components are constant, because they are local coordinates. As an example; my nose will always be in the middle and front of my head and my right ear will always stay at the right of my nose, irrespectively of how I move my head. Again, we can write these coordinates in a column vector with the superscript notation for the triad of reference, in this case the body-fixed triad $\mathcal{B}$,

$$^{\mathcal{B}}\boldsymbol{p} = \begin{pmatrix} p_{b1} \\ p_{b2} \\ p_{b3} \end{pmatrix}. \tag{18.8}$$

We like to keep track of the position of point P using components expressed in the inertial triad, because Newton's second law of motion is most conveniently expressed in an inertial triad. To determine

these components, we take the expression of the inertial-triad projections (18.4) and substitute for the vector $\boldsymbol{p}$ the expression in terms of the body-fixed projections (18.6),

$$p_i = \left(p_{b1}\hat{\boldsymbol{b}}_1 + p_{b2}\hat{\boldsymbol{b}}_2 + p_{b3}\hat{\boldsymbol{b}}_3 \right) \cdot \hat{\boldsymbol{n}}_i, \qquad i = 1 \dots 3, \tag{18.9}$$

or in an even more compact notation where we have reordered the right-hand side,

$$p_{ni} = \left(\hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{b}}_j \right) p_{bj}, \qquad i,\, j = 1 \dots 3. \tag{18.10}$$

We then rewrite this in matrix-vector form as

$$\begin{pmatrix} p_{n1} \\ p_{n2} \\ p_{n3} \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{b}}_1 \cdot \hat{\boldsymbol{n}}_1 & \hat{\boldsymbol{b}}_2 \cdot \hat{\boldsymbol{n}}_1 & \hat{\boldsymbol{b}}_3 \cdot \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{b}}_1 \cdot \hat{\boldsymbol{n}}_2 & \hat{\boldsymbol{b}}_2 \cdot \hat{\boldsymbol{n}}_2 & \hat{\boldsymbol{b}}_3 \cdot \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{b}}_1 \cdot \hat{\boldsymbol{n}}_3 & \hat{\boldsymbol{b}}_2 \cdot \hat{\boldsymbol{n}}_3 & \hat{\boldsymbol{b}}_3 \cdot \hat{\boldsymbol{n}}_3 \end{pmatrix} \begin{pmatrix} p_{b1} \\ p_{b2} \\ p_{b3} \end{pmatrix}. \tag{18.11}$$

This linear transformation maps the coordinates of point P with respect to point C in terms of the body-fixed triad $\mathcal{B}$ into the coordinates with respect to C expressed in the inertially fixed triad $\mathcal{N}$. The 3-by-3 matrix, formed by the nine dot-products of the 2-times-3 base vectors (which are all unique combinations), is what we call the rotation matrix[1] $\mathbf{R}$, defined by,

$$R_{ij} = \hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{b}}_j, \qquad i,\, j = 1 \dots 3. \tag{18.12}$$

With the superscript notation for the two triads $\mathcal{N}$ and $\mathcal{B}$, we can write this transformation from body-fixed to inertial components as

$${}^{\mathcal{N}}\boldsymbol{p} = {}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}\, {}^{\mathcal{B}}\boldsymbol{p}, \tag{18.13}$$

where the rotation matrix ${}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}$ is defined by the nine dot products of the inertial triad $\mathcal{N}$ base vectors $\hat{\boldsymbol{n}}_i$ and the body-fixed triad $\mathcal{B}$ base vectors $\hat{\boldsymbol{b}}_j$,

$${}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}} = R_{ij} = \hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{b}}_j, \qquad i,\, j = 1 \dots 3. \tag{18.14}$$

The inverse transformation of the components, from inertial triad $\mathcal{N}$ to body-fixed triad $\mathcal{B}$,

$${}^{\mathcal{B}}\boldsymbol{p} = {}^{\mathcal{B}}\mathbf{R}_{\mathcal{N}}\, {}^{\mathcal{N}}\boldsymbol{p}, \tag{18.15}$$

[1] Note that this rotation matrix $\mathbf{R}$ is the transpose of the rotation matrix $\mathbf{C}$ as derived in Chapter 3. To avoid any ambiguity or confusion, we will usually indicate the involved triads by sub- and superscript.

can easily be derived by starting from expressions (18.7) and substitution of the vector $\boldsymbol{p}$ expressed in the inertial triad (18.2), resulting in

$$p_{bi} = (p_{n1}\hat{\boldsymbol{n}}_1 + p_{n2}\hat{\boldsymbol{n}}_2 + p_{n3}\hat{\boldsymbol{n}}_3) \cdot \hat{\boldsymbol{b}}_i = \left(\hat{\boldsymbol{b}}_i \cdot \hat{\boldsymbol{n}}_j\right) p_j, \qquad i,\, j = 1 \ldots 3, \tag{18.16}$$

or in matrix–vector form,

$$\begin{pmatrix} p_{b1} \\ p_{b2} \\ p_{b3} \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{b}}_1 \cdot \hat{\boldsymbol{n}}_1 & \hat{\boldsymbol{b}}_1 \cdot \hat{\boldsymbol{n}}_2 & \hat{\boldsymbol{b}}_1 \cdot \hat{\boldsymbol{n}}_3 \\ \hat{\boldsymbol{b}}_2 \cdot \hat{\boldsymbol{n}}_1 & \hat{\boldsymbol{b}}_2 \cdot \hat{\boldsymbol{n}}_2 & \hat{\boldsymbol{b}}_2 \cdot \hat{\boldsymbol{n}}_3 \\ \hat{\boldsymbol{b}}_3 \cdot \hat{\boldsymbol{n}}_1 & \hat{\boldsymbol{b}}_3 \cdot \hat{\boldsymbol{n}}_2 & \hat{\boldsymbol{b}}_3 \cdot \hat{\boldsymbol{n}}_3 \end{pmatrix} \begin{pmatrix} p_{n1} \\ p_{n2} \\ p_{n3} \end{pmatrix}. \tag{18.17}$$

From the above, we conclude that the inverse transformation is determined by

$${}^{\mathcal{B}}\mathbf{R}_{\mathcal{N}} = R_{ij} = \hat{\boldsymbol{b}}_i \cdot \hat{\boldsymbol{n}}_j, \qquad i,\, j = 1 \ldots 3. \tag{18.18}$$

If we compare this with the regular rotation matrix ${}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}$ (18.14), we see that the indices i and j are interchanged, and apparently the inverse is equal to the transpose of the regular transformation. In other words,

$$(R^{-1})_{ij} = R_{ji} \qquad \text{or} \qquad \mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}. \tag{18.19}$$

We say that the rotation matrix $\mathbf{R}$ is an *orthogonal matrix*. Orthogonal matrices have the characteristics that $\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$ or $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{E}$, and the eigenvalues λ of such a matrix are all such that $|\lambda| = 1$.

In general, a three-by-three matrix, like the rotation matrix $\mathbf{R}$, has nine entries. However, the orthogonality condition $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{E}$ imposes six constraints on these nine entries: Six, because the off-diagonal conditions in $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{E}$ are identical expressions, as $\mathbf{R}\mathbf{R}^{\mathrm{T}}$ is symmetric. However, with nine entries and six constraints, this only leaves room for three independent parameters to describe the rotation matrix.

We conclude that a 3×3 rotation matrix can uniquely be determined by three parameters, or in other words, rotation in space can be described by three parameters. One may be tempted to jump to the conclusion that these are the three rotations about the three orthogonal axes, or the three base vectors. However, this is a very confusing view, which we will explain in the following section.

Example 18.1 Let us investigate the parameterization of a rotation matrix for a two-dimensional case. In that case, the rotation matrix $\mathbf{R}$ is 2×2 matrix, so there are 4 entries. Moreover, $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{E}$ imposes three constraints. This means that we can parameterize $\mathbf{R}$ with only $4 - 3 = 1$ parameter, for which we choose the angle φ, which is the rotation about an

axis which is perpendicular to the plane.

In two dimensions, such a rotation over an angle φ is shown in Figure 18.3.

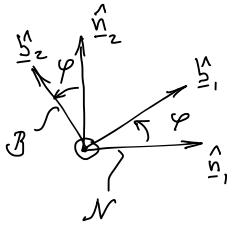


Figure 18.3: Rotation in 2-D.

Then, the unit base vectors spanning the two triads $\mathcal{N}$ and $\mathcal{B}$, all expressed in components in the inertial triad $\mathcal{N}$, are

$$\hat{\boldsymbol{n}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \hat{\boldsymbol{n}}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{18.20E}$$

$$\hat{\boldsymbol{b}}_1 = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \qquad \hat{\boldsymbol{b}}_2 = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}. \tag{18.21E}$$

The rotation matrix mapping the components of a vector from the body-fixed triad $\mathcal{B}$ to the inertial triad $\mathcal{N}$ is then

$$R_{ij} = \hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{b}}_j = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \tag{18.22E}$$

and its inverse

$$(R^{-1})_{ij} = \hat{\boldsymbol{b}}_i \cdot \hat{\boldsymbol{n}}_j = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = (R^{\mathrm{T}})_{ij}. \tag{18.23E}$$

We can take the idea of parameterizing a rotation matrix further in a thought experiment with more than three dimensions. In a four-dimensional space, the rotation matrix has four times four, so 16 entries. How many parameters are needed to describe the rotation matrix in this four-dimensional space? In other words, how many independent rotations describe the attitude of a rigid body in a four-dimensional space? The number of constraints we have on the four-by-four rotation matrix is ten, so the number of parameters to describe this four-by-four rotation matrix is $16 - 10 = 6$. In other words, in a four-dimensional space, the orientation of a rigid body is described by six parameters. That may sound nonintuitive, some people would answer four. However, two base vectors span a plane, and about the axis perpendicular to that plane, we have a rotation. In two-dimensional space, you can construct one plane, since there are only two base vectors. In three dimensions,

we have three base vectors and one can make three unique combinations of these three base vectors, 1–2, 1–3 and 2–3, and therefore there are three rotations. In four dimensions, with four base vectors, one can create six combinations, 1–2, 1–3, 1–4, 2–3, 2–4, and 3–4, and we have six rotations, etc.

18.2 Euler Angles

One way to parameterize the rotation matrix $\mathbf{R}$ in three dimensions is by means of *Euler angles*. We define a body-fixed triad $\mathcal{B}$, associated with coordinate axes x, y and z, and an inertial triad $\mathcal{N}$, associated with coordinate axes X, Y and Z. The recipe for Euler angles is then as follows:

1. Align the body-fixed triad $\mathcal{B}$ with the inertial triad $\mathcal{N}$,
2. Rotation about the Z-axis by an angle φ,
3. Rotation about the rotated X-axis by an angle θ,
4. Rotation about the rotated Z-axis by an angle ψ.

These Euler angles are called Z-X-Z or 3–1–3 angles. The three angles are sometimes also designated as: precession-nutation-spin. These names can be traced back to the study of the motion of spinning tops, where spin or large angular speed occurs about the rotated Z-axis.

An insightful way of representing the concept of Euler angles is by the so-called *cans-in-series* [62], as shown in Figure 18.4: Euler angles can be visualized as a physically realizable mechanism, by placing three hinges in series. On purpose, the figure shows the three hinges out of center, to depict the intermediate triads. In reality, the centers of the three hinges should coincide, so the dimensions are negligible. So, one can think of the cans-in-series drawing as a sort of exploded view.

We now will derive the rotation matrix parameterized by these Z-X-Z Euler angles. The rotation matrix ${}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}$ maps the components of a vector $\boldsymbol{x}$ from the body-fixed triad ${}^{\mathcal{B}}\boldsymbol{x}$ into the components in the inertial triad ${}^{\mathcal{N}}\boldsymbol{x}$, as in

$$ {}^{\mathcal{N}}\boldsymbol{x} = {}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}\, {}^{\mathcal{B}}\boldsymbol{x}. \tag{18.24} $$

We will use the strategy of "divide and conquer" to obtain this rotation matrix, i.e. we look at each individual rotation first, and then combine these three individual rotation matrices to generate the final rotation matrix.

Two intermediate triads are introduced: the triad $\mathcal{F}$, which is associated with the coordinate system $x''y''z''$, and the triad $\mathcal{G}$, which is associated with the coordinate system $x'y'z'$, such that the chain of triads is $\mathcal{N}$–$\mathcal{F}$–$\mathcal{G}$–$\mathcal{B}$, see Figure 18.5. The first rotation is about the Z-axis ($\hat{\boldsymbol{n}}_3 = \hat{\boldsymbol{f}}_3$) by an angle φ, and the rotation matrix that maps the components from $\mathcal{F}$ to $\mathcal{N}$ is,

$$
{}^{\mathcal{N}}\boldsymbol{x} = {}^{\mathcal{N}}\mathbf{R}_{\mathcal{F}}(\varphi)\,{}^{\mathcal{F}}\boldsymbol{x}, \tag{18.25}
$$

$$
\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix}. \tag{18.26}
$$

The second rotation is about the x'-axis ($\hat{\boldsymbol{f}}_1 = \hat{\boldsymbol{g}}_1$) by an angle θ, and the mapping from the triad $\mathcal{G}$ to the triad $\mathcal{F}$ is

$$
{}^{\mathcal{F}}\boldsymbol{x} = {}^{\mathcal{F}}\mathbf{R}_{\mathcal{G}}(\theta)\,{}^{\mathcal{G}}\boldsymbol{x}, \tag{18.27}
$$

$$
\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}. \tag{18.28}
$$

Finally, the third rotation is about the z''-axis ($\hat{\boldsymbol{g}}_3 = \hat{\boldsymbol{b}}_3$) by an angle ψ, and the mapping from the triad $\mathcal{B}$ to the triad $\mathcal{G}$ is

$$
{}^{\mathcal{G}}\boldsymbol{x} = {}^{\mathcal{G}}\mathbf{R}_{\mathcal{B}}(\psi)\,{}^{\mathcal{B}}\boldsymbol{x} \tag{18.29}
$$

$$
\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{18.30}
$$

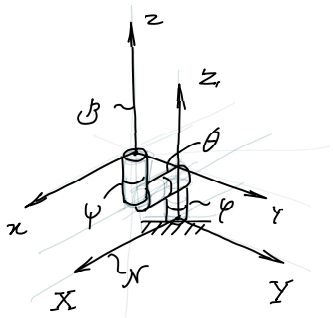


Figure 18.4: Cans-in-series representation of a sequence of rotations described by Z-X-Z Euler angles.

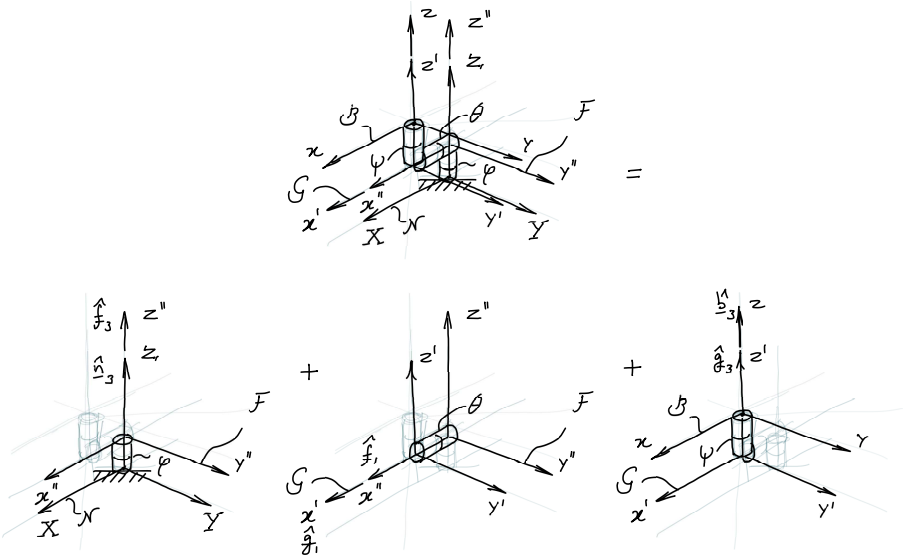


Figure 18.5: Cans-in-series representation of a the sequence of rotations described by Z-X-Z Euler angles, including the intermediate triads and coordinate systems.

Substitution of (18.29) in (18.27), and then (18.27) in (18.25) results in the expression

$$
\begin{aligned}
{}^{\mathcal{N}}\boldsymbol{x} &= {}^{\mathcal{N}}\mathbf{R}_{\mathcal{F}}(\varphi)\,{}^{\mathcal{F}}\mathbf{R}_{\mathcal{G}}(\theta)\,{}^{\mathcal{G}}\mathbf{R}_{\mathcal{B}}(\psi)\,{}^{\mathcal{B}}\boldsymbol{x} && (18.31)\\
&= {}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}\,{}^{\mathcal{B}}\boldsymbol{x}. && (18.32)
\end{aligned}
$$

From this we conclude that the final rotation matrix is the product of the three elementary rotation matrices in the following order:

$$
\begin{aligned}
{}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}} &= {}^{\mathcal{N}}\mathbf{R}_{\mathcal{F}}(\varphi)\,{}^{\mathcal{F}}\mathbf{R}_{\mathcal{G}}(\theta)\,{}^{\mathcal{G}}\mathbf{R}_{\mathcal{B}}(\psi) && (18.33)\\
&= \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} && (18.34)\\
&= \begin{pmatrix} \cos\varphi\cos\psi - \sin\varphi\cos\theta\sin\psi & -\cos\varphi\sin\psi - \sin\varphi\cos\theta\cos\psi & \sin\varphi\sin\theta \\ \sin\varphi\cos\psi + \cos\varphi\cos\theta\sin\psi & -\sin\varphi\sin\psi + \cos\varphi\cos\theta\cos\psi & -\cos\varphi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix} && (18.35)
\end{aligned}
$$

It should be stressed that the order of the individual rotations is important. For instance, if we change the order of the first two steps,

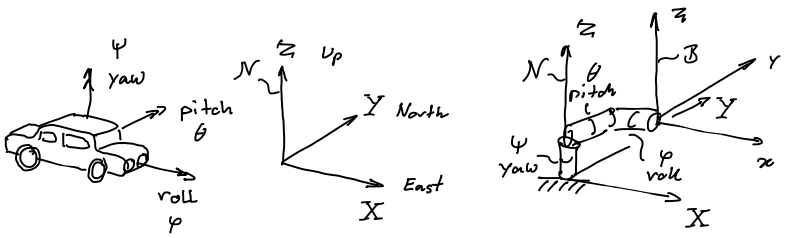


Figure 18.7: Coordinate system for a car with yaw-pitch-roll Euler angles for the attitude.

we will obtain a completely different result. This is also clear if we look at the expression of the final rotation matrix (18.33), which is the matrix product of three matrices. We know from matrix algebra that in general the order of multiplication is important, $\mathbf{AB} \neq \mathbf{BA}$. In other words, matrix multiplication is non-commutative, and so, therefore, the individual rotations about the coordinate axes are non-commutative.

One might be tempted to see the three Euler angles as a vector, but since they are non-commutative, this is not possible.

◯ **Example 18.2** If the Euler angles were vectors, the order would not matter and they could be added "head to tail" as shown in Figure 18.6.

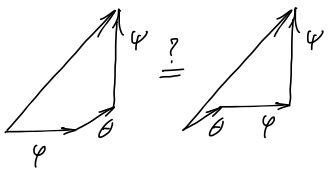


Figure 18.6: Illustration of Euler angles added "head to tail" as if they were vectors, which they are *not*.

However, the order does matter and hence, Euler angles are *not* vectors.

If we choose a different order of rotation, we have a different set of Euler angles. There are in fact $3 \times 2 \times 2 = 12$ different sets of Euler angles. Besides the precession-nutation-spin or Z-X-Z Euler angles, as shown above, the most common one in use is the yaw-pitch-roll, or Z-Y-X. An example of the yaw-pitch-roll Euler angles is shown in Figure 18.7. The rotation matrix for the yaw-pitch-roll Euler angles can easily be composed from the individual rotation matrices for the rotations about the X-, Y-, and Z-axis, respectively. The rotation matrices for rotation about the X- and Z-axis are already given by (18.27) and (18.25). The rotation matrix for the pitch motion, which

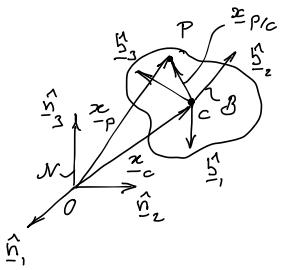


Figure 18.8: A rigid body with center of mass C, point P, body-fixed triad $\mathcal{B}$ and the point's position vectors with respect to the origin O of a coordinate system associated with inertial triad $\mathcal{N}$.

is a rotation about the Y-axis ($\hat{\boldsymbol{f}}_2 = \hat{\boldsymbol{g}}_2$) associated with an $\mathcal{F}$-triad by an angle θ to obtain directions of the $\mathcal{G}$-triad, is given by

$$^{\mathcal{F}}\boldsymbol{x} = {}^{\mathcal{F}}\mathbf{R}_{\mathcal{G}}(\theta)\,{}^{\mathcal{G}}\boldsymbol{x}, \tag{18.36}$$

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}. \tag{18.37}$$

With the individual rotation matrices about the X- (18.27), Y- (18.36), and Z-axis (18.25), the combined rotation matrix for any combination of Euler angles can be constructed.

Problem 18.1 Determine the rotation matrix ${}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}$ for the yaw-pitch-roll (Z-Y-X, or ψ-θ-φ) Euler angles.

18.3 Angular Velocities and Euler Angles

In this section, we will derive the relations between angular velocities and Euler angles and their time derivatives. The starting point is the general expression for the position of a point P within the rigid body as being the vector sum of the position of the CoM of the body and the position of P with respect to the CoM within the body,

$$\boldsymbol{x}_P = \boldsymbol{x}_C + \boldsymbol{x}_{P/C}, \tag{18.38}$$

as shown in Figure 18.8. Expressed in terms of its components with respect to the inertial triad $\mathcal{N}$, the position vector is

$$^{\mathcal{N}}\boldsymbol{x}_P = {}^{\mathcal{N}}\boldsymbol{x}_C + {}^{\mathcal{N}}\boldsymbol{x}_{P/C} \tag{18.39}$$

$$= {}^{\mathcal{N}}\boldsymbol{x}_C + {}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}\,{}^{\mathcal{B}}\boldsymbol{x}_{P/C}. \tag{18.40}$$

In the second expression we have introduced the rotation matrix ${}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}$ and the body-fixed coordinates ${}^{\mathcal{B}}\boldsymbol{x}_{P/C}$ of the position of P within the body with respect to C, because these coordinates are constant for a rigid body.

The velocities of point P follow as

$$
\begin{aligned}
{}^{\mathcal{N}}\dot{\boldsymbol{x}}_P &= {}^{\mathcal{N}}\dot{\boldsymbol{x}}_C + {}^{\mathcal{N}}\dot{\boldsymbol{x}}_{P/C} && (18.41)\\
&= {}^{\mathcal{N}}\dot{\boldsymbol{x}}_C + {}^{\mathcal{N}}\dot{\mathbf{R}}_{\mathcal{B}}\,{}^{\mathcal{B}}\boldsymbol{x}_{P/C} + {}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}\cancelto{0}{{}^{\mathcal{B}}(\dot{\boldsymbol{x}}_{P/C})_{\mathcal{B}}} && (18.42)
\end{aligned}
$$

We will focus on the rotational part only, and disregard the motion of the CoM. The relative velocity of P with respect to C is

$$
{}^{\mathcal{N}}\dot{\boldsymbol{x}}_{P/C} = {}^{\mathcal{N}}\dot{\mathbf{R}}_{\mathcal{B}}\,{}^{\mathcal{B}}\boldsymbol{x}_{P/C}, \tag{18.43}
$$

where ${}^{\mathcal{N}}\dot{\boldsymbol{x}}_{P/C}$ and ${}^{\mathcal{B}}\boldsymbol{x}_{P/C}$ are expressed in two different triads. We express everything in the inertial triad by substitution of the inverse transformation ${}^{\mathcal{B}}\boldsymbol{x}_{P/C} = ({}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}})^{\mathrm{T}}\,{}^{\mathcal{N}}\boldsymbol{x}_{P/C}$ in (18.43), which results in

$$
{}^{\mathcal{N}}\dot{\boldsymbol{x}}_{P/C} = {}^{\mathcal{N}}\dot{\mathbf{R}}_{\mathcal{B}}({}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}})^{\mathrm{T}}\,{}^{\mathcal{N}}\boldsymbol{x}_{P/C}. \tag{18.44}
$$

Since everything is now expressed in one triad, we will drop the triad indication $\mathcal{N}$ and shorten the notation for the rotation matrix by defining $\mathbf{R} := {}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}$, which results in the velocity expression

$$
\dot{\boldsymbol{x}}_{P/C} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\boldsymbol{x}_{P/C}. \tag{18.45}
$$

Apparently the matrix $\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$ expresses the effect of change in orientation with respect to time, or in other words the effect of *angular velocities*. To derive this, we start from the rotation matrix constraints $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{E}$, and differentiate these with respect to time,

$$
\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} + \mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}} = \mathbf{0}. \tag{18.46}
$$

We substitute

$$
\mathbf{W} := \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} \tag{18.47}
$$

and see the identities,

$$
\mathbf{W} + \mathbf{W}^{\mathrm{T}} = \mathbf{0} \rightarrow \mathbf{W}^{\mathrm{T}} = -\mathbf{W}, \tag{18.48}
$$

which expresses that the matrix $\mathbf{W}$ is skew-symmetric. In general, a 3×3 skew-symmetric matrix has only three independent parameters, which we can write as

$$\mathbf{W} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}. \tag{18.49}$$

We have written this matrix in such a form that it resembles the cross product of two vectors, written as a matrix–vector multiplication using the tilde matrix (see Appendix A.3), so that with the definition of the angular velocity vector $\boldsymbol{\omega} := (a,\, b,\, c)^{\mathrm{T}}$ of the rigid body and a 3-D vector $\boldsymbol{x} = (x, y, z)^{\mathrm{T}}$, we can write

$$\mathbf{W}\boldsymbol{x} = \tilde{\boldsymbol{\omega}}\boldsymbol{x} = \boldsymbol{\omega} \times \boldsymbol{x}. \tag{18.50}$$

From (18.47) and (18.50), we conclude that the time derivative of the rotation matrix can be written as the cross product of the angular velocity vector $\boldsymbol{\omega}$ and the columns of the rotation matrix,

$$\dot{\mathbf{R}} = \tilde{\boldsymbol{\omega}}\mathbf{R}. \tag{18.51}$$

The infinitesimally small changes of the base vectors can only be perpendicular to the base vectors. Therefore the base vectors do not change length, only direction.

Using these concepts for the velocities (18.44), $\dot{\boldsymbol{x}}_{P/C} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\boldsymbol{x}_{P/C}$, we rewrite these as,

$$\dot{\boldsymbol{x}}_{P/C} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\boldsymbol{x}_{P/C} = \tilde{\boldsymbol{\omega}}\boldsymbol{x}_{P/C} = \boldsymbol{\omega} \times \boldsymbol{x}_{P/C}. \tag{18.52}$$

Note that $\boldsymbol{\omega}$ is shorthand for the velocity vector $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ of triad $\mathcal{B}$ with respect to triad $\mathcal{N}$,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}. \tag{18.53}$$

We now aim to determine the components of this vector,

$${}^{\mathcal{N}}\boldsymbol{\omega} = \begin{pmatrix} \omega_{n1} & \omega_{n2} & \omega_{n3} \end{pmatrix}^{\mathrm{T}}, \tag{18.54}$$

by expressing them in terms of the rotation matrix and its time derivate from

$$\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = \tilde{\boldsymbol{\omega}}. \tag{18.55}$$

The elements of the rotation matrix can be written as functions of the three Z-X-Z Euler angles (φ-θ-ψ), as in (18.35). So, following (18.55),

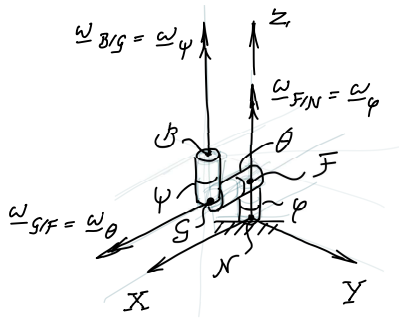


Figure 18.9: Angular velocity vectors of the Euler angle derivatives within the sequence of triads $\mathcal{N}$–$\mathcal{F}$–$\mathcal{G}$–$\mathcal{B}$.

also the components of $\boldsymbol{\omega}$ can be expressed as functions of the Euler angles and their time derivatives ($\dot{\varphi}$, $\dot{\theta}$, $\dot{\psi}$).

We calculate the time derivative $\dot{\mathbf{R}}$ of the rotation matrix by first taking the partial derivatives of $\mathbf{R}$ with respect to the Euler angles φ-θ-ψ times the time derivatives of these angles. Substituting this in (18.55) gives $\tilde{\boldsymbol{\omega}}$,

$$\frac{\partial \mathbf{R}}{\partial \varphi}\mathbf{R}^{\mathrm{T}}\dot{\varphi} + \frac{\partial \mathbf{R}}{\partial \theta}\mathbf{R}^{\mathrm{T}}\dot{\theta} + \frac{\partial \mathbf{R}}{\partial \psi}\mathbf{R}^{\mathrm{T}}\dot{\psi} = \tilde{\boldsymbol{\omega}}. \tag{18.56}$$

These expressions become lengthy and prone to error. Moreover, since $\tilde{\boldsymbol{\omega}}$ is skew-symmetric, calculating all components is more than twice the amount of work necessary. Still, it is also useful to derive three zeros on the diagonal, because these can be used for checking correctness.

Although Euler angles are not vectors, angular velocities are. Therefore, we can apply all the regular vector operations such as commutative addition etc. The three angular velocity vectors corresponding to the Euler angle derivatives are depicted in Figure 18.9. To determine the angular velocity vector, we may add the three individual velocity vectors, as in

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{G}/\mathcal{F}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{G}}. \tag{18.57}$$

We use abbreviated notations as

$$\boldsymbol{\omega}_{\varphi} := \boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}, \quad \boldsymbol{\omega}_{\theta} := \boldsymbol{\omega}_{\mathcal{G}/\mathcal{F}}, \quad \boldsymbol{\omega}_{\psi} := \boldsymbol{\omega}_{\mathcal{B}/\mathcal{G}}. \tag{18.58}$$

The angular velocity vectors are most easily expressed in different triads:

$${}^{\mathcal{N}}\boldsymbol{\omega}_{\varphi} = \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix}, \quad {}^{\mathcal{F}}\boldsymbol{\omega}_{\theta} = \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix}, \quad {}^{\mathcal{G}}\boldsymbol{\omega}_{\psi} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}. \tag{18.59}$$

As the vectors are all expressed in different triads, appropriate rotations need to be applied to be able to add them in (18.57) and to express them in the inertial triad $\mathcal{N}$,

$$ {}^{\mathcal{N}}\boldsymbol{\omega} = {}^{\mathcal{N}}\boldsymbol{\omega}_{\varphi} + {}^{\mathcal{N}}\mathbf{R}_{\mathcal{F}}\,{}^{\mathcal{F}}\boldsymbol{\omega}_{\theta} + {}^{\mathcal{N}}\mathbf{R}_{\mathcal{F}}\,{}^{\mathcal{F}}\mathbf{R}_{\mathcal{G}}\,{}^{\mathcal{G}}\boldsymbol{\omega}_{\psi}. \tag{18.60} $$

Substitution of the rotation matrices and angular velocities, combined with expanding and reordering of terms results in the following expressions for the angular velocities ${}^{\mathcal{N}}\boldsymbol{\omega}$, expressed in the inertial triad in terms of the Euler angles and their time derivatives:

$$ {}^{\mathcal{N}}\boldsymbol{\omega} = \begin{pmatrix} \omega_{n1} \\ \omega_{n2} \\ \omega_{n3} \end{pmatrix} = \begin{pmatrix} 0 & \cos\varphi & \sin\varphi\sin\theta \\ 0 & \sin\varphi & -\cos\varphi\sin\theta \\ 1 & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} =: \mathbf{A}\dot{\boldsymbol{a}}. \tag{18.61} $$

Here we have written the time derivatives of the three Euler angles as a column vector $\dot{\boldsymbol{a}} = (\dot{\varphi}, \dot{\theta}, \dot{\psi})^{\mathrm{T}}$. However, this is *not* a vector, but just an ordered set of algebraic numbers. This can be compared to the way we often write down generalized coordinates, which also looks like a vector, although it is not.

To determine the angular velocity vector expressed in the body-fixed triad ${}^{\mathcal{B}}\boldsymbol{\omega}$, there are two routes. The first is to apply the rotation matrix ${}^{\mathcal{B}}\mathbf{R}_{\mathcal{N}}$ to (18.61), as in

$$ {}^{\mathcal{B}}\boldsymbol{\omega} = {}^{\mathcal{B}}\mathbf{R}_{\mathcal{N}}\,{}^{\mathcal{N}}\boldsymbol{\omega} = {}^{\mathcal{B}}\mathbf{R}_{\mathcal{N}}\mathbf{A}\dot{\boldsymbol{a}} = \mathbf{B}\dot{\boldsymbol{a}}. \tag{18.62} $$

The second route is to use the same approach as above, and to transform all algebraic angular velocities to the body-fixed triad $\mathcal{B}$, as in

$$ {}^{\mathcal{B}}\boldsymbol{\omega} = {}^{\mathcal{B}}\boldsymbol{\omega}_{\psi} + {}^{\mathcal{B}}\mathbf{R}_{\mathcal{G}}\,{}^{\mathcal{G}}\boldsymbol{\omega}_{\theta} + {}^{\mathcal{B}}\mathbf{R}_{\mathcal{G}}\,{}^{\mathcal{G}}\mathbf{R}_{\mathcal{F}}\,{}^{\mathcal{F}}\boldsymbol{\omega}_{\varphi}. \tag{18.63} $$

We note that ${}^{\mathcal{G}}\boldsymbol{\omega}_{\theta} = {}^{\mathcal{F}}\boldsymbol{\omega}_{\theta}$ and ${}^{\mathcal{F}}\boldsymbol{\omega}_{\psi} = {}^{\mathcal{G}}\boldsymbol{\omega}_{\psi}$, because the axis of rotation is identical in the two triads.

Both approaches must lead to the same result for the angular velocities expressed in the body-fixed triad $\mathcal{B}$ in terms of the Euler angles and their time derivatives,

$$ {}^{\mathcal{B}}\boldsymbol{\omega} = \begin{pmatrix} \omega_{b1} \\ \omega_{b2} \\ \omega_{b3} \end{pmatrix} = \begin{pmatrix} \sin\psi\sin\theta & \cos\psi & 0 \\ \cos\psi\sin\theta & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \mathbf{B}\dot{\boldsymbol{a}}. \tag{18.64} $$

For a rigid body, the state of the orientation or attitude is described by the three Euler angles and the three angular speeds,

$$ \boldsymbol{X} = \begin{pmatrix} \varphi & \theta & \psi & \omega_{b1} & \omega_{b2} & \omega_{b3} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} \boldsymbol{a}^{\mathrm{T}} & {}^{\mathcal{B}}\boldsymbol{\omega}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}. \tag{18.65} $$

The change in orientation state over time is described by the time derivative of the orientation state,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X} = \begin{pmatrix} \dot{\varphi} & \dot{\theta} & \dot{\psi} & \dot{\omega}_{b1} & \dot{\omega}_{b2} & \dot{\omega}_{b3} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} \dot{\boldsymbol{a}}^{\mathrm{T}} & {}^{\mathcal{B}}\dot{\boldsymbol{\omega}}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}, \tag{18.66}$$

where we made use of the fact that the transport theorem simplifies to

$${}^{\mathcal{B}}(\dot{\boldsymbol{\omega}})_{\mathcal{N}} = {}^{\mathcal{B}}(\dot{\boldsymbol{\omega}})_{\mathcal{B}} + {}^{\mathcal{B}}\boldsymbol{\omega} \times {}^{\mathcal{B}}\boldsymbol{\omega} = \begin{pmatrix} \dot{\omega}_{b1} & \dot{\omega}_{b2} & \dot{\omega}_{b3} \end{pmatrix}^{\mathrm{T}}. \tag{18.67}$$

The second part of (18.66) can be determined from the Euler equations of motion,

$$\sum {}^{\mathcal{B}}\boldsymbol{M}_C = {}^{\mathcal{B}}\mathbf{I}_C \, {}^{\mathcal{B}}\dot{\boldsymbol{\omega}} + {}^{\mathcal{B}}\boldsymbol{\omega} \times \left({}^{\mathcal{B}}\mathbf{I}_C \, {}^{\mathcal{B}}\boldsymbol{\omega} \right). \tag{18.68}$$

From this we can solve for the angular accelerations,

$${}^{\mathcal{B}}\dot{\boldsymbol{\omega}} = \left({}^{\mathcal{B}}\mathbf{I}_C \right)^{-1} \left(\sum {}^{\mathcal{B}}\boldsymbol{M}_C - {}^{\mathcal{B}}\boldsymbol{\omega} \times \left({}^{\mathcal{B}}\mathbf{I}_C \, {}^{\mathcal{B}}\boldsymbol{\omega} \right) \right), \tag{18.69}$$

and subsequently, by using numerical integration, we obtain the angular velocities for the next time step,

$${}^{\mathcal{B}}\boldsymbol{\omega}_{n+1} = {}^{\mathcal{B}}\boldsymbol{\omega}_n + \int_0^h {}^{\mathcal{B}}\dot{\boldsymbol{\omega}} \, \mathrm{d}t. \tag{18.70}$$

The first part of (18.66), so the Euler angles for the next time step, result from numerically integrating the time derivatives of the Euler angles, which we can find from the inverse of the kinematic relations (18.64),

$$\dot{\boldsymbol{a}} = \mathbf{B}^{-1} \, {}^{\mathcal{B}}\boldsymbol{\omega}, \tag{18.71}$$

which gives written out:

$$\begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \frac{1}{\sin\theta} \begin{pmatrix} \sin\psi & \cos\psi & 0 \\ \cos\psi\sin\theta & -\sin\psi\sin\theta & 0 \\ -\sin\psi\cos\theta & -\cos\psi\cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} \omega_{b1} \\ \omega_{b2} \\ \omega_{b3} \end{pmatrix}. \tag{18.72}$$

However, here a problem arises, that is, the matrix $\mathbf{B}$ is *singular* for $\sin\theta = 0$, which is when $\theta = 0 + k\pi$. So, we are unable to determine the time derivatives of the Euler angles from the angular speeds when $\theta = 0 + k\pi$. This singular configuration is usually called *gimbal lock*.

Let us have a closer look at the singular configuration, as shown in Figure 18.10. Here, $\theta = 0$ and $\psi = 0$ (the latter condition is not

necessary for a singularity, but makes things easier to visualize). For the angular velocities (18.64), we now have

$$\begin{pmatrix} \omega_{b1} \\ \omega_{b2} \\ \omega_{b3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}, \tag{18.73}$$

thus,

$$\dot{\theta} = \omega_{b1}, \qquad \omega_{b2} = 0, \qquad \dot{\varphi} + \dot{\psi} = \dot{\omega}_{b3}. \tag{18.74}$$

Clearly, we are unable to determine what $\dot{\varphi}$ and $\dot{\psi}$ are, because the two rotations are perfectly aligned. We call this gimbal lock, because if one builds such a physical systems, one would use gimbals and they would be hard to move in this configuration.

An alternative way of looking at it is like this: in the singular configuration one is not able to describe a rotational speed about the y-axis, because all hinges have their rotational axes in the zx–plane. A way to avoid the local singularity is to re-define the rotations by adding or subtracting 90° to/from θ, as

$$\theta^* = \theta + \frac{\pi}{2}. \tag{18.75}$$

Then, the configuration is as in Figure 18.11 and no two axes are aligned. However, eventually this system can also run into a gimbal lock, thus the proposed solution is only a local solution to the problem. Another major disadvantage of such a re-definition is that the state variable θ is not smooth anymore, which hurts the numerical integration process, because that usually relies on smooth functions for making

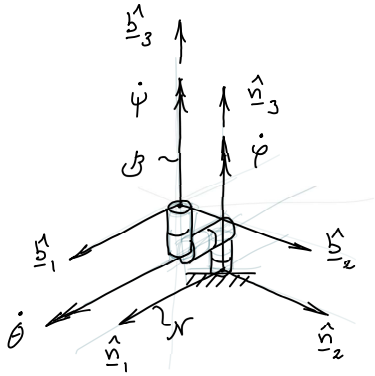


Figure 18.10: Singular configuration or gimbal lock for a set of Euler angles.

error estimates, convergence tests etc. Any type of Euler angles suffers from gimbal lock, because whatever axes and order are chosen, there is always the chance that the middle hinge puts the three axes of rotation in one plane.

18.4 Rotation Parameters

One way to parameterize the rotation matrix is by three Euler angles. Unfortunately, Euler angles exhibit a singularity when the middle angle aligns the rotation axes of the first and the third rotation, the so-called gimbal lock.

It can be proved that any parameterization with three parameters of the rotation matrix, or three-dimensional rotation group, in mathematics denoted by SO(3), eventually encounters such a singularity.

Fortunately, there is a solution to the problem, by using four instead of three parameters. Of course, the four parameters show a dependence, which we will express in terms of a constraint on the parameters. Since we can by now easily deal with constraints in multibody dynamic systems, we achieve singularity-free rotational parameters. We will present two methods, the axis–angle approach and Euler parameters.

18.4.1 Axis–Angle Parameters

The basis for the axis-angle representation of rotations is Euler's theorem on rotations in space, which reads (loosely translated from [15]):

> "*Any rotation in three-dimensional space can be represented by a rotation about a fixed axis at a given angle.*"

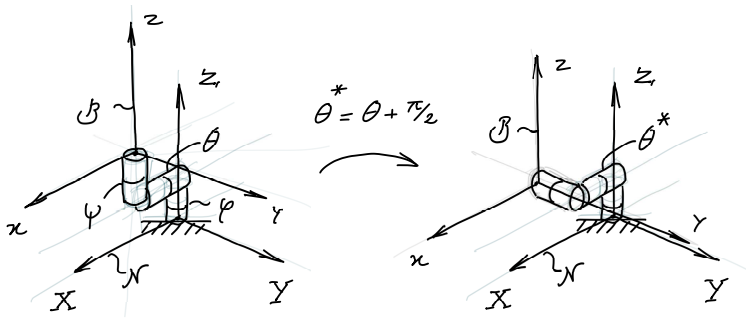


Figure 18.11: Redefined θ to avoid singularity or gimbal lock.

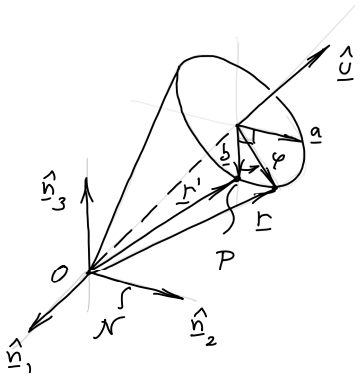


Figure 18.12: Rotation about a direction described by the unit vector $\hat{\boldsymbol{u}}$, by an angle φ, where point P moves from $\boldsymbol{r}'$ to $\boldsymbol{r}$ over a cone.

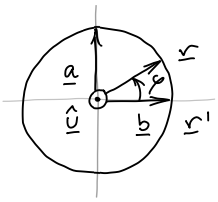


Figure 18.13: Top view of the "cone".

In Figure 18.12, we track the change in position of point P from $\boldsymbol{r}'$ to $\boldsymbol{r}$ as a result of a rotation about a direction described by the unit vector $\hat{\boldsymbol{u}}$, centered in O, by an angle φ. All vectors are algebraic vectors expressed in the inertial triad $\mathcal{N}$.

To express the change from $\boldsymbol{r}'$ to $\boldsymbol{r}$ in terms of the rotational parameters $\hat{\boldsymbol{u}}$ and φ, we look in the plane of the base of the "cone", as shown in Figure 18.13. In this projection, we create an orthogonal coordinate system $\boldsymbol{a}$, $\boldsymbol{b}$, such that they span the plane.

These vectors can be formed from the original position vector $\boldsymbol{r}'$ and the rotation axis $\hat{\boldsymbol{u}}$, as in

$$\boldsymbol{a} = \hat{\boldsymbol{u}} \times \boldsymbol{r}', \tag{18.76}$$

$$\boldsymbol{b} = \boldsymbol{a} \times \hat{\boldsymbol{u}} \tag{18.77}$$

$$= \left(\hat{\boldsymbol{u}} \times \boldsymbol{r}'\right) \times \hat{\boldsymbol{u}}. \tag{18.78}$$

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ do not have unit length. With the use of these two orthogonal vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and the angle φ, we can express $\boldsymbol{r}$ in terms of

$\boldsymbol{r}'$, as in

$$\begin{aligned}\boldsymbol{r} &= \boldsymbol{r}' - \boldsymbol{b}\,(1-\cos\varphi) + \boldsymbol{a}\sin\varphi && (18.79)\\ &= \boldsymbol{r}' - (1-\cos\varphi)\left(\hat{\boldsymbol{u}}\times\boldsymbol{r}'\right)\times\hat{\boldsymbol{u}} + \sin\varphi\left(\hat{\boldsymbol{u}}\times\boldsymbol{r}'\right) && (18.80)\\ &= \boldsymbol{r}' + (1-\cos\varphi)\,\hat{\boldsymbol{u}}\times\left(\hat{\boldsymbol{u}}\times\boldsymbol{r}'\right) + \sin\varphi\left(\hat{\boldsymbol{u}}\times\boldsymbol{r}'\right). && (18.81)\end{aligned}$$

Next, we use again the $\tilde{\mathbf{u}}$ (n-tilde) notation from Appendix A.3, to write the cross product as a matrix-vector multiplication, resulting in

$$\boldsymbol{r} = (\mathbf{E} + (1-\cos\varphi)\,\tilde{\mathbf{u}}\tilde{\mathbf{u}} + \sin\varphi\,\tilde{\mathbf{u}})\,\boldsymbol{r}'. \tag{18.82}$$

From this we conclude that the rotation matrix that transforms the body-triad components ${}^{\mathcal{B}}\boldsymbol{r}$ to the inertial-triad components ${}^{\mathcal{N}}\boldsymbol{r}$, expressed in terms of the (axis $\hat{\boldsymbol{u}}$, angle φ) is

$${}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}(\hat{\boldsymbol{u}},\,\varphi) = \mathbf{E} + (1-\cos\varphi)\,\tilde{\mathbf{u}}\tilde{\mathbf{u}} + \sin\varphi\,\tilde{\mathbf{u}}. \tag{18.83}$$

The inverse transformation must be equal to the transpose of the original transformation, ${}^{\mathcal{B}}\mathbf{R}_{\mathcal{N}} = ({}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}})^{\mathrm{T}}$. However, the axis–angle representation also gives a direct and insightful way to determine the inverse transformation, by rotating backwards, $\varphi \to -\varphi$ or by flipping the axis of rotation, $\hat{\boldsymbol{u}} \to -\hat{\boldsymbol{u}}$. If we substitute either of these two in the original transformation (18.83), we get

$${}^{\mathcal{B}}\mathbf{R}_{\mathcal{N}}(\hat{\boldsymbol{u}},\varphi) = \mathbf{E} + (1-\cos\varphi)\,\tilde{\mathbf{u}}\tilde{\mathbf{u}} - \sin\varphi\,\tilde{\mathbf{u}}, \tag{18.84}$$

which is clearly identical to $({}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}})^{\mathrm{T}}$ (because $\tilde{\mathbf{u}}^{\mathrm{T}} = -\tilde{\mathbf{u}}$).

The relation between the angular velocities $\boldsymbol{\omega}$ and the axis–angle parameters ($\hat{\boldsymbol{u}}$, φ) and their time derivatives ($\dot{\hat{\boldsymbol{u}}}$, $\dot{\varphi}$) can be determined in a straight-forward manner from $\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = \tilde{\boldsymbol{\omega}}$, as

$$\frac{\partial\mathbf{R}}{\partial\hat{u}_1}\mathbf{R}^{\mathrm{T}}\dot{\hat{u}}_1 + \frac{\partial\mathbf{R}}{\partial\hat{u}_2}\mathbf{R}^{\mathrm{T}}\dot{\hat{u}}_2 + \frac{\partial\mathbf{R}}{\partial\hat{u}_3}\mathbf{R}^{\mathrm{T}}\dot{\hat{u}}_3 + \frac{\partial\mathbf{R}}{\partial\varphi}\mathbf{R}^{\mathrm{T}}\dot{\varphi} = \tilde{\boldsymbol{\omega}}. \tag{18.85}$$

After some algebraic manipulation and interpretation [22], we find the angular velocities expressed in the inertial triad in terms of the axis and angle and their time derivatives,

$$\boldsymbol{\omega} = \dot{\varphi}\,\hat{\boldsymbol{u}} + \sin\varphi\,\dot{\hat{\boldsymbol{u}}} + (1-\cos\varphi)\,\hat{\boldsymbol{u}}\times\dot{\hat{\boldsymbol{u}}}. \tag{18.86}$$

We see that the angular velocity is constructed from three parts, the first along the axis of rotation and the other two parts perpendicular to the axis of rotation, because the axis is a unit vector and $\hat{\boldsymbol{u}}\cdot\dot{\hat{\boldsymbol{u}}} = 0$. If

we take the dot product of the angular velocity vector $\boldsymbol{\omega}$ with the axis of rotation $\hat{\boldsymbol{u}}$, we see that the last two terms vanish and that the result is

$$\boldsymbol{\omega} \cdot \hat{\boldsymbol{u}} = \dot{\varphi}. \tag{18.87}$$

So, the projection of the angular velocity vector onto the axis of rotation is the time derivative of the rotation angle, which we intuitively understand. This is already the first part of the inverse relation between the angular velocity and the time derivatives of the angle and the axis.

The second part, the change of the axis with time, $\dot{\hat{\boldsymbol{u}}}$, is somewhat more complex, but after some algebraic manipulation [22], we obtain the following result:

$$\dot{\hat{\boldsymbol{u}}} = \tfrac{1}{2} \left(\boldsymbol{\omega} \times \hat{\boldsymbol{u}} + \cot(\varphi/2)\, \hat{\boldsymbol{u}} \times (\boldsymbol{\omega} \times \hat{\boldsymbol{u}}) \right). \tag{18.88}$$

There appears to be a singularity for $\varphi = 0$, but this is no real problem, because for $\varphi = 0$, the axis of rotation $\hat{\boldsymbol{u}}$ and the angular velocity vector $\boldsymbol{\omega}$ are parallel.

Finally, in using these axis–angle parameter representations of the rotation matrix, we have to enforce the constraint on unit length of the axis $\hat{\boldsymbol{u}}$ at all times, expressed by

$$C_n = \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{u}} - 1 = 0. \tag{18.89}$$

The expression for the rate of change of the axis of rotation (18.88) already fulfills this constraint. However, when numerically integrating we have to make sure that the resulting axis does not drift. Applying the coordinate projection method locally, on the estimated axis $\hat{\tilde{\boldsymbol{u}}}$, is effectively the same as renormalizing the axis, as in $\hat{\boldsymbol{u}} = \hat{\tilde{\boldsymbol{u}}}/||\hat{\tilde{\boldsymbol{u}}}||$. The latter method has the preference because of its simple implementation.

18.4.2 Euler Parameters or Quaternions

The axis–angle representation for the rotation matrix shows a singularity at $\varphi = 0$, and the parameters are not uniform. They have different physical dimensions, unit length and angle. Moreover, in determining the rotation matrix, we have to calculate the sine and cosine of the angle, which is computationally costly. A solution is to combine the axis and the sine and cosine of the half angle in a new set of parameters, called *Euler parameters*.

To introduce the Euler parameters, we start from the trigonometric double-angle formulae,

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha, \tag{18.90}$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha, \tag{18.91}$$

and with $\varphi = \alpha/2$, we can write

$$\sin\varphi = 2\sin\left(\tfrac{1}{2}\varphi\right)\cos\left(\tfrac{1}{2}\varphi\right), \tag{18.92}$$

$$1 - \cos\varphi = 2\sin^2\left(\tfrac{1}{2}\varphi\right). \tag{18.93}$$

Substitution in the axis–angle representation of the rotation matrix (18.81) results in

$$\boldsymbol{r} = \boldsymbol{r}' + 2\sin^2\left(\tfrac{1}{2}\varphi\right)\left(\hat{\boldsymbol{u}} \times \left(\hat{\boldsymbol{u}} \times \boldsymbol{r}'\right)\right) + 2\cos\left(\tfrac{1}{2}\varphi\right)\sin\left(\tfrac{1}{2}\varphi\right)\left(\hat{\boldsymbol{u}} \times \boldsymbol{r}'\right). \tag{18.94}$$

Next, we introduce the four *Euler parameters* as a scalar q_0 and a vector $\boldsymbol{q}$, with

$$q_0 := \cos\left(\tfrac{1}{2}\varphi\right), \qquad \boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} := \sin\left(\tfrac{1}{2}\varphi\right)\hat{\boldsymbol{u}}. \tag{18.95}$$

These four parameters clearly have a dependency, namely that the overall length should be one,

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \tag{18.96}$$

Substitution of the four Euler parameter expressions in (18.94) results in

$$\boldsymbol{r} = \boldsymbol{r}' + 2\boldsymbol{q} \times \left(\boldsymbol{q} \times \boldsymbol{r}'\right) + 2q_0\left(\boldsymbol{q} \times \boldsymbol{r}'\right). \tag{18.97}$$

This can be written in matrix–vector form with the help of the tilde notation for the cross product, resulting in

$$\boldsymbol{r} = \left(\mathbf{E} + 2\tilde{\mathbf{q}}\tilde{\mathbf{q}} + 2q_0\tilde{\mathbf{q}}\right)\boldsymbol{r}'. \tag{18.98}$$

Here the vectors $\boldsymbol{r}'$ and $\boldsymbol{r}$ identify the same point P on the rigid body, the only difference is before and after the rotation. If the body-fixed triad $\mathcal{B}$ was initially aligned to the inertial triad $\mathcal{N}$, then we can identify $\boldsymbol{r}' = {}^{\mathcal{B}}\boldsymbol{r}$ and $\boldsymbol{r} = {}^{\mathcal{N}}\boldsymbol{r}$. We conclude that the rotation matrix which

transforms the body-triad components ${}^{\mathcal{B}}\boldsymbol{r}$ to the inertial-triad components ${}^{\mathcal{N}}\boldsymbol{r}$, expressed in terms of the four Euler parameters $(q_0, \boldsymbol{q})$, is given by

$$ {}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}(q_0, \boldsymbol{q}) = \mathbf{E} + 2\tilde{\mathbf{q}}\tilde{\mathbf{q}} + 2q_0\,\tilde{\mathbf{q}}. \tag{18.99} $$

Expanding the matrix products and making use of the unit length constraint on the Euler parameters $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$, results in

$$ {}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}} = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2\,(q_1q_2 - q_0q_3) & 2\,(q_1q_3 + q_0q_2) \\ 2\,(q_2q_1 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2\,(q_2q_3 - q_0q_1) \\ 2\,(q_3q_1 - q_0q_2) & 2\,(q_3q_2 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}. \tag{18.100} $$

The rotation matrix is bilinear in the Euler parameters, and we do not have any sine or cosine terms, which makes the calculation very efficient. In the next section, on quaternions, we will see an even nicer way to determine this rotation matrix, as the product of two matrices, linear in the Euler parameters.

The inverse of the rotation matrix is of course equal to the transpose, $\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$. However, the inverse can also be found directly, by rotating backwards $\varphi \to -\varphi$ or by flipping the rotation axis $\boldsymbol{q} \to -\boldsymbol{q}$, and the inverse of the rotation matrix becomes,

$$ \mathbf{R}^{-1} = \mathbf{E} + 2\,(-\tilde{\mathbf{q}})\,(-\tilde{\mathbf{q}}) + 2q_0\,(-\tilde{\mathbf{q}}) \tag{18.101} $$

$$ = \mathbf{E} + 2\tilde{\mathbf{q}}\tilde{\mathbf{q}} - 2q_0\tilde{\mathbf{q}}. \tag{18.102} $$

By applying the transpose, we arrive at the same result,

$$ \mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}} = \mathbf{E}^{\mathrm{T}} + 2\,(\tilde{\mathbf{q}}\tilde{\mathbf{q}})^{\mathrm{T}} + 2q_0\tilde{\mathbf{q}}^{\mathrm{T}} \tag{18.103} $$

$$ = \mathbf{E} + 2\tilde{\mathbf{q}}\tilde{\mathbf{q}} - 2q_0\tilde{\mathbf{q}}, \tag{18.104} $$

where we used $\tilde{\mathbf{q}}^{\mathrm{T}} = -\tilde{\mathbf{q}}$.

Euler parameters are a special class of *quaternions* [62, 1, 2, 37], namely unit quaternions. By using quaternion algebra, the derivation of the rotation matrix and the time derivates related to the angular velocities become a lot easier. This is treated in Appendix E.

Example 18.3 We will derive the rotation matrix for a rotation about the Z-axis by an angle φ, by means of Euler parameters, which are for this rotation

$$ q_0 = \cos\left(\tfrac{\varphi}{2}\right), \qquad \boldsymbol{q} = \sin\left(\tfrac{\varphi}{2}\right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{18.105E} $$

Substitution in the rotation matrix (18.100) results in

$$\mathbf{R} = \begin{pmatrix} \cos^2\left(\frac{\varphi}{2}\right) - \sin^2\left(\frac{\varphi}{2}\right) & -2\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right) & 0 \\ 2\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right) & \cos^2\left(\frac{\varphi}{2}\right) - \sin^2\left(\frac{\varphi}{2}\right) & 0 \\ 0 & 0 & \cos^2\left(\frac{\varphi}{2}\right) + \sin^2\left(\frac{\varphi}{2}\right) \end{pmatrix}. \tag{18.106E}$$

With the trigonometric relations

$$2\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right) = \sin\varphi, \tag{18.107E}$$

$$\cos^2\left(\frac{\varphi}{2}\right) - \sin^2\left(\frac{\varphi}{2}\right) = \cos\varphi, \tag{18.108E}$$

$$\cos^2\left(\frac{\varphi}{2}\right) + \sin^2\left(\frac{\varphi}{2}\right) = 1, \tag{18.109E}$$

the rotation matrix becomes

$$\mathbf{R} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{18.110E}$$

which matches the previous result, as derived in the case where we used Euler angles (18.25).

The angular velocity vector can also be expressed in terms of the Euler parameters and their time derivatives. We start again from the expressions $\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = \tilde{\boldsymbol{\omega}}$, which can be expanded to

$$\frac{\partial \mathbf{R}}{\partial q_0}\mathbf{R}^{\mathrm{T}}\dot{q}_0 + \frac{\partial \mathbf{R}}{\partial q_1}\mathbf{R}^{\mathrm{T}}\dot{q}_1 + \frac{\partial \mathbf{R}}{\partial q_2}\mathbf{R}^{\mathrm{T}}\dot{q}_2 + \frac{\partial \mathbf{R}}{\partial q_3}\mathbf{R}^{\mathrm{T}}\dot{q}_3 = \tilde{\boldsymbol{\omega}}. \tag{18.111}$$

After simplification and making use of the unit length characteristic $q_i q_i = 1$, the following set of equations for the three angular velocities, expressed in the inertial triad $\mathcal{N}$ are obtained:

$${}^{\mathcal{N}}\boldsymbol{\omega} = \begin{pmatrix} \omega_{n1} \\ \omega_{n2} \\ \omega_{n3} \end{pmatrix} = 2\begin{pmatrix} -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{pmatrix}\begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix}. \tag{18.112}$$

Problem 18.2 Derive the expressions for the angular velocities in terms of the Euler parameters and their time derivates (18.112) from (18.111).

To determine the inverse relation, the time derivatives of the Euler parameters in terms of the angular velocities and the Euler parameters, we first need an extra equation. Since the Euler parameters are constrained by $q_i q_i = 1$, also the time derivatives are constrained:

$q_i q_i = 1 \Rightarrow 2\dot{q}_i q_i = 0$. This constraint is added to the set of three equations to obtain the full set of four equations,

$$\begin{pmatrix} 0 \\ \omega_{n1} \\ \omega_{n2} \\ \omega_{n3} \end{pmatrix} = 2 \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = 2\bar{\mathbf{Q}}^{\mathrm{T}} \begin{pmatrix} \dot{q}_0 \\ \dot{\boldsymbol{q}} \end{pmatrix}. \quad (18.113)$$

The naming of the leading matrix by $\bar{\mathbf{Q}}^{\mathrm{T}}$ may seem ad hoc here, but we use this notation because of the definition (E.14) used in quaternion algebra, as presented in Appendix E.

From the equations (18.113), we can solve for the time derivative of the Euler parameters, which results in

$$\begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{pmatrix} \begin{pmatrix} 0 \\ \omega_{n1} \\ \omega_{n2} \\ \omega_{n3} \end{pmatrix} = \frac{1}{2}\bar{\mathbf{Q}} \begin{pmatrix} 0 \\ \boldsymbol{\omega} \end{pmatrix}. \quad (18.114)$$

We observe that the inverse (which is the transpose, because $\bar{\mathbf{Q}}\bar{\mathbf{Q}}^{\mathrm{T}} = \mathbf{E}$) shows no singularities, opposed to the case were we used Euler angles or axis–angle parameters. This, and the facts that all four Euler parameters have the same physical dimension, and that the rotation matrix is bilinear in the Euler parameters, demonstrates the extreme usefulness of Euler parameters to describe orientations. Moreover, by using quaternion algebra, which is the topic of Appendix E, we will be able to easily combine successive rotations by combining the Euler parameters from the individual rotations.

The expressions for the Euler parameter rates, in terms of the Euler parameters and angular velocity ${}^{\mathcal{B}}\boldsymbol{\omega}$ expressed in the body triad $\mathcal{B}$, are found in the same manner as above. However, we now start from the expressions $\mathbf{R}^{\mathrm{T}}\dot{\mathbf{R}} = {}^{\mathcal{B}}\tilde{\boldsymbol{\omega}}$. This results in

$$\begin{pmatrix} 0 \\ {}^{\mathcal{B}}\boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} 0 \\ \omega_{b1} \\ \omega_{b2} \\ \omega_{b3} \end{pmatrix} = 2 \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{pmatrix} \begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = 2\mathbf{Q}^{\mathrm{T}} \begin{pmatrix} \dot{q}_0 \\ \dot{\boldsymbol{q}} \end{pmatrix}, \quad (18.115)$$

and the inverse expressions for the time derivatives of the Euler parameters in terms of the angular velocities expressed in the body triad $\mathcal{B}$

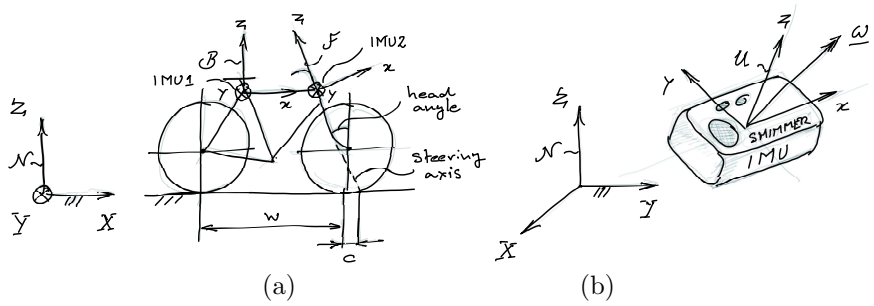
and the Euler parameters are

$$\begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \begin{pmatrix} 0 \\ \omega_{b1} \\ \omega_{b2} \\ \omega_{b3} \end{pmatrix} = \frac{1}{2} \mathbf{Q} \begin{pmatrix} 0 \\ {}^{\mathcal{B}}\boldsymbol{\omega} \end{pmatrix}. \quad (18.116)$$

In Appendix E, we introduce the concept of quaternions and demonstrate the usefulness of the quaternion algebra in determining the rotation matrix, combining Euler parameters from individual rotations for successive rotations, and the relations between the angular velocities and the time derivatives of the Euler parameters.

18.5 Problems

Problem 18.3 A group of Bachelor students measure the motions of a bicycle. To that end, the bicycle is equipped with two Inertial Measurement Units (IMUs), one on the rear frame and one on the front steering assembly, see Figure 18.14a.



(a) (b)

Figure 18.14: Experimental setup, (a): Bicycle equipped with two IMUs, in the neutral configuration. (b): IMU with reference triads.

We describe the orientation of the bicycle rear frame by means of "yaw-roll-pitch" Euler angles,

The output of one IMU is the orientation of its body-fixed triad $\mathcal{U}$ with respect to the inertial triad $\mathcal{N}$ by means of a rotation matrix ${}^{\mathcal{N}}\mathbf{R}_{\mathcal{U}}$, together with the angular velocities with respect to the inertial triad $\mathcal{N}$, about the IMU triad axes, expressed in the IMU body-fixed triad, ${}^{\mathcal{U}}\boldsymbol{\omega}$, in rad/s, see Figure 18.14b. IMU1 is mounted on the rear frame and aligned with the rear-frame triad $\mathcal{B}$. IMU2 is mounted on the steering assembly and aligned with the steering-assembly triad $\mathcal{F}$, see Figure 18.14a.

At some instant in time, the output of IMU1 is,

$$\mathbf{R}_1 = \begin{pmatrix} 0.8660 & -0.4924 & 0.0868 \\ 0.5000 & 0.8529 & -0.1504 \\ 0 & 0.1736 & 0.9848 \end{pmatrix}, \boldsymbol{\omega}_1 = \begin{pmatrix} 0.5000 \\ 0.3257 \\ 1.2803 \end{pmatrix},$$

and the output of the IMU2,

$$\mathbf{R}_2 = \begin{pmatrix} 0.6970 & -0.6965 & -0.1702 \\ 0.6401 & 0.7114 & -0.2900 \\ 0.3231 & 0.0932 & 0.9418 \end{pmatrix}, \boldsymbol{\omega}_2 = \begin{pmatrix} 0.9077 \\ 0.0940 \\ 2.0781 \end{pmatrix}.$$

The wheelbase of the bicycle is $w = 1.02$ m, the *head angle*, that is the angle between the steering axis and the z-axis of the rear frame, is 18°, and the trail of the front wheel is $c = 0.08$ m.
Please address the following questions:

a. Make a drawing of the Euler angles describing the orientation of the rear frame with respect to the inertial triad by means of the so-called *cans-in-series*. Clearly indicate the inertial triad $\mathcal{N}$, the body-fixed triad $\mathcal{B}$, and the three Euler angles (ψ, φ, θ).
b. Make a drawing of the steering angle describing the orientation of the steering assembly with respect to the body-fixed triad of the rear frame, using a so-called *can*. Clearly indicate the the body-fixed triad of the rear frame $\mathcal{B}$, the steering assembly triad $\mathcal{F}$, the steering axis and head angle, and the steering angle δ.
c. What would be an easy check to see if matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ are proper rotation matrices? Do that check.
d. Determine the yaw, roll, and pitch angle in degrees of the rear frame, (ψ, φ, θ), from the rotation matrix $\mathbf{R}_1$.
e. Determine the steer angle δ, in degrees, from the rotation matrix $\mathbf{R}_1$ and $\mathbf{R}_2$.
f. One of the student has some doubts on the alignment of IMU2. Determine the actual head angle in degrees by which IMU2 is mounted on the steering assembly from the rotation matrices $\mathbf{R}_1$ and $\mathbf{R}_2$.
g. Determine the angular rates for the yaw, roll, and pitch angles of the rear frame in rad/s, $(\dot{\psi}, \dot{\varphi}, \dot{\theta})$.
h. Determine the steer angle rate $\dot{\delta}$, in rad/s, of the steering assembly.
i. Estimate the forward speed in m/s of the bicycle, based on the yaw rate, the steer angle, and the wheelbase, using a simple bicycle model.
j. In Figure 18.15 you see two tracks made by a bicycle in the snow. Which track is made by the front wheel, and which way did the bicycle go, left-to-right or right-to-left?

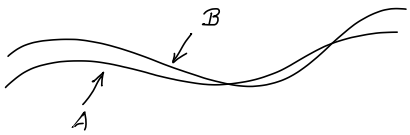


Figure 18.15: Two tracks made by a bicycle in the snow.

Problem 18.4 Consider the rotary motion of a torque-free satellite in deep space (no gravity, no drag). At a certain instant in time, which we call $t = 0$, the orientation of the body-fixed triad $\mathcal{B}$, expressed in the inertial triad $\mathcal{N}$, is given by ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_1 = (0.768,\, 0.024,\, 0.640)^{\mathrm{T}}$, ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_2 = (-0.424,\, 0.768,\, 0.480)^{\mathrm{T}}$, and ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_3 = (-0.480,\, -0.640,\, 0.600)^{\mathrm{T}}$.

a. Determine, for $t = 0$, the rotation matrix ${}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}$ that transforms the body-fixed components ${}^{\mathcal{B}}\boldsymbol{x}$ into the inertial triad components ${}^{\mathcal{N}}\boldsymbol{x}$, as in ${}^{\mathcal{N}}\boldsymbol{x} = {}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}\, {}^{\mathcal{N}}\boldsymbol{x}$.
b. Determine, for $t = 0$, from this matrix ${}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}$ the associated Euler angles (Z-X-Z) φ, θ, and ψ.

The initial angular velocity vector at $t = 0$, expressed in the inertial triad, is given by ${}^{\mathcal{N}}\boldsymbol{\omega} = (7.67952,\, 0.23936,\, 6.40060)^{\mathrm{T}}$ rad/s.

c. Determine, for $t = 0$, the angular velocities ${}^{\mathcal{B}}\boldsymbol{\omega}$ in the body-fixed triad.
d. Determine, for $t = 0$, the rate of change of the Euler angles $(\dot{\varphi},\, \dot{\theta},\, \dot{\psi})$.

The satellite is modeled by a rectangular box with mass $m = 50\,\mathrm{kg}$ and dimensions $l_x = 0.4\,\mathrm{m}$, $l_y = 1.2\,\mathrm{m}$ and $l_z = 0.3\,\mathrm{m}$ in the body-fixed coordinate system.

e. Determine the inertia tensor ${}^{\mathcal{B}}\boldsymbol{I}$ in the body-fixed triad.

Next, we want to show the motion of the satellite.

f. Write down the Euler equations of motion for the rigid body, as well as the state equations $\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y})$. As state variables $\boldsymbol{y}$, use the Euler angles (φ, θ, ψ) and the components of the angular velocity vector ${}^{\mathcal{B}}\boldsymbol{\omega}$ expressed in the body-fixed triad $\mathcal{B}$.
g. Show the motion of the satellite as a function of time by numerical integration of the state equations for 60 seconds. Plot in one figure the Euler angles (φ, θ, ψ) as a function of time and in a second figure the three components of the angular velocity vector ${}^{\mathcal{B}}\boldsymbol{\omega}$. Discuss the results.
h. The first Euler angle grows linearly with time, therefore plot the other two Euler angles, θ and ψ, as a function of time in one figure. To

invetigate what really happens, make a second figure with a 3-D plot of the trajectory in the inertial space ($\mathcal{N}$) of point P with position vector ${}^{\mathcal{B}}\boldsymbol{p} = (l_x/2, 0, 0)^{\mathrm{T}}$ of the body, for $t = 0...60\,\mathrm{s}$. Discuss the results. What happens around $\theta = 0 \pm k\pi$, $k = 0 \ldots n$?

i. Plot in one figure the three components of the angular velocity vector ${}^{\mathcal{N}}\boldsymbol{\omega}$ expressed in $\mathcal{N}$ as functions of time, and plot in a second figure the three components of the angular momentum vector ${}^{\mathcal{N}}\boldsymbol{H}_C$ expressed in $\mathcal{N}$ as functions of time. Discuss the results.

j. Which invariants (invariant with respect to time) can you use to check your time-series results? Plot these invariants as a function of time. Are they really invariant? Explain.

▶

Problem 18.5 A simple mechanical model of the human (left) arm consists of two rigid bodies connected by three hinges. The inertial triad $\mathcal{N}$ is, seen from a human perspective looking straight ahead North, the Z-axis up, the X-axis North, and the Y-axis West. The arm is an open-loop structure with, starting from the torso (the origin O), a revolute joint with an angle α about the Y-axis, a revolute joint with an angle β about the X-axis, the upper arm (body 1) with length $d = 0.3\,\mathrm{m}$ in the negative Z-direction, a revolute joint with an angle γ about the Y-axis, and finally the forearm including the hand (body 2) with length $e = 0.375\,\mathrm{m}$ in the plus X-direction. The location of the imaginary hand at the endpoint is now $\boldsymbol{r}_H = (e, 0, -d)^{\mathrm{T}}$ with the angles α, β, and γ all equal to zero. The CoM of the upper arm is located at a distance $d/2$ from the shoulder, and it has a mass of $m_1 = 1.9\,\mathrm{kg}$, whereas the CoM of the forearm (including the hand) is located at a distance $4e/10$ from the elbow, and it has a mass of $m_2 = 1.5\,\mathrm{kg}$. The body-fixed triad of the upper arm is denoted by $\mathcal{U}$ whereas the body-fixed triad of the forearm is denoted by $\mathcal{F}$. In the depicted configuration from above both triads coincided with the inertial triad $\mathcal{N}$. The mass moments of inertia of the upper arm at the CoM are ${}^{\mathcal{U}}(I_{xx}, I_{yy}, I_{zz})_1 = (0.015, 0.014, 0.002)\mathrm{kg\,m^2}$, whereas those of the forearm are ${}^{\mathcal{F}}(I_{xx}, I_{yy}, I_{zz})_2 = (0.001, 0.019, 0.019)\mathrm{kg\,m^2}$. These are principal values about the principal axes, so there are no off-diagonal terms(^{a}). We assume gravity to work in the negative Z-direction with a field strength of $g = 9.81\,\mathrm{N/kg}$.

In this problem, we will derive the equations of motion for the system expressed in terms of the independent generalized coordinates $\boldsymbol{q} = (\alpha, \beta, \gamma)$ and their time derivatives, using the TMT-method.

a. Make a sketch of the model, use cans-in-series to depict the revolute joints, and mark out the origin O including the inertial triad, the elbow location E, the location of the hand H, and the locations of the CoMs.

b. Write down the expressions for the CoM coordinates of the two bodies, $x_i = (x_1, y_1, z_1, x_2, y_2, z_2)$, expressed in terms of the generalized

coordinates $\boldsymbol{q}$ and system parameters, $x_i = T_i(q_j)$.

c. For the initial configuration, where all angles are zero, determine the three revolute joint torques, which are the three generalized forces $\boldsymbol{Q} = (M_\alpha, M_\beta, M_\gamma)$, necessary to maintain this configuration (static equilibrium). Check this with a pencil and paper calculation.

d. Write down the expressions for the angular velocities of the two bodies expressed in their individual body-fixed triads as a function of the generalized coordinates, their time derivatives, and the system parameters. Write them in a linear-in-speed form $\omega_i = B_{ij}(q_k)\dot{q}_j$.

e. Now derive the equations of motion for the arm in terms of the generalized coordinates $\boldsymbol{q}$, their time derivatives and the system parameters using the TMT-method. Remember that you have to transform both the Newton as well as the Euler equations of motion for the two bodies. Do not write the resulting equations of motion out term-by-term but just write down the symbols and steps to form these equations of motion.

f. Code these equations of motion in a MATLAB$^{\text{TM}}$ program and check your result from the calculation of some simple configurations where you can predict the resulting accelerations of generalized coordinates.

g. For the initial configuration (all angles zero) calculate the 3×3 reduced generalized mass-matrix $\bar{\boldsymbol{M}}$.

h. Now picture a ball-catching posture described by the angles $(\alpha, \beta, \gamma) = (-70°, 70°, -30°)$. Determine the three revolute joint torques necessary to maintain this posture (static equilibrium).

i. Check your result by means of a forward dynamic analysis of the system for a time period of 5 seconds (copy the static equilibrium torques from item h and paste them as applied torques in the equations of motion). Clearly, after initially being at rest, the arms start moving. Explain why this happens.

j. What happens when you start your 5 second simulation from a different equilibrium posture, for instance from $(\alpha, \beta, \gamma) = (30°, 20°, 60°)$? Explain the results. (First calculate the static equilibrium torques to maintain this new posture and then copy and paste these in the equations of motion for the forward dynamic analysis).

▶

[a] All anthropomorphic data is taken from: [10]

Problem 18.6 Consider the rotational motion of a torque-free satellite in deep space (no gravity, no drag). The body-fixed triad $\mathcal{B}$ is initially aligned with the inertial triad $\mathcal{N}$, see Figure 18.16.

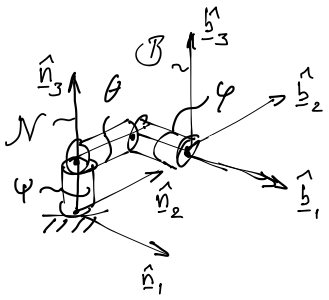


Figure 18.16: Inertial triad $\mathcal{N}$ and body-fixed triad $\mathcal{B}$.

After some time the orientation of the body has changed such that the direction vectors of the body-fixed triad expressed in the inertial triad are:

$$
\begin{aligned}
{}^{\mathcal{N}}\hat{\boldsymbol{b}}_{\mathbf{1}} &= (0.612372, 0.353553, -0.707107)^{\mathrm{T}},\\
{}^{\mathcal{N}}\hat{\boldsymbol{b}}_{\mathbf{2}} &= (0.280330, 0.739199, 0.612372)^{\mathrm{T}},\\
{}^{\mathcal{N}}\hat{\boldsymbol{b}}_{\mathbf{3}} &= (0.739199, -0.573223, 0.353553)^{\mathrm{T}}.
\end{aligned}
$$

The angular speed of the body at that moment in time, expressed in the body-fixed triad, is ${}^{\mathcal{B}}\boldsymbol{\omega} = (1,1,1)^{\mathrm{T}}$ rad/s. For the description of the orientation of the body we use Euler Angles of the type yaw-pitch-roll, which is the sequence 3-2-1, where the associated angles are denoted by ψ, θ, and φ. Determine for this configuration the roll rate, $\dot{\varphi}$, of the body.[a] ▷

[a] *Hint*: First you have to determine the values of Euler angles from the values of the rotated base vectors and the sequence of rotation. The rotated base vector form the columns of the rotation matrix ${}_{\mathcal{N}}\mathbf{R}_{\mathcal{B}}$. And you equate this to the rotation matrix in terms of ψ, θ, φ from the sequence of rotation ${}_{\mathcal{N}}\mathbf{R}_{\mathcal{B}} = \mathbf{R}(\psi)\mathbf{R}(\theta)\mathbf{R}(\varphi)$. Check: We found that $\psi = \pi/6$ and $\theta = \pi/4$. Next you derive the expressions for the body-fixed angular speed in terms of the Euler angles and the rate of change of the Euler angles, from which you then solve for the Euler angle rates.

Problem 18.7 A simple mechanical model of a 3-D robot manipulator consists of two rigid bodies connected by three revolute joints (Figure 18.17). The space-fixed coordinate system O-XYZ is with Z-up, Y-North, and X-East. The robot arm is an open-loop structure starting from the base O, a revolute joint with angle α about the Z-axis, a revolute joint with angle β about the Y-axis, the first arm with length l_1 in the positive Z-direction, a revolute joint with an angle γ about the X-axis, and finally the second arm with length l_2 in the negative Z-direction. With all angles α, β, and γ equal to zero, the location of the end effector B is $(0, 0, l_1 - l_2)$. In the middle of

each arm, there is a concentrated mass, m_1 resp. m_2. We neglect the mass moments of inertia of the arms. Gravity acts in the negative Z-direction.

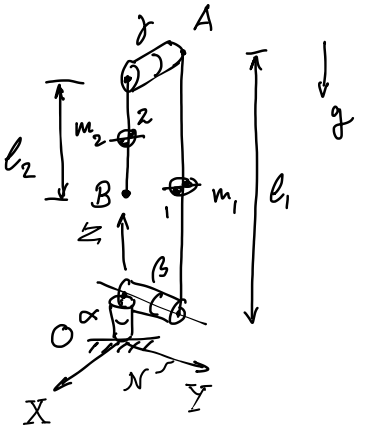


Figure 18.17: Model of a 3-D robot manipulator.

At some moment in time, the electric torque motors at the revolute joints are keeping the system at rest in the configuration $\alpha = 45°$, $\beta = 30°$, and $\gamma = 60°$. Suddenly, the voltage supply is cut off, the joint torques drop to zero, and the system starts to move. Determine the angular acceleration in the second revolute joint, $\ddot{\beta}$.[a]

For parameters, use: $l_1 = 0.4\,\mathrm{m}$, $l_2 = 0.3\,\mathrm{m}$, $m_1 = 5\,\mathrm{kg}$, $m_2 = 3\,\mathrm{kg}$, and $g = 9.81\,\mathrm{N/kg}$. ▷

[a] *Hint*: In the powered rest configuration, the center of mass of the first arm should be located at $(X, Y, Z) = (0.0707, 0.0707, 0.1732)\mathrm{m}$, and the second arm should be located at $(X, Y, Z) = (0.0230, 0.2068, 0.2815)\mathrm{m}$. In the reduced mass matrix M_{red}, the entries for $M_{\mathrm{red}}(\beta, \gamma)$ and $M_{\mathrm{red}}(\gamma, \beta)$ are always zero. When the system is at rest, the generalized force Q_α is always zero, that is the gravity forces will never exert a torque about the first revolute joint.

Problem 18.8 A simple mechanical model of a 3-D robot manipulator consists of three rigid bodies connected by three revolute joints, see Figure 18.18. The space-fixed coordinate system O-XYZ is with Z-up. The robot arm is an open-loop structure with, starting from the base O, a revolute joint with angle α about the Z-axis, the first arm with length L_1 in the Y-direction, a revolute joint with an angle β about the X-axis, the second arm with length L_2 in the Z-direction, a revolute joint with an angle γ about the Y-axis, and finally the third arm with length L_3 in the X-direction. With all angles α, β, and γ equal to zero, the location of the end effector C is (L_3, L_1, L_2). In the middle of each arm, there is a concentrated mass

m_1, respectively m_2 and m_3. We neglect the mass moments of inertia of the arms. Gravity acts in the negative Z-direction.

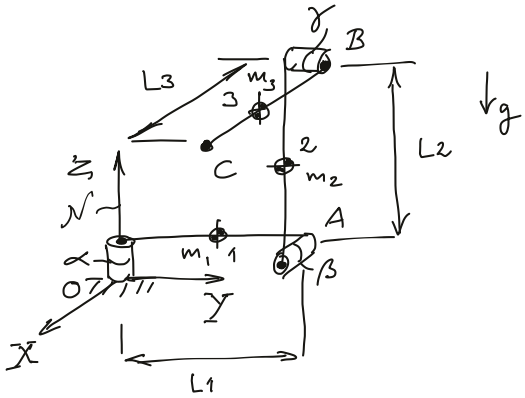


Figure 18.18: Model of a 3-D robot manipulator.

At some moment in time, the electric torque motors at the revolute joints are keeping the system at rest in the configuration $\alpha = 45°$, $\beta = 30°$ and $\gamma = 60°$. Suddenly the voltage supply is cut off, the joint torques drop to zero, and the system starts to move. Determine the angular acceleration in the second revolute joint, $\ddot{\beta}$.[a]

For parameter values use: $L_1 = 0.4\,\text{m}$, $L_2 = 0.3\,\text{m}$, $L_3 = 0.2\,\text{m}$, $m_1 = 5\,\text{kg}$, $m_2 = 3\,\text{kg}$, $m_3 = 1\,\text{kg}$, and $g = 9.81\,\text{N/kg}$. ▷

[a] *Hint*: In the powered rest configuration, the center of mass of the second arm should be located at $(x, y, z) = (-0.2298, 0.2298, 0.1299)\text{m}$, and the center of mass of the third arm should be located at $(x, y, z) = (-0.1720, 0.2428, 0.1848)\text{m}$. In the generalized mass matrix M_{gen} the entries for $M_{\text{gen}}(\beta, \gamma)$ and $M_{\text{gen}}(\gamma, \beta)$ are always zero. When the system is at rest, the generalized force Q_α is always zero, that is the gravity forces will never exert a torque about the first revolute joint.

Problem 18.9 Consider a 3-D pendulum mounted on a rotating disk, see Figure 18.19. The disk can rotate relative to the inertial triad $\mathcal{N}$ about the Z-axis through an angle α. The pendulum is mounted at a distance R from the disk center by means of two revolute joints in series. The first revolute joint is about the X-axis through an angle β and the second revolute joint is about the Y-axis through an angle γ. Attached to the CoM of the pendulum is a body-fixed triad $\mathcal{B}$ which is initially aligned with the inertial triad $\mathcal{N}$. The pendulum has a length L, with the CoM in the middle. The system is moving under the action of gravity and some applied forces.

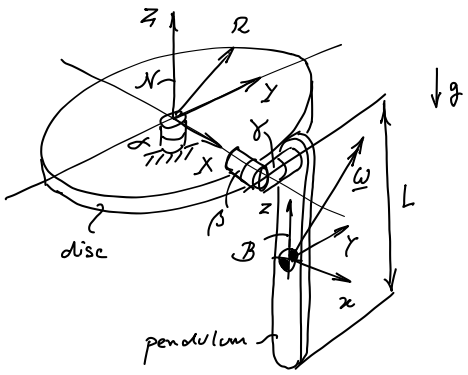


Figure 18.19: 3D pendulum on a rotating disk.

After some time the system is in a configuration described by the angles α, β, and γ. The pendulum then has angular velocity vector $\boldsymbol{\omega}$ with components expressed in the body-fixed triad $\mathcal{B}$ as ω_x, ω_y, and ω_z. Determine in this state the rate of change of the angle gamma, $\dot{\gamma}$.[a] Gravity acts in the negative Z-direction with field strength g.

The configuration of the system and the pendulum angular velocities are: $\alpha = 10^\circ$, $\beta = 20^\circ$, and $\gamma = -30^\circ$, ${}^{\mathcal{B}}\omega_x = 2\,\text{rad/s}$, ${}^{\mathcal{B}}\omega_y = 3\,\text{rad/s}$, and ${}^{\mathcal{B}}\omega_z = 3\,\text{rad/s}$.

For system parameters use: $R = 0.6\,\text{m}$, $L = 1.2\,\text{m}$, $g = 9.81\,\text{N/kg}$. ▷

[a] *Hint*: For each revolute joint, write the angle's rate of change as an angular velocity vector. Next, transform these angular velocity vectors to the body-fixed triad $\mathcal{B}$ and add them up to obtain the angular velocity of the pendulum expressed in $\mathcal{B}$. Invert the linear relation between the angular velocities and the rate of change of the three revolute joint angles. Substitute the values of the angles and the angular velocities to find the rate of change of the three revolute joint angles. For this configuration, we calculated $\dot{\alpha} = 3.828\,99\,\text{rad/s}$.

Problem 18.10 Consider a 3-D pendulum disk mounted to a rotating table, see Figure 18.20. The table, body 1, can rotate relative to the inertial coordinate system with triad $\mathcal{N}$, about the Z-axis through an angle α. The pendulum disk, body 2, is mounted at a distance R from the table center A in point B, by means of two revolute joints in series. The first revolute joint is about the Y-axis by an angle β and the second revolute joint is about the negative Z-axis by an angle γ. The table has mass m_1 and mass moment of inertia $I_{1_{ZZ}}$ at the CoM in A. The CoM of the pendulum disk is at C, a distance L from the attachment point B, with mass m_2 and mass moment of inertia $I_{2_{ZZ}}$ at the CoM (the thin rod from B to C is assumed massless and

the table and disk are very thin such that $I_{i_{XX}} = I_{i_{YY}} = I_{i_{ZZ}}/2$). Attached to the CoM of the pendulum disk 2 is a body-fixed triad $\mathcal{D}$ which is initially aligned with the inertial triad $\mathcal{N}$. In this initial configurations the angles α, β, and γ are all zero.

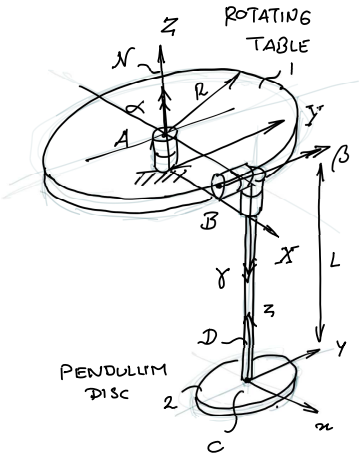


Figure 18.20: 3D pendulum disk on a rotating table.

The system is moving under the action of gravity. After some time the system is in a configuration described by the angles α, β, and γ. The pendulum then has angular velocity vector $\boldsymbol{\omega}$ with components expressed in the body-fixed triad $\mathcal{D}$ as ω_x, ω_y, and ω_y. Calculate in this state the rate of change of the angle γ, that is, $\dot{\gamma}$ in rad/s[a]. Gravity acts in the negative Z-direction with field strength g.

The configuration of the system and the pendulum disk angular velocities are: $\alpha = 30^\circ$, $\beta = 10^\circ$, $\gamma = 80^\circ$, $\omega_x = -21.0428\,\text{rad/s}$, $\omega_y = 1.594\,49\,\text{rad/s}$, $\omega_z = -21.1823\,\text{rad/s}$.

For system parameters use: $R = 0.6\,\text{m}$, $m_1 = 3\,\text{kg}$, $I_{1_{ZZ}} = 0.5\,\text{kg}\,\text{m}^2$, $L = 1.2\,\text{m}$, $m_2 = 2\,\text{kg}$, $I_{2_{ZZ}} = 0.01\,\text{kg}\,\text{m}^2$, and $g = 9.81\,\text{N/kg}$. ▷

[a] *Hint*: For each revolute joint, write the rate of change of the angle as an angular velocity vector. Next, transform each of these angular velocity vectors to the body-fixed triad $\mathcal{D}$ and add them up to obtain the angular velocity of the pendulum expressed in $\mathcal{D}$. Invert the linear relation between the angular velocities and the rate of change of the three revolute joint angles. Substitute the values of the angles and the angular velocities to find the rate of change of the three revolute joint angles. For this configuration, we find $\dot{\alpha} = 12.0000\,\text{rad/s}$.

Problem 18.11 A thrilling ride on an ejection seat reverse bungee is facilitated by two telescopic columns mounted on a semitrailer and a two-

seat spherical vehicle connected to the upper parts of the columns by means of two bungee cords (see Figure 18.21). The passenger unit is held to the base of the ride by means of a strong magnet that keeps the bungee cords stretched. Once the command is given by the operator from the control panel, the magnet releases the vehicle, launching it with an acceleration of $4.8g$. During the launch, the vehicle loops freely around its axis, making the passengers scream with fear and satisfaction about this new experience.

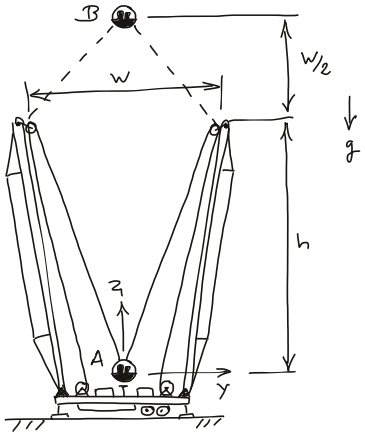


Figure 18.21: An ejection seat reverse bungee fair attraction.

The vehicle is modeled as a rigid sphere with radius $r = 1\,\mathrm{m}$, total mass $m = 420\,\mathrm{kg}$, and inertia tensor $\boldsymbol{I} = \mathrm{diag}(170,\, 120,\, 140)\mathrm{kg\,m^2}$ at the center of mass (CoM). The CoM is located at $\boldsymbol{c} = (-0.01,\, 0.01,\, -0.1)\mathrm{m}$ relative to the sphere's center.

The two bungee cords are connected at a distance $(0,\, r,\, 0)$ left and right of the sphere center, and they are modeled as linear-elastic cords with a relative material damping of 10% (in the case of a simple linear mass-spring-damper system for the cords and vehicle). The stffness k and unstretched length l_0 of the cords are designed such that the vehicle has a maximal acceleration of $4.8g$ at launch, and a maximum launch height of $h + w/2$.

Air drag is modeled by a force vector $\boldsymbol{D}$ applied at the CoM of the body, according to

$$\boldsymbol{D} = -\frac{1}{2}\rho A c_d |\boldsymbol{v}|\boldsymbol{v}, \tag{18.117P}$$

with the specific mass of the air $\rho = 1.25\,\mathrm{kg/m^3}$, the frontal area $A = \pi r^2$, the drag coefficient $c_\mathrm{d} = 0.5$, and the velocities $\boldsymbol{v}$ of the CoM of the vehicle. The dimensions of the supporting columns are $h = 25\,\mathrm{m}$ and $w = 18\,\mathrm{m}$, gravity is $g = 9.81\,\mathrm{N/kg}$.

a. Determine the stiffness k and unstretched length l_0 for each bungee cord from a simple linear mass-spring model (in a first approximation we neglect the damping). With this stiffness, determine the damping c for each bungee cord from a simple linear mass-spring-damper model.
b. Determine the motion of the vehicle by numerical integration of the equations of motion over a time period of at least 30 seconds. Give a clear representation of this motion in a number of graphs of your own choice. A minimal set should include the time history of all state variables and all state derivatives, and the forces in the bungee cords. Use Euler parameters for the description of the orientation of the body. Express the constrained equations of motion in the DAE form, and use as state variables the position of the CoM of the vehicle expressed in the global coordinate system, the four Euler parameters, the velocity of the CoM expressed in the global coordinate system, and the angular velocity of the vehicle expressed in the local coordinate system of the vehicle, as in $\boldsymbol{x} = (x,\, y,\, z,\, \lambda_0,\, \lambda_1,\, \lambda_2,\, \lambda_3,\, v_x,\, v_y,\, v_z,\, \omega'_x,\, \omega'_y,\, \omega'_z)^{\mathrm{T}}$. After each numerical integration step, renormalize the four Euler parameters according to $\lambda_i := \lambda_i / \sqrt{(\lambda_j \lambda_j)}$, to "stabilize" the constraint. Since the standard ODE solvers in MATLAB$^{\mathrm{TM}}$ have problems with this approach, it is advised to use a standard single-step numerical integrator like RK4 with fixed step-size.
c. Make an estimate on the accuracy (global error) of your solution.
d. Create a cumulative plot of the gravity potential energy V_{g}, the elastic energy V_{e}, the translational kinetic energy T_{t}, and the rotational kinetic energy T_{r} for the duration of the motion. The first line is V_{g}, the second line is $V_{\mathrm{g}} + V_{\mathrm{e}}$, the third line is $V_{\mathrm{g}} + V_{\mathrm{e}} + T_{\mathrm{t}}$, and the last line is $V_{\mathrm{g}} + V_{\mathrm{e}} + T_{\mathrm{t}} + T_{\mathrm{r}}$. The zero-potential energy level is assumed to be at launch. Discuss the graph.
e. Clearly the apex at $h + w/2$ is not reached. Change the properties of the cords such that you do reach the apex with an acceleration at launch of $a_{\max} = 4.8g$ (and keep a relative damping of 10%). Report the values of k, l_0 and c, and show the proper motion in a number of plots. Redo the energy plot.
f. Does the vehicle make a looping (= one full rotation of 360° about the global Y-axis)? Explain how you determine this and demonstrate it in a plot.
g. If one could use so-called generalized Euler angles to describe the orientation of the vehicle, and one could choose between the order of rotation Z–X–Y or Z–Y–X, which one would you prefer, and why?[a]

[a] *Hint*: draw the cans-in-series for each option.

19 Complex 3-D Multibody Systems

In this chapter, we will combine all knowledge and skills from the previous chapters to derive the constrained equations of motion for a complex three-dimensional multibody system with idealized holonomic and non-holonomic constraints.

19.1 The Bicycle Model

A bicycle can be modeled as a complex three-dimensional multibody system, using rigid bodies and constraints [45]. We model the bicycle with four rigid bodies: two wheels, the frame and the handlebar assembly, connected by three revolute joints as shown in Figure 19.1. The revolute joints and the ground contact can be expressed in terms of holonomic constraints on the system. The wheels have knife-edged no-slip contact with the ground, which can be expressed as non-holonomic constraints on the system.

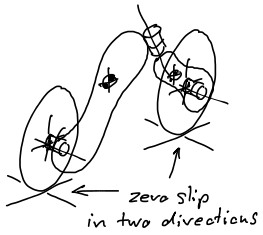


Figure 19.1: A bicycle.

We start with counting the number of degrees of freedom in the configuration space and in the velocity space:

$$
\begin{array}{rcrll}
4 \text{ bodies} & \Rightarrow & 4 \times 6 & = 24 & \text{coordinates,} \\
3 \text{ revolute joints} & \Rightarrow & 3 \times 5 & = 15 & \text{constraints,} \\
2 \text{ contact points} & \Rightarrow & 2 \times 1 & = 2 & \text{constraints on coordinates,} \\
 & & 2 \times 2 & = 4 & \text{constraints on velocities.}
\end{array}
$$

Thus, in total there are 24 coordinates, 17 constraints on coordinates (and velocities), and 4 constraints on velocities only. This leaves us with $24 - 17 = 7$ independent coordinates, and $24 - 21 = 3$ degrees of freedom in the velocities, the so-called mobility number.

Next, we identify the individual rigid bodies and number them as in Figure 19.2. We define a space-fixed coordinate system with associated triad $\mathcal{N}$, having axes xyz and an origin O at the rear contact point at time $t = 0$. Furthermore, a number of points of interest are labelled alphabetically.

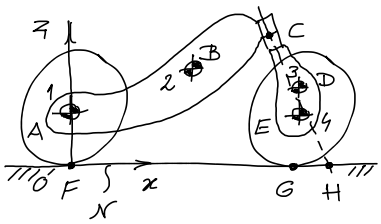


Figure 19.2: The bicycle in a space-fixed coordinate system with associated triad $\mathcal{N}$, together with some points of interest.

19.2 Equations of Motion

The equations of motion for a single unconstrained rigid body with center of mass G and body-fixed triad $\mathcal{B}$, as shown in Figure 19.3, are (Newton and Euler respectively)

$$\sum \boldsymbol{F} = m\dot{\boldsymbol{v}}_G, \tag{19.1}$$

$$\sum {}^{\mathcal{B}}\boldsymbol{M}_G = {}^{\mathcal{B}}\mathbf{I}_G \, {}^{\mathcal{B}}\dot{\boldsymbol{\omega}} + {}^{\mathcal{B}}\boldsymbol{\omega} \times \left({}^{\mathcal{B}}\mathbf{I}_G \, {}^{\mathcal{B}}\boldsymbol{\omega} \right). \tag{19.2}$$

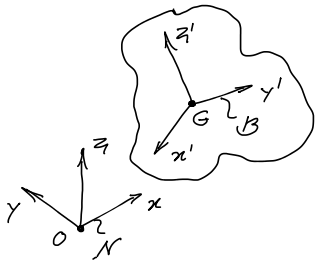


Figure 19.3: Unconstrained rigid body in an inertial triad $\mathcal{N}$, with a body-fixed coordinate system $x'y'z'$ that is associated with triad $\mathcal{B}$ and has its origin in the center of mass G.

19.2.1 State Variables

The state of the system is described by the kinematic configuration and the velocities. For each rigid body, the configuration is described by the position of the CoM $\boldsymbol{x}_G$ and the body's orientation. To describe the orientation $\boldsymbol{p}$, one can use either Euler angles, Cardan angles, Euler parameters, or some other parameterization. Each rigid body's velocities are the linear velocities of the CoM $\boldsymbol{v}_G$ and the angular velocities ${}^{\mathcal{B}}\boldsymbol{\omega}$ of the rigid body, expressed in the moving body-fixed triad $\mathcal{B}$. All together, for one rigid body, they form the state variables $(\boldsymbol{x}_G, \boldsymbol{p}, \boldsymbol{v}_G, {}^{\mathcal{B}}\boldsymbol{\omega})$.

To describe the orientation of the bicycle, we will use the following Euler angles as depicted in Figure 19.4:

$$\boldsymbol{p} = \begin{pmatrix} \varphi & \theta & \psi \end{pmatrix}^{\mathrm{T}}, \tag{19.3}$$

where

- φ is about the z-axis (heading, steer, or yaw),
- θ is about the rotated x-axis (bank, lean, or roll),
- ψ is about the rotated y-axis (pitch).

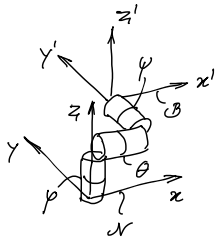


Figure 19.4: The Euler angles used for the bicycle model.

Why do we choose these particular angles? The answer to that question can be found if we look at a wheel as shown in Figure 19.5.

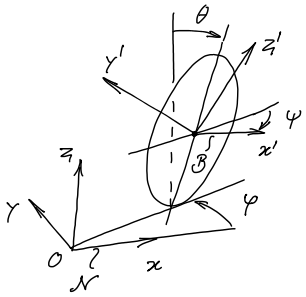


Figure 19.5: The bicycle wheel.

The last rotation, the pitch ψ, is the largest one. Therefore, this is the last angle in the Euler angle sequence. The singular configuration is at a bank angle of $\theta = \pm 90\,^\circ$, which we try to avoid in cycling.

19.2.2 Rotation Matrix and Angular Velocities

For the bicycle model, we first derive the rotation matrices for the chosen set of Euler angles:

$$^{\mathcal{N}}\boldsymbol{x} = \mathbf{R}_\varphi \mathbf{R}_\theta \mathbf{R}_\psi \, ^{\mathcal{B}}\boldsymbol{x}, \tag{19.4}$$

with

$$\mathbf{R}_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{19.5}$$

$$\mathbf{R}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \tag{19.6}$$

$$\mathbf{R}_\psi = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix}. \tag{19.7}$$

Next, we derive the angular velocities:

$$^{\mathcal{N}}\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \dot\varphi \end{pmatrix} + \mathbf{R}_\varphi \begin{pmatrix} \dot\theta \\ 0 \\ 0 \end{pmatrix} + \mathbf{R}_\varphi \mathbf{R}_\theta \begin{pmatrix} 0 \\ \dot\psi \\ 0 \end{pmatrix} \tag{19.8}$$

$$= {}^{\mathcal{N}}\mathbf{A}(\boldsymbol{p})\dot{\boldsymbol{p}}, \tag{19.9}$$

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 & \cos\varphi & -\sin\varphi\cos\theta \\ 0 & \sin\varphi & \cos\varphi\cos\theta \\ 1 & 0 & \sin\theta \end{pmatrix} \begin{pmatrix} \dot\varphi \\ \dot\theta \\ \dot\psi \end{pmatrix}, \tag{19.10}$$

or in the body-fixed triad

$$^{\mathcal{B}}\boldsymbol{\omega} = \begin{pmatrix} 0 \\ \dot\psi \\ 0 \end{pmatrix} + \mathbf{R}_\psi^{\mathrm{T}} \begin{pmatrix} \dot\theta \\ 0 \\ 0 \end{pmatrix} + \mathbf{R}_\psi^{\mathrm{T}} \mathbf{R}_\theta^{\mathrm{T}} \begin{pmatrix} 0 \\ 0 \\ \dot\varphi \end{pmatrix} \tag{19.11}$$

$$= {}^{\mathcal{B}}\mathbf{A}(\boldsymbol{p})\dot{\boldsymbol{p}}, \tag{19.12}$$

$$\begin{pmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{pmatrix} = \begin{pmatrix} -\sin\psi\cos\theta & \cos\psi & 0 \\ \sin\theta & 0 & 1 \\ \cos\psi\cos\theta & \sin\psi & 0 \end{pmatrix} \begin{pmatrix} \dot\varphi \\ \dot\theta \\ \dot\psi \end{pmatrix}. \tag{19.13}$$

The inverse mapping is

$$\dot{\boldsymbol{p}} = \left({}^{\mathcal{B}}\mathbf{A}(\boldsymbol{p})\right)^{-1} {}^{\mathcal{B}}\boldsymbol{\omega}, \tag{19.14}$$

$$\begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \frac{1}{\cos\theta} \begin{pmatrix} -\sin\psi & 0 & \cos\psi \\ \cos\theta\cos\psi & 0 & \cos\theta\sin\psi \\ \sin\theta\sin\psi & \cos\theta & -\sin\theta\cos\psi \end{pmatrix} \begin{pmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{pmatrix}, \tag{19.15}$$

which has singularities at $\theta = \frac{1}{2}\pi \pm k\pi$.

19.2.3 Kinematic Differential Equations

The kinematic differential equations are

$$\dot{\boldsymbol{x}}_{Gi} = \boldsymbol{v}_{Gi}, \tag{19.16}$$

$$\dot{\boldsymbol{p}}_i = \left({}^{\mathcal{B}i}\boldsymbol{A}(\boldsymbol{p}_i)\right)^{-1} {}^{\mathcal{B}i}\boldsymbol{\omega}_i, \tag{19.17}$$

for each body $i = 1 \dots n$.

In our bicycle model, we have 4 bodies, so $n = 4$.

19.2.4 Constrained Equations of Motion

The constrained equations of motion become

$$\begin{aligned} &\begin{pmatrix} m_1\mathbf{E}_3 & 0 & 0 & 0 & \cdots & \\ 0 & {}^{\mathcal{B}1}\mathbf{I}_{G1} & 0 & 0 & \cdots & \\ 0 & 0 & m_2\mathbf{E}_3 & 0 & \cdots & \mathbf{C}^{\mathrm{T}} \\ 0 & 0 & 0 & {}^{\mathcal{B}2}\mathbf{I}_{G2} & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ & & \mathbf{C} & & & \mathbf{0} \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{v}}_{G1} \\ {}^{\mathcal{B}1}\dot{\boldsymbol{\omega}}_1 \\ \dot{\boldsymbol{v}}_{G2} \\ {}^{\mathcal{B}2}\dot{\boldsymbol{\omega}}_2 \\ \vdots \\ \boldsymbol{\lambda} \end{pmatrix} \\ &\qquad = \begin{pmatrix} \sum \boldsymbol{F}_1 \\ \sum {}^{\mathcal{B}1}\boldsymbol{M}_1 - {}^{\mathcal{B}1}\boldsymbol{\omega}_1 \times \left({}^{\mathcal{B}1}\mathbf{I}_{G1}\, {}^{\mathcal{B}1}\boldsymbol{\omega}_1\right) \\ \sum \boldsymbol{F}_2 \\ \sum {}^{\mathcal{B}2}\boldsymbol{M}_2 - {}^{\mathcal{B}2}\boldsymbol{\omega}_2 \times \left({}^{\mathcal{B}2}\mathbf{I}_{G2}\, {}^{\mathcal{B}2}\boldsymbol{\omega}_2\right) \\ \vdots \\ \boldsymbol{g} \end{pmatrix}. \end{aligned} \tag{19.18}$$

Here the matrix $\mathbf{C}$ is the Jacobian matrix $C_{h,j}$ of the (holonomic) constraints $C_h = C_h(\boldsymbol{x}_{Gj}, \boldsymbol{p}_j)$, together with the Jacobian matrix C_n of the (non-holonomic) velocity constraints $\dot{C}_n = C_n \cdot (\dot{\boldsymbol{x}}_{Gj}, \dot{\boldsymbol{p}}_j)$. The convective acceleration terms are collected in the vector $\boldsymbol{g}$, and can be derived

from the holonomic constraint equations as $C_{h,kl} \cdot (\dot{\boldsymbol{x}}_{Gj}, \dot{\boldsymbol{p}}_j)_k \cdot (\dot{\boldsymbol{x}}_{Gj}, \dot{\boldsymbol{p}}_j)_l$ and for the non-holonomic constraints as $C_{n,k} \cdot (\dot{\boldsymbol{x}}_{Gj}, \dot{\boldsymbol{p}}_j) \cdot (\dot{\boldsymbol{x}}_{Gj}, \dot{\boldsymbol{p}}_j)_k$. We will see, further down, that it is usually more convenient to write these in terms of the body-fixed angular speeds ${}^{\mathcal{B}i}\boldsymbol{\omega}_i$, instead of the time derivatives of the Eulers $\dot{\boldsymbol{p}}_i$.

Our bicycle model has 21 constraints, so the Lagrange multiplier vector is $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{21})^{\mathrm{T}}$. The constraints are formulated in the next section.

19.3 Constraints

19.3.1 Revolute Joint

In a revolute joint as shown in Figure 19.6, the two separate bodies have a coinciding point. This decreases the number of degrees of freedom by three (the three translations). Moreover, the two revolute joint axes of the bodies have to stay parallel, resulting in another reduction by two degrees of freedom. Therefore, a revolute joint removes five degrees of freedom, and we have to formulate five constraint equations for this type of joint.

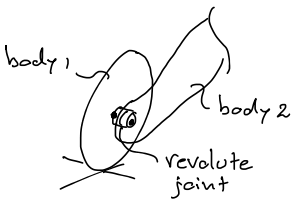


Figure 19.6: Revolute joint between the rear wheel and the bicycle frame.

The constraint equations for a revolute joint are formulated as follows. First, we look at the two coinciding points (see Figure 19.7).

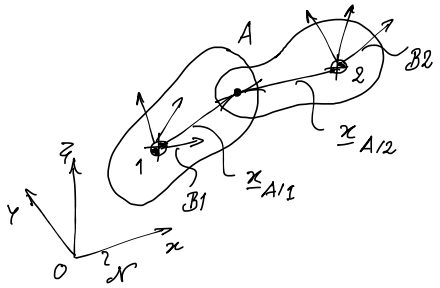


Figure 19.7: Bodies with a coinciding point.

We can write

$$\boldsymbol{C} = \boldsymbol{x}_{A2} - \boldsymbol{x}_{A1} = \begin{pmatrix} C_x \\ C_y \\ C_z \end{pmatrix} \tag{19.19}$$

$$= \boldsymbol{x}_2 + \boldsymbol{x}_{A/2} - \boldsymbol{x}_1 - \boldsymbol{x}_{A/1} \tag{19.20}$$

$$= \boldsymbol{x}_2 + \mathbf{R}_2\, {}^{\mathcal{B}2}\boldsymbol{x}_{A/2} - \boldsymbol{x}_1 - \mathbf{R}_1\, {}^{\mathcal{B}1}\boldsymbol{x}_{A/1} = \mathbf{0}, \tag{19.21}$$

with constant position vectors ${}^{\mathcal{B}1}\boldsymbol{x}_{A/1}$ and ${}^{\mathcal{B}2}\boldsymbol{x}_{A/2}$ and rotation matrices

$$\mathbf{R}_1 = {}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}1}(\boldsymbol{p}_1), \tag{19.22}$$

$$\mathbf{R}_2 = {}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}2}(\boldsymbol{p}_2). \tag{19.23}$$

The constraints on the velocities for the revolute joint are then

$$\dot{\boldsymbol{C}} = \dot{\boldsymbol{x}}_2 + \boldsymbol{\omega}_2 \times \boldsymbol{x}_{A/2} - \dot{\boldsymbol{x}}_1 - \boldsymbol{\omega}_1 \times \boldsymbol{x}_{A/1} \tag{19.24}$$

$$= \boldsymbol{v}_2 + \boldsymbol{\omega}_2 \times \boldsymbol{x}_{A/2} - \boldsymbol{v}_1 - \boldsymbol{\omega}_1 \times \boldsymbol{x}_{A/1} = \mathbf{0}. \tag{19.25}$$

Furthermore, the constraints on the accelerations are

$$\ddot{\boldsymbol{C}} = \dot{\boldsymbol{v}}_2 + \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{x}_{A/2} - \dot{\boldsymbol{v}}_1 - \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{x}_{A/1} + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \boldsymbol{x}_{A/2}) - \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \boldsymbol{x}_{A/1}) = \mathbf{0}. \tag{19.26}$$

These equations are the acceleration constraint equations we have to add to the equations of motion.

However, we like to work with body-fixed ${}^{\mathcal{B}}\boldsymbol{\omega}$ instead of inertial $\boldsymbol{\omega}$ (wherever we have omitted a superscript about the reference triad, we assume that the vector is expressed in the inertial triad $\mathcal{N}$, with vector components xyz). Therefore, the constraints in the velocities need to be re-written to

$$\dot{\boldsymbol{C}} = \boldsymbol{v}_2 + \mathbf{R}_2 \left({}^{\mathcal{B}2}\boldsymbol{\omega}_2 \times {}^{\mathcal{B}2}\boldsymbol{x}_{A/2}\right) - \boldsymbol{v}_1 - \mathbf{R}_1 \left({}^{\mathcal{B}1}\boldsymbol{\omega}_1 \times {}^{\mathcal{B}1}\boldsymbol{x}_{A/1}\right) = \mathbf{0}. \tag{19.27}$$

We can write this as a matrix–vector expression by using the tilde matrix for the cross-product as

$$\dot{\boldsymbol{C}} = \boldsymbol{v}_2 - \mathbf{R}_2\, {}^{\mathcal{B}2}\tilde{\mathbf{x}}_{A/2}\, {}^{\mathcal{B}2}\boldsymbol{\omega}_2 - \boldsymbol{v}_1 + \mathbf{R}_1\, {}^{\mathcal{B}1}\tilde{\mathbf{x}}_{A/1}\, {}^{\mathcal{B}1}\boldsymbol{\omega}_1 = \mathbf{0}, \tag{19.28}$$

or

$$\dot{\boldsymbol{C}} = \begin{pmatrix} -\mathbf{E} & \mathbf{R}_1\, {}^{\mathcal{B}1}\tilde{\mathbf{x}}_{A/1} & \mathbf{E} & -\mathbf{R}_2\, {}^{\mathcal{B}2}\tilde{\mathbf{x}}_{A/2} \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1 \\ {}^{\mathcal{B}1}\boldsymbol{\omega}_1 \\ \boldsymbol{v}_2 \\ {}^{\mathcal{B}2}\boldsymbol{\omega}_2 \end{pmatrix} \tag{19.29}$$

$$= \mathbf{C} \begin{pmatrix} \boldsymbol{v}_1 & {}^{\mathcal{B}1}\boldsymbol{\omega}_1 & \boldsymbol{v}_2 & {}^{\mathcal{B}2}\boldsymbol{\omega}_2 \end{pmatrix}^{\mathrm{T}} = \mathbf{0}. \tag{19.30}$$

Which shows the expressions for the Jacobian matrix of the constraints $\mathbf{C}$ for the revolute joint. The constraints on the accelerations are

$$\dot{\boldsymbol{C}} = \begin{pmatrix} -\mathbf{E} & \mathbf{R}_1 \, {}^{\mathcal{B}1}\tilde{\mathbf{x}}_{A/1} & \mathbf{E} & -\mathbf{R}_2 \, {}^{\mathcal{B}2}\tilde{\mathbf{x}}_{A/2} \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{v}}_1 \\ {}^{\mathcal{B}1}\dot{\boldsymbol{\omega}}_1 \\ \dot{\boldsymbol{v}}_2 \\ {}^{\mathcal{B}2}\dot{\boldsymbol{\omega}}_2 \end{pmatrix} + \boldsymbol{g} \tag{19.31}$$

$$= \mathbf{C} \begin{pmatrix} \dot{\boldsymbol{v}}_1 & {}^{\mathcal{B}1}\dot{\boldsymbol{\omega}}_1 & \dot{\boldsymbol{v}}_2 & {}^{\mathcal{B}2}\dot{\boldsymbol{\omega}}_2 \end{pmatrix}^{\mathrm{T}} + \boldsymbol{g} = \mathbf{0}, \tag{19.32}$$

where the convective terms $\boldsymbol{g}$ can be calculated from the state variables as

$$\boldsymbol{g} = \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \boldsymbol{x}_{A/2}) - \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \boldsymbol{x}_{A/1}). \tag{19.33}$$

Next, we look at the constraint of two revolute joint axes being parallel as depicted in Figure 19.8.

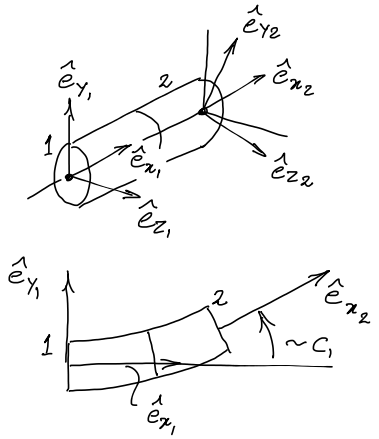


Figure 19.8: Revolute joint axes.

We want the two revolute joint axes $\hat{\boldsymbol{e}}_{x1}$ and $\hat{\boldsymbol{e}}_{x2}$ to be parallel or, in other words, $\hat{\boldsymbol{e}}_{x2} \perp \hat{\boldsymbol{e}}_{y1}$ and $\hat{\boldsymbol{e}}_{x2} \perp \hat{\boldsymbol{e}}_{z1}$. This is written in constraint form as (see also Figure 19.8)

$$C_1 = \hat{\boldsymbol{e}}_{y1}^{\mathrm{T}} \cdot \hat{\boldsymbol{e}}_{x2} = 0, \tag{19.34}$$

$$C_2 = \hat{\boldsymbol{e}}_{z1}^{\mathrm{T}} \cdot \hat{\boldsymbol{e}}_{x2} = 0. \tag{19.35}$$

We can imagine C_1 like "bending" about ${}^{\mathcal{B}1}\hat{\boldsymbol{e}}_{z1}$ and C_2 like "bending" about ${}^{\mathcal{B}1}\hat{\boldsymbol{e}}_{y1}$.

Alternatively, with $\mathbf{R}_1 = \mathbf{R}(\boldsymbol{p}_1)$ and $\mathbf{R}_2 = \mathbf{R}(\boldsymbol{p}_2)$ and referring to the body-fixed prime vectors, which are constant:

$$C_1 = \left(\mathbf{R}_1\, {}^{\mathcal{B}1}\hat{\boldsymbol{e}}_{y1}\right)^{\mathrm{T}} \cdot \left(\mathbf{R}_2\, {}^{\mathcal{B}2}\hat{\boldsymbol{e}}_{x2}\right) = {}^{\mathcal{B}1}\hat{\boldsymbol{e}}_{y1}^{\mathrm{T}} \cdot \mathbf{R}_1^{\mathrm{T}}\mathbf{R}_2\, {}^{\mathcal{B}2}\hat{\boldsymbol{e}}_{x2} = 0, \quad (19.36)$$

$$C_2 = \left(\mathbf{R}_1\, {}^{\mathcal{B}1}\hat{\boldsymbol{e}}_{z1}\right)^{\mathrm{T}} \cdot \left(\mathbf{R}_2\, {}^{\mathcal{B}2}\hat{\boldsymbol{e}}_{x2}\right) = {}^{\mathcal{B}1}\hat{\boldsymbol{e}}_{z1}^{\mathrm{T}} \cdot \mathbf{R}_1^{\mathrm{T}}\mathbf{R}_2\, {}^{\mathcal{B}2}\hat{\boldsymbol{e}}_{x2} = 0. \quad (19.37)$$

This means for velocities:

$$\dot{C}_1 = \dot{\hat{\boldsymbol{e}}}_{y1}^{\mathrm{T}} \cdot \hat{\boldsymbol{e}}_{x2} + \hat{\boldsymbol{e}}_{y1}^{\mathrm{T}} \cdot \dot{\hat{\boldsymbol{e}}}_{x2} \quad (19.38)$$

$$= \left(\boldsymbol{\omega}_1 \times \hat{\boldsymbol{e}}_{y1}\right)^{\mathrm{T}} \cdot \hat{\boldsymbol{e}}_{x2} + \hat{\boldsymbol{e}}_{y1}^{\mathrm{T}} \cdot \left(\boldsymbol{\omega}_2 \times \hat{\boldsymbol{e}}_{x2}\right) = 0. \quad (19.39)$$

However, we prefer the body-fixed coordinate system, so an expression in ${}^{\mathcal{B}}\boldsymbol{\omega}$:

$$\dot{C}_1 = \left(\mathbf{R}_1 \left({}^{\mathcal{B}1}\boldsymbol{\omega}_1 \times {}^{\mathcal{B}1}\hat{\boldsymbol{e}}_{y1}\right)\right)^{\mathrm{T}} \hat{\boldsymbol{e}}_{x2} + \hat{\boldsymbol{e}}_{y1}^{\mathrm{T}} \left(\mathbf{R}_2 \left({}^{\mathcal{B}2}\boldsymbol{\omega}_2 \times {}^{\mathcal{B}2}\hat{\boldsymbol{e}}_{x2}\right)\right) \quad (19.40)$$

$$= -\hat{\boldsymbol{e}}_{x2}^{\mathrm{T}} \cdot \mathbf{R}_1\, {}^{\mathcal{B}1}\tilde{\mathbf{e}}_{y1}\, {}^{\mathcal{B}1}\boldsymbol{\omega}_1 - \hat{\boldsymbol{e}}_{y1}^{\mathrm{T}} \cdot \mathbf{R}_2\, {}^{\mathcal{B}2}\tilde{\mathbf{e}}_{y2}\, {}^{\mathcal{B}2}\boldsymbol{\omega}_2 = 0. \quad (19.41)$$

Apparently, the Jacobian matrix "$\mathbf{C}$" is given by

$$\frac{\partial \dot{C}_1}{\partial\, {}^{\mathcal{B}1}\boldsymbol{\omega}_1} = -\hat{\boldsymbol{e}}_{x2}^{\mathrm{T}}\mathbf{R}_1\, {}^{\mathcal{B}1}\tilde{\mathbf{e}}_{y1}, \quad (19.42)$$

$$\frac{\partial \dot{C}_1}{\partial\, {}^{\mathcal{B}2}\boldsymbol{\omega}_2} = -\hat{\boldsymbol{e}}_{y1}^{\mathrm{T}}\mathbf{R}_2\, {}^{\mathcal{B}2}\tilde{\mathbf{e}}_{x2}. \quad (19.43)$$

Finally, the constraint on the accelerations is:

$$\begin{aligned}\ddot{C}_1 =& \left(\dot{\boldsymbol{\omega}}_1 \times \hat{\boldsymbol{e}}_{y1}\right)^{\mathrm{T}} \hat{\boldsymbol{e}}_{x2} + \hat{\boldsymbol{e}}_{y1}^{\mathrm{T}} \left(\dot{\boldsymbol{\omega}}_2 \times \hat{\boldsymbol{e}}_{x2}\right) \\ &+ \left(\boldsymbol{\omega}_1 \times \left(\boldsymbol{\omega}_1 \times \hat{\boldsymbol{e}}_{y1}\right)\right)^{\mathrm{T}} \hat{\boldsymbol{e}}_{x2} + \left(\boldsymbol{\omega}_1 \times \hat{\boldsymbol{e}}_{y1}\right)^{\mathrm{T}} \left(\boldsymbol{\omega}_2 \times \hat{\boldsymbol{e}}_{x2}\right) \\ &+ \left(\boldsymbol{\omega}_1 \times \hat{\boldsymbol{e}}_{y1}\right)^{\mathrm{T}} \left(\boldsymbol{\omega}_2 \times \hat{\boldsymbol{e}}_{x2}\right) + \hat{\boldsymbol{e}}_{y1}^{\mathrm{T}} \left(\boldsymbol{\omega}_2 \times \left(\boldsymbol{\omega}_2 \times \hat{\boldsymbol{e}}_{x2}\right)\right) \\ =& \left(\dot{\boldsymbol{\omega}}_1 \times \hat{\boldsymbol{e}}_{y1}\right)^{\mathrm{T}} \hat{\boldsymbol{e}}_{x2} + \hat{\boldsymbol{e}}_{y1}^{\mathrm{T}} \left(\dot{\boldsymbol{\omega}}_2 \times \hat{\boldsymbol{e}}_{x2}\right) + g_1 = 0,\end{aligned} \quad (19.44)$$

with g_1 the convective terms, which belong to the right-hand side of the equations of motion.

Of course

$$\frac{\partial \ddot{C}_1}{\partial \dot{\boldsymbol{\omega}}_1} = \frac{\partial \dot{C}_1}{\partial \boldsymbol{\omega}_1}, \quad (19.45)$$

so this Jacobian matrix (here a vector) is always the same as the one from the velocities.

The same procedure can be applied to the second constraint, C_2, to derive the corresponding constraint equations for the velocities and the accelerations.

19.3.2 Wheel Contact

For the constraints at the wheel contact point we look at the wheel as shown in Figure 19.9.

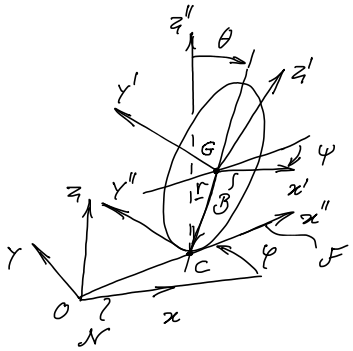


Figure 19.9: Wheel in contact with the ground.

The velocity of the contact point C is

$$\dot{\boldsymbol{x}}_C = \dot{\boldsymbol{x}}_G + \boldsymbol{\omega} \times \boldsymbol{r}. \tag{19.46}$$

"No slip" means that this velocity is zero, $\dot{\boldsymbol{x}}_C = 0$. In fact, we have one contact condition,

$$z_G = r \cos\theta, \tag{19.47}$$

and two zero-slip conditions,

$$\dot{x}_C = 0, \tag{19.48}$$

$$\dot{y}_C = 0. \tag{19.49}$$

However, we like to express the zero-slip constraints in a local triad $\mathcal{F}$ with components $x''y''z''$, which only includes the yaw angle φ of the wheel, and not the roll and pitch angle, so that we obtain

$$\text{longitudinal slip} \qquad \dot{x}''_C \neq 0, \tag{19.50}$$

$$\text{lateral slip} \qquad \dot{y}''_C \neq 0, \tag{19.51}$$

and

$${}^{\mathcal{F}}\dot{\boldsymbol{x}}_C = {}^{\mathcal{F}}\dot{\boldsymbol{x}}_G + {}^{\mathcal{F}}\boldsymbol{\omega} \times {}^{\mathcal{F}}\boldsymbol{r}. \tag{19.52}$$

The two right-hand side summands evaluate to ${}^{\mathcal{F}}\dot{\boldsymbol{x}}_G = \mathbf{R}_\varphi^{\mathrm{T}} \dot{\boldsymbol{x}}_G$ and $\left({}^{\mathcal{F}}\boldsymbol{\omega} \times {}^{\mathcal{F}}\boldsymbol{r}\right) = \mathbf{R}_\theta \mathbf{R}_\psi \left({}^{\mathcal{B}}\boldsymbol{\omega} \times {}^{\mathcal{B}}\boldsymbol{r}\right)$. With reference to Figure 19.10, we

obtain for the instantaneous radius vector $\boldsymbol{r}$ pointing at the contact point C, expressed in the body-fixed triad $\mathcal{B}$:

$$ {}^{\mathcal{B}}\boldsymbol{r} = \begin{pmatrix} r\sin\psi \\ 0 \\ -r\cos\psi \end{pmatrix}. \tag{19.53} $$

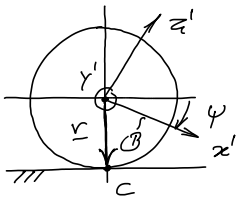


Figure 19.10: Instantaneous radius vector $\boldsymbol{r}$ pointing at the contact point C, together with the body-fixed triad $\mathcal{B}$ and the pitch angle ψ.

Evaluating ${}^{\mathcal{F}}\dot{\boldsymbol{x}}_C$ (preferably by using some software capable of doing symbolic calculations, e.g. MATLAB$^{\mathrm{TM}}$) simplifies to:

$$ \begin{pmatrix} \dot{x}''_C \\ \dot{y}''_C \\ \dot{z}''_C \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 & 0 & -r & 0 \\ -\sin\varphi & \cos\varphi & 0 & r\cos\theta\cos\psi & 0 & r\cos\theta\sin\psi \\ 0 & 0 & 1 & r\sin\theta\cos\psi & 0 & r\sin\theta\sin\psi \end{pmatrix} \begin{pmatrix} \dot{x}_G \\ \dot{y}_G \\ \dot{z}_G \\ \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{pmatrix}. \tag{19.54} $$

The last row is the constraint $\dot{z} + r\dot{\theta}\sin\theta = 0$ (one can check this with $\dot{\theta}$ from $\dot{p} = ({}^{\mathcal{B}}\mathbf{A}(\boldsymbol{p}))^{-1}\,{}^{\mathcal{B}}\boldsymbol{\omega}$ in terms of ${}^{\mathcal{B}}\boldsymbol{\omega}$, i.e. $\dot{\theta} = \omega_{x'}\cos\psi + \omega_{z'}\sin\psi$), which originates from the *holonomic* contact condition

$$ C_1 = Z - r\cos\theta = 0. \tag{19.55} $$

The first two rows are the two *non-holonomic* constraints on the slip:

$$ \text{longitudinal} \qquad \dot{C}_2 = \dot{x}\cos\varphi + \dot{y}\sin\varphi - r\omega_{y'} = 0, \tag{19.56} $$

$$ \text{lateral} \qquad \dot{C}_3 = -\dot{x}\sin\varphi + \dot{y}\cos\varphi + r\omega_{x'}\cos\theta\cos\psi + r\omega_{z'}\cos\theta\sin\psi = 0. \tag{19.57} $$

Now, we only have to find the convective terms for the wheel con-

straint. First, we substitute $\boldsymbol{\omega}$ in terms of $\boldsymbol{p}$ and $\dot{\boldsymbol{p}}$ in the $\dot{\boldsymbol{C}}$ equations:

$$\begin{pmatrix}\dot{C}_1\\ \dot{C}_2\\ \dot{C}_3\end{pmatrix} = \begin{pmatrix}0 & 0 & 1 & 0 & r\sin\theta & 0\\ \cos\varphi & \sin\varphi & 0 & -r\sin\theta & 0 & -r\\ -\sin\varphi & \cos\varphi & 0 & 0 & r\cos\theta & 0\end{pmatrix}\begin{pmatrix}\dot{x}_G\\ \dot{y}_G\\ \dot{z}_G\\ \dot{\varphi}\\ \dot{\theta}\\ \dot{\psi}\end{pmatrix} = \mathbf{0}. \tag{19.58}$$

Next, we differentiate with respect to time and disregard all second-derivative terms. This delivers the convective terms

$$\begin{pmatrix}g_1\\ g_2\\ g_3\end{pmatrix} = \begin{pmatrix}r\dot{\theta}^2\cos\theta\\ -\dot{x}\dot{\varphi}\sin\varphi + \dot{y}\dot{\varphi}\cos\varphi - r\dot{\varphi}\dot{\theta}\cos\theta\\ -\dot{x}\dot{\varphi}\cos\varphi - \dot{y}\dot{\varphi}\sin\varphi - r\dot{\theta}^2\sin\theta\end{pmatrix}. \tag{19.59}$$

Finally, substituting $\dot{\boldsymbol{p}} = ({}^{\mathcal{B}}\mathbf{A}(\boldsymbol{p}))^{-1}\,{}^{\mathcal{B}}\boldsymbol{\omega}$ provides $\boldsymbol{g}$ in terms of the state variables $\boldsymbol{p}$, $\boldsymbol{v}$, and ${}^{\mathcal{B}}\boldsymbol{\omega}$.

19.4 Numerical Integration

We will use the coordinate projection method to minimize the constraint errors after each function evaluation during the numerical integration. In this scheme, we rely on the Moore–Penrose pseudo-inverse $\mathbf{C}^+$, the Jacobian matrix of the constraints $\mathbf{C}$ as in $\mathbf{C}^+ = \mathbf{C}^{\mathrm{T}}(\mathbf{C}\mathbf{C}^{\mathrm{T}})^{-1}$. Here, the Jacobian matrix is $\mathbf{C} = \partial\boldsymbol{C}/\partial(\boldsymbol{x}_i, \boldsymbol{p}_i)$, the partial derivatives of the constraints with respect to the state variables $\boldsymbol{x}_i$ and $\boldsymbol{p}_i$. This is not equal to the Jacobian matrix $\mathbf{C}$ which we have used in the constraint equations of motion, since there we had the partial derivatives of the velocities with respect to the state variables $\boldsymbol{v}_i$ and ${}^{\mathcal{B}}\boldsymbol{\omega}_i$, as in $\partial\dot{\boldsymbol{C}}/\partial(\boldsymbol{v}_i, {}^{\mathcal{B}}\boldsymbol{\omega}_i)$. Given the latter, we can easily transform to the former with the matrices ${}^{\mathcal{B}}\mathbf{A}(\boldsymbol{p}_i)$, since $\partial\,{}^{\mathcal{B}}\boldsymbol{\omega}_i/\partial\dot{\boldsymbol{p}}_i = {}^{\mathcal{B}}\mathbf{A}(\boldsymbol{p}_i)$, and so

$$\frac{\partial\boldsymbol{C}}{\partial(\boldsymbol{x}_i, \boldsymbol{p}_i)} = \frac{\partial\dot{\boldsymbol{C}}}{\partial(\boldsymbol{v}_i, {}^{\mathcal{B}}\boldsymbol{\omega}_i)} \cdot {}^{\mathcal{B}}\mathbf{A}(\boldsymbol{p}_i). \tag{19.60}$$

Example 19.1 To illustrate the method, we investigate the "two points coincide" constraint. For the velocity constraint equations, we derived

$$\dot{\boldsymbol{C}} = \begin{pmatrix}-\mathbf{E} & \mathbf{R}_1\,{}^{\mathcal{B}}\tilde{\mathbf{x}}_{A/1} & \mathbf{E} & -\mathbf{R}_2\,{}^{\mathcal{B}}\tilde{\mathbf{x}}_{A/2}\end{pmatrix}\begin{pmatrix}\boldsymbol{v}_1 & {}^{\mathcal{B}}\boldsymbol{\omega}_1 & \boldsymbol{v}_2 & {}^{\mathcal{B}}\boldsymbol{\omega}_2\end{pmatrix}^{\mathrm{T}}, \tag{19.61E}$$

which transforms to

$$\Delta \boldsymbol{C} = \begin{pmatrix} -\mathbf{E} & \mathbf{R}_1 \, {}^{\mathcal{B}}\tilde{\mathbf{x}}_{A/1} \, {}^{\mathcal{B}}\mathbf{A}_1 & \mathbf{E} & -\mathbf{R}_2 \, {}^{\mathcal{B}}\tilde{\mathbf{x}}_{A/2} \, {}^{\mathcal{B}}\mathbf{A}_2 \end{pmatrix} \begin{pmatrix} \Delta \boldsymbol{x}_1 \\ \Delta \boldsymbol{p}_1 \\ \Delta \boldsymbol{x}_2 \\ \Delta \boldsymbol{p}_2 \end{pmatrix} . \quad \text{(19.62E)}$$

Note that this coordinate projection method only operates on the holonomic constraints, which are in our bicycle problem a total of 17 constraints.

Finally, we have to determine speeds $\boldsymbol{v}_i$ and ${}^{\mathcal{B}}\boldsymbol{\omega}_i$ that fulfill the velocity constraints. At this point, we include the non-holonomic constraints. To minimize velocity constraint errors, we use the coordinate projection method again, but since we work in velocity space, we can use the Jacobian matrix "$\mathbf{C}$" as derived for the constrained EoM. Since the equations are linear in the speeds, this is a one-step iteration.

19.5 Problems

Problem 19.1 Dynamic Analysis of the Motion of a Bicycle: A bicycle is modeled by four rigid bodies: the rear wheel, the frame with a rigid rider attached, the fork with the handle bars assembly forming the front frame, and the front wheel. The configuration of the bicycle will be described in a space-fixed Cartesian coordinate system O-xyz.

The rear wheel and the front wheel are in contact with a horizontal flat plane, such that we assume no slip in both the longitudinal and the lateral directions of the wheel. The rear wheel is connected to the frame by a revolute joint with the principal joint axis oriented in the y–direction. The (rear) frame is connected to the front frame by a revolute joint with the principal joint axis in the xz-plane. The front wheel is connected to the front frame by a revolute joint with the principal axis in the y-direction.

The rear wheel and front wheel are identical, with a diameter of 0.7 m, a mass of 2 kg, and principal moments of inertia $\mathbf{J} = (J_{xx}, J_{yy}, J_{zz}) = (0.12, 0.24, 0.12)\text{kg m}^2$. The wheel base is 1.2 m. The center of mass of the frame with a rigid rider attached is located at (0.55, 0, 0.5)m, with a mass of 96 kg, and it has principal moments of inertia $\mathbf{J} = (16.8, 24, 5.2)\text{kg m}^2$. The center of mass of the front frame is located at (1.0, 0, 0.7)m with a mass of 1 kg and zero moments of inertia. The steering revolute joint is located at (0.9, 0, 0.9)m, pointing in the (0.36, 0, −0.90) direction. Assume a gravity field strength of $g = 10\,\text{N/kg}$.

In the initial configuration, the bicycle is in the upright position, with the handle bars pointing straight ahead in the x-direction.

a. Derive the equations of motion for the bicycle according to the method

described in this chapter.

b. Simulate the motion of the bicycle for 10 seconds. The initial conditions are an upright position with the steering straight ahead, an initial forward speed of 4.2 m/s, an initial lean angular velocity for the whole assembly of −0.5 rad/s, and zero steering velocity. Give a clear representation of the motion in a number of graphs of your own choice. Representative answers are shown in Figure 19.11.

c. Use your simulation to address one of the following questions or one of your own:

   1) Below 3.5 m/s, this bicycle is unstable. Is it stable for all speeds above?
   2) Does the bicycle have any stable motions besides upright straight ahead?
   3) By changing mass and geometry parameters, what is the slowest speed for which you can find a stable bicycle?
   4) A folklore in bicycle science is that the wheel reaction forces in the contact point are in the plane of the wheel. Is this true?

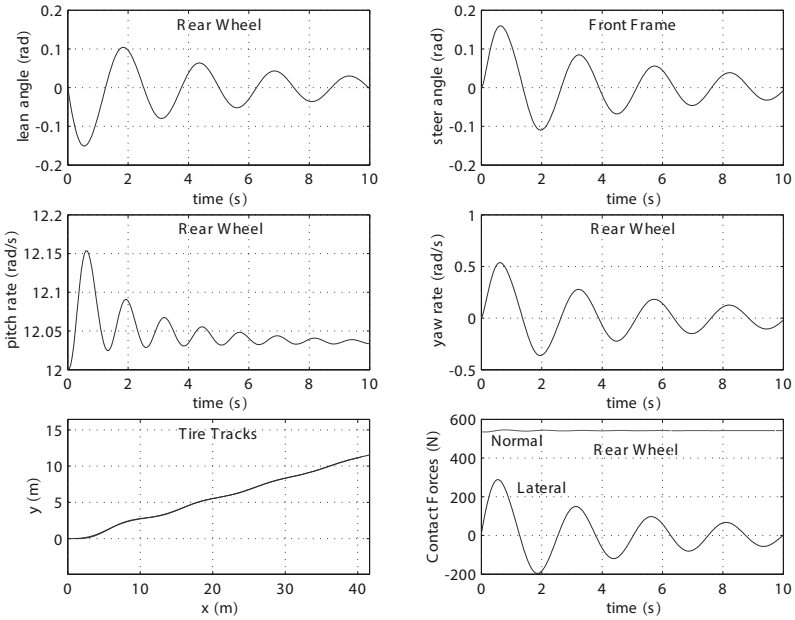


Figure 19.11: Time series of the simulation of a bicycle with the initial conditions: an upright position with the steering straight ahead, an initial forward speed of 4.2 m/s, an initial lean angular velocity for the whole assembly of −0.5 rad/s, and zero steering velocity.

Appendices

A Vector and Matrix Algebra

A.1 Vector Addition

Vectors are added (Figure A.1) by adding their individual components. For the addition of vectors, equal rules apply as for scalars summation.

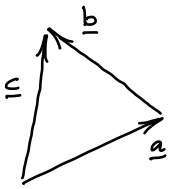


Figure A.1: Vector addition: The vector $\boldsymbol{c}$ is found by adding $\boldsymbol{a}$ and $\boldsymbol{b}$.

Example A.1 The meaning of vector addition is particularly intuitive for position vectors: Assume for example that the vector $\boldsymbol{r}_{A/O}$ specifies the location of a point A with respect to another point O, and that the vector $\boldsymbol{r}_{C/A}$ specifies the location of a point C with respect to A. Then, the position vector $\boldsymbol{r}_{C/O}$, which points from O to C, is given by $\boldsymbol{r}_{C/O} = \boldsymbol{r}_{A/O} + \boldsymbol{r}_{C/A}$.

Problem A.1 In Figure A.1, assume the components of $\boldsymbol{c}$ and $\boldsymbol{a}$ are known. How do you calculate $\boldsymbol{b}$?

Problem A.2 If you know $\boldsymbol{r}_{C/O}$, how do you calculate $\boldsymbol{r}_{O/C}$?

A.2 Multiplication of a Vector with a Scalar

Multiplication of a vector with a scalar means multiplication of each single vector component by this scalar. This operation changes a vector's magnitude, but *not* its direction. So, scalar multiplication only "scales" a vector.

Accordingly, one can represent any vector by a multiplication of a scalar value (its magnitude) and a unit direction vector. This is particularly helpful when solving problems where the direction of a vector

is known, but its magnitude is not. This situation occurs frequently in mechanical systems.

◯ **Problem A.3** Consider again Figure A.1, and assume we are in a 2-dimensional space (so in $\mathbb{R}^2$). Also assume that we know the vector $\boldsymbol{a}$ including its magnitude and direction, but for the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ we only know their directions, given by the unit vectors $\hat{\boldsymbol{e}}_b$ and $\hat{\boldsymbol{e}}_c$, respectively. We are interested in setting up the equations that allow calculating the respective magnitudes b and c of the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$. ▷►

◯ **Problem A.4** Consider Problem A.3, now assume the vectors are defined in $\mathbb{R}^3$. Will there always be a solution for b and c? If not, what is/are the condition(s) on the given variables such that a solution exists?

A.3 Cross Product and Tilde Matrix

The *cross product* $\boldsymbol{a}\times\boldsymbol{b}$ is a vector perpendicular to both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The cross product of two vectors is the zero vector if both vectors have the same or the exact opposite direction (i.e. the vectors are linearly dependent). The direction of the cross product is determined by the right-hand rule (Figure A.2).

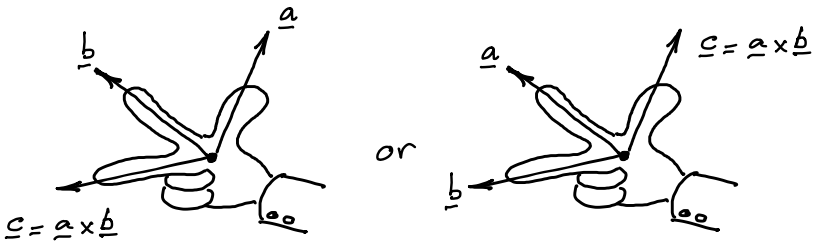


Figure A.2: Right-hand rule.

The magnitude of the cross product equals the area of the parallelogram with the vectors as sides, see Figure A.3. So, the magnitude of a cross product $\boldsymbol{c}$ of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}, \tag{A.1}$$

can be calculated using:

$$|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}| \sin(\theta), \tag{A.2}$$

where $\theta \in [0, \pi]$ is the angle enclosed by the two vectors.

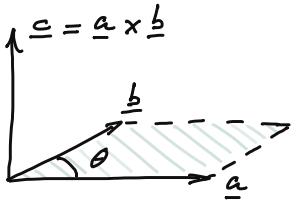


Figure A.3: The magnitude of the cross product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ describes the area of the parallelogram, spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$, which are separated by angle θ.

Problem A.5 What is the cross product of a vector with itself?

The cross product is used very often in dynamic calculations, for example for moments, angular momenta etc.

When writing the components of the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as

$$\boldsymbol{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}, \tag{A.3}$$

the cross product is defined as:

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}. \tag{A.4}$$

From its definition, it becomes evident that the cross product of two vectors exists only in three-dimensional spaces. The cross product is *anti-commutative*, meaning that $\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{a})$.

Problem A.6 Use the definition of the cross product to show that $\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{a})$.

Using the *tilde matrix* $\tilde{\mathbf{a}}$, the definition of the cross product, (A.4), can be re-written as a matrix product, namely

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \tilde{\mathbf{a}}\boldsymbol{b}, \tag{A.5}$$

In case we are interested in writing the cross product in terms of a tilde matrix function of vector $\boldsymbol{b}$, we make use of the anti-commutative property $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$, so $\boldsymbol{a} \times \boldsymbol{b} = -\tilde{\mathbf{b}}\boldsymbol{a}$. Also, we can recognize that the tilde matrix is skew-symmetric, so $\tilde{\mathbf{b}}^{\mathrm{T}} = -\tilde{\mathbf{b}}$. This means that alternatively we can write $\boldsymbol{a} \times \boldsymbol{b} = \tilde{\mathbf{b}}^{\mathrm{T}}\boldsymbol{a}$.

◯ **Problem A.7** What is the relationship between the transpose $\tilde{\mathbf{a}}^{\mathrm{T}}$ of the tilde matrix and its negative $-\tilde{\mathbf{a}}$?

A.4 Dot Product

The *dot product* of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, as illustrated in Figure A.4, is defined as

$$c = \boldsymbol{a} \cdot \boldsymbol{b} := |\boldsymbol{a}||\boldsymbol{b}| \cos(\theta), \tag{A.6}$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Note that the dot product results in a scalar value c, so it is also called the *scalar product*.

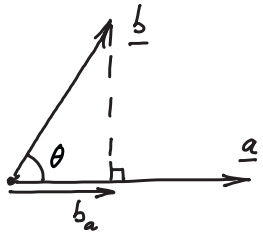


Figure A.4: Dot product.

From this formulation and Figure A.4, a very important property can be seen: The *scalar projection* of $\boldsymbol{b}$ onto $\boldsymbol{a}$, i.e. the component of vector $\boldsymbol{b}$ in direction of $\boldsymbol{a}$, is the signed magnitude $b_{\boldsymbol{a}}$:

$$b_{\boldsymbol{a}} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|}. \tag{A.7}$$

So, to calculate the component of a vector $\boldsymbol{b}$ in a particular direction, one takes the scalar product of $\boldsymbol{b}$ with a unit vector (so a vector having a length equal to 1) in that particular direction. It also implies that if two vectors are orthogonal (perpendicular), their dot product is zero.

In contrast, the *vector projection* of vector $\boldsymbol{b}$ onto $\boldsymbol{a}$ is a vector $\boldsymbol{b}_{\boldsymbol{a}}$:

$$\boldsymbol{b}_{\boldsymbol{a}} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|} \cdot \frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|^2} \cdot \boldsymbol{a}. \tag{A.8}$$

◯ **Problem A.8** Consider a vector that points in the direction of the x-axis. Use the scalar product to calculate the scalar projection of this vector onto the y-axis.

When using the same components as in (A.3), the dot product resolves to

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z. \tag{A.9}$$

So, another way to write the dot product is via matrix notation:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^{\mathrm{T}} \boldsymbol{b}. \tag{A.10}$$

Note that the dot product does not only exist in $\mathbb{R}^3$, but also in any other dimension. More broadly in $\mathbb{R}^n$, it is defined as

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{n} a_i b_i. \tag{A.11}$$

In contrast to the cross product, the dot product is commutative (which can be seen directly from (A.9)):

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}. \tag{A.12}$$

◯ **Problem A.9** This Problem is about the top of corks and skewers of Problem 1.8 on page 12. Calculate the distance h from point Q to the ground as a function of the vector $\boldsymbol{r}_{Q/P}$, using a scalar product. ►

◯ **Problem A.10** Try calculating dot products and cross products by using MATLAB™.

A.5 Basic Matrix Definitions

Among other interpretations, matrices can be seen as *linear transformations* from one space to another: When a vector $\boldsymbol{x} \in \mathbb{R}^n$ is pre-multiplied by an $m \times n$-matrix $\mathbf{A}$, the result is another vector $\boldsymbol{y} \in \mathbb{R}^m$, mapped to the n-dimensional *image* space of $\mathbf{A}$. This space is spanned by the column vectors of $\mathbf{A}$.

The *rank* of $\mathbf{A}$ is the same as the dimension of the column space or image. It is important to note that the rank can be lower than n, because it could be that some columns are *linearly dependent*, meaning that they can be constructed by linear combinations of other columns. In that case, the columns cannot span an n-dimensional space.

◯ **Problem A.11** What is the rank of an $n \times n$ identity matrix?

◯ **Problem A.12** What is the maximum rank that a 3×4-matrix can have? Provide a reasoning for your answer.

◯ **Problem A.13** What is the rank of the tilde matrix of an arbitrary nonzero vector? Interpret this result by answering the question: How many possibilities are there for vector $\boldsymbol{b}$ in (A.1) if $\boldsymbol{c}$ and $\boldsymbol{a}$ are given?

There are some matrices with a special structure, for example *square* matrices, where $m = n$. Within this subgroup, there are for example *diagonal* matrices, where all values except for those on the diagonal are zero, or *upper triangular* matrices (also called *right triangular* matrices), where all values below the diagonal are zero.

The *transpose* $\mathbf{A}^{\mathrm{T}}$ of an arbitrary matrix $\mathbf{A}$ is another matrix that contains the columns of $\mathbf{A}$ as rows. For *symmetric* matrices, $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$, which means that all entries are mirror-symmetric with respect to the diagonal.

◯ **Problem A.14** Can non-square matrices be symmetric?

The *trace* of a square matrix is defined as the sum of all diagonal elements of this matrix.

A critical characteristic of a square matrix $\mathbf{A}$ is its *determinant* $|\mathbf{A}|$. The determinant is a scalar value and can be calculated and interpreted as described in Chapter 3 of [40].

◯ **Problem A.15** Calculate the determinant of a diagonal matrix.

◯ **Problem A.16** Calculate the determinant of an upper triangular matrix.

◯ **Problem A.17** If you know the determinant of a matrix $\mathbf{A}$, how can you find the determinant of $\mathbf{A}^{\mathrm{T}}$?

Using its determinant, one can also calculate the *inverse* of a square matrix, as described in Sections 2 and 3 of [40]. When pre- or post-multiplying a matrix $\mathbf{A}$ with its inverse $\mathbf{A}^{-1}$, one obtains the identity matrix. However, not all square matrices can be inverted. Matrices that have an inverse are called *invertible* or *nonsingular* matrices, while those without an inverse are called *singular* matrices.

Consider a square matrix that has *deficient* rank, so where the rank is lower than the number of columns. In that case, mapping a vector $\boldsymbol{x}$ to its counterpart $\boldsymbol{y}$ in the image space, via $\mathbf{A}\boldsymbol{x} = \boldsymbol{y}$ means compressing the original dimension of $\boldsymbol{x}$ to a lower dimensionality.

Example A.2 Consider a vector $\boldsymbol{x} = (1,\, 2,\, 3)^{\mathrm{T}}$ in 3-D that is mapped onto a paper plane via pre-multiplication with a matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We notice that this matrix only has rank 2. So, even though it also has three components, the vector $\boldsymbol{y} = \mathbf{A}\boldsymbol{x} = (1,\, 2,\, 0)^{\mathrm{T}}$ is merely a two-dimensional projection of the original vector $\boldsymbol{x}$. If we try to reconstruct the original vector from its shadow, we will not succeed, because there is an infinite number of possible original vectors, for example $(1,\, 2,\, 0)^{\mathrm{T}}$ or $(1,\, 2,\, 325)^{\mathrm{T}}$. That means there cannot be a unique inverse $\mathbf{A}^{-1}$ such that one can be sure that $\mathbf{A}^{-1}\boldsymbol{y} = \boldsymbol{x}$.

Problem A.18 What can you say about the inverse of a matrix that has a determinant of 0?

For a matrix that has a rank lower than its number of columns, there is always a set of vectors that must all map to the same image, namely to the zero vector. The set of these vectors is called the *nullspace* or *kernel* of a matrix.

Problem A.19 What is the relationship for a matrix between its number of columns, the dimension of its kernel, and its rank?

Problem A.20 Use the MATLAB$^{\mathrm{TM}}$ command `null` to calculate the nullspace of the matrix $\mathbf{A}$ from Example A.2. Does the result match your expectations?

Definiteness of a symmetric matrix is an important property that can be employed for example for stability analysis. If a matrix $\mathbf{A}$ is positive definite, it means that for any nonzero vector $\boldsymbol{x}$, the scalar $\boldsymbol{x}^{\mathrm{T}}\mathbf{A}\boldsymbol{x}$ is positive. If $\mathbf{A}$ is negative definite, the same scalar will be always negative. Semi-definiteness extends to the case where $\boldsymbol{x}^{\mathrm{T}}\mathbf{A}\boldsymbol{x}$ may also be zero for some vectors $\boldsymbol{x}$.

Problem A.21 Consider the two-dimensional identity matrix. Can you make a statement about its definiteness?

Problem A.22 Consider a general two-dimensional diagonal matrix $\mathbf{D}$ with only negative entries on the diagonal. Can you make a general statement about the definiteness of such a matrix?

A.6 Eigenvectors and Eigenvalues

A vector $\boldsymbol{a}$ that does not change its direction when it is pre-multiplied by a square matrix $\mathbf{C}$ (which need not be the identity matrix) is called an *eigenvector* of this matrix. The multiplication factor needed to represent the change in length is called the associated *eigenvalue* λ:

$$\mathbf{C}\boldsymbol{a} = \lambda \boldsymbol{a}. \tag{A.13}$$

If we are interested in finding an eigenvalue for a particular matrix, we can re-write (A.13) as

$$(\mathbf{C} - \lambda \mathbf{E})\boldsymbol{a} = \mathbf{0}, \tag{A.14}$$

where $\mathbf{E}$ is the identity matrix of the same dimension as $\mathbf{C}$.

For example in $\mathbb{R}^3$, this equation system, together with for example the condition that $\boldsymbol{a} = \hat{\boldsymbol{a}}$ has unit length, provides four equations for four unknowns: The components of $\hat{\boldsymbol{a}}$ and the scalar λ.

If the matrix $\mathbf{C}$ has full rank, we will find as many independent eigenvectors with nonzero eigenvalues as there are dimensions. Remark: It is possible that some or all vectors are complex, in which case also their associated eigenvalues are complex. These solutions always occur in complex conjugate pairs. Algebra of complex numbers is not required in the following; for more information, please consult Appendix H of [70] or Appendix C of [57].

A commonly employed method to solve the equation system is to first establish the determinant of the matrix $\mathbf{C} - \lambda\mathbf{E}$, which delivers the so-called *characteristic polynomial* in λ:

$$\det(\mathbf{C} - \mathbf{E}\lambda) = 0. \tag{A.15}$$

So, finding the n eigenvalues means finding the n roots of this n-degree polynomial. Afterwards, for each of these roots, one finds the associated eigenvector from (A.14).

Problem A.23 Computing eigenvalues by hand can be much work. However, checking if a vector is an eigenvector is usually quicker. Show that $\begin{pmatrix}1\\1\\1\end{pmatrix}$ is an eigenvector of $\begin{pmatrix}1&2&3\\3&1&2\\2&3&1\end{pmatrix}$.

Problem A.24 Calculate the eigenvalues of a general diagonal matrix.

It is possible that some eigenvalues have a *multiplicity* larger than 1. If the characteristic polynomial contains factors $(p - \lambda_i)^{k_i}$, then k_i is

called the *algebraic multiplicity* of the eigenvalue λ_i. Note that it could be that there are less than k_i eigenvectors associated with a particular eigenvalue λ_i. In that case, the *geometric* multiplicity of that eigenvalue is lower, in fact it equals the number of independent eigenvectors for the particular eigenvalue. It will always be at least one.

◯ **Problem A.25** Use the MATLAB^{TM} command `eig` to calculate the eigenvectors and eigenvalues of the matrices given in Example A.2 and in Problem A.23. Do the results match your expectations?

Further useful properties of eigenvalues are that the determinant of a matrix equals the product of all eigenvalues, and the trace of the matrix equals the sum of all eigenvalues.

◯ **Problem A.26** Check plausibility of your answer to Problem A.24 using the determinant and the trace.

◎ **Problem A.27** Calculate the eigenvalues of a general upper triangular matrix. Check plausibility of your answer using the determinant and the trace.

If all its eigenvalues are positive, a symmetric matrix is *positive definite*. If eigenvalues are only non-negative (meaning some could be zero), the matrix is positive *semidefinite*. In analogy, if all eigenvalues are negative (nonpositive), a symmetric matrix is *negative (semi-)definite*.

◯ **Problem A.28** Check your answer to Problem A.21 using eigenvalues.

A.7 Eigendecomposition of a matrix

Any *diagonalizable* square matrix $\mathbf{A}$ can be decomposed into the form

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}, \tag{A.16}$$

where the matrix $\mathbf{Q}$ contains the eigenvectors of matrix $\mathbf{A}$, and the matrix $\mathbf{\Lambda}$ is a diagonal matrix with the eigenvalues of $\mathbf{A}$ on the diagonal.

This will be shown in the following.

First, we express the n-by-n matrix $\mathbf{Q}$ in terms of its column vectors

$$\mathbf{Q} = \begin{pmatrix} \boldsymbol{q}_1 & \dots & \boldsymbol{q}_n \end{pmatrix}, \tag{A.17}$$

If these are all eigenvectors of $\mathbf{A}$, we know from Appendix A.6 that

$$\mathbf{A}\boldsymbol{q}_i = \lambda_i \boldsymbol{q}_i, \quad i = 1 \dots n, \tag{A.18}$$

with scalar λ_i.

We can write this in matrix format:

$$\mathbf{A} \cdot \begin{pmatrix} \boldsymbol{q}_1 & \dots & \boldsymbol{q}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 \boldsymbol{q}_1 & \dots & \lambda_n \boldsymbol{q}_n . \end{pmatrix} \tag{A.19}$$

By defining the diagonal matrix Λ as

$$\mathbf{\Lambda} := \begin{pmatrix} \lambda_1 & & \mathbf{0} \\ & \dots & \\ \mathbf{0} & & \lambda_n \end{pmatrix}, \tag{A.20}$$

we can re-write (A.19) to

$$\mathbf{A} \cdot \begin{pmatrix} \boldsymbol{q}_1 & \dots & \boldsymbol{q}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{q}_1 & \dots & \boldsymbol{q}_n \end{pmatrix} \cdot \mathbf{\Lambda}, \tag{A.21}$$

or in short

$$\mathbf{AQ} = \mathbf{Q\Lambda}. \tag{A.22}$$

If the matrix $\mathbf{Q}$ is invertible, we obtain the intended relationship (A.16).

A matrix $\mathbf{A}$ that is square but not diagonalizable is a *defective* matrix.

All symmetric matrices are diagonalizable: They possess three real eigenvectors that are mutually orthogonal, such that $\mathbf{Q}$ is invertible.

Diagonalization will be useful for the inertia tensor (see Section 5.2.3).

Problem A.29 Diagonalize the matrix

$$\mathbf{A} = \begin{pmatrix} 7 & -12 \\ 4 & -7 \end{pmatrix}. \tag{A.23P}$$

▷

Problem A.30 Show that this matrix is defective:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{A.24P}$$

Problem A.31 Diagonalize the matrix

$$\mathbf{A} = \begin{pmatrix} 10 & 2 & -2 \\ 2 & 17 & 0 \\ -2 & 0 & 17 \end{pmatrix} \tag{A.25P}$$

For ease of calculation, you may use the MATLAB$^{\text{TM}}$ command `eig`. Check your result by re-calculating matrix $\mathbf{A}$ from $\mathbf{Q}$ and $\mathbf{\Lambda}$.

B Differentiation and Integration

B.1 Derivatives with Respect to Time

Many variables that describe mechanical systems (such as distances or angles) change over time and so are time-dependent.

To denote an infinitesimally small change in a variable, we use the operator d. For example, an infinitesimal increment in the vector $\boldsymbol{r}$ would be denoted as $\mathrm{d}\boldsymbol{r}$.

To represent the rate of change of a quantity with respect to time, one calculates the quotient of this infinitesimal increment and an increment in time $\mathrm{d}t$, which is the derivative with respect to time. To denote the derivative of a vector $\boldsymbol{r}$ or a scalar r with respect to time t, we use the abbreviated notations,

$$\dot{\boldsymbol{r}} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)}{\Delta t}, \text{ and } \dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t}, \tag{B.1}$$

respectively. Note that the time derivative of constants, so quantities that do not vary over time, is zero.

The above are all absolute derivatives with respect to time, observed with respect to a system of reference that is not rotating. Section 4.2.2 of Chapter 4, extends this topic to relative derivatives, such as as $(\dot{\boldsymbol{r}})_{\mathcal{F}}$.

B.2 Product Rule

The product rule of differentiation is applied when an expression contains a multiplication of several functions. The derivative of the scalar function $h(t) = f(t) \cdot g(t)$ with respect to time is:

$$\dot{h} = \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(f(t)g(t)\right) = \frac{\mathrm{d}f(t)}{\mathrm{d}t}g(t) + f(t)\frac{\mathrm{d}g(t)}{\mathrm{d}t}. \tag{B.2}$$

Problem B.1 Take the time derivative of $h(t) = t^2$ by rewriting h as $h(t) = f(t)g(t)$, where $f(t) = g(t) = t$ and applying the product rule.

Problem B.2 Take the time derivative of $h(t) = t^3$ by rewriting h as $h(t) = f(t)g(t)$, where $f(t) = t^2$, $g(t) = t$ and applying the product rule.

Problem B.3 Show (B.2) from the definition of the time derivative in (B.1).

For the scalar product of two vector functions, so $h_1(t) = \boldsymbol{f}(t)^{\mathrm{T}}\boldsymbol{g}(t)$, the rule can equally be applied:

$$\dot{h}_1 = \frac{\mathrm{d}h_1}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{f}(t)^{\mathrm{T}}\boldsymbol{g}(t)\right) = \dot{\boldsymbol{f}}(t)^{\mathrm{T}}\boldsymbol{g}(t) + \boldsymbol{f}(t)^{\mathrm{T}}\dot{\boldsymbol{g}}(t) \tag{B.3}$$

as well as for the cross product $\boldsymbol{h}_2(t) = \boldsymbol{f}(t) \times \boldsymbol{g}(t)$:

$$\dot{\boldsymbol{h}}_2 = \frac{\mathrm{d}\boldsymbol{h}_2}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{f}(t) \times \boldsymbol{g}(t)\right) = \dot{\boldsymbol{f}}(t) \times \boldsymbol{g}(t) + \boldsymbol{f}(t) \times \dot{\boldsymbol{g}}(t) \tag{B.4}$$

Problem B.4 Use the product rule of differentiation to show that $\int \ddot{\boldsymbol{r}}^{\mathrm{T}}\dot{\boldsymbol{r}}\,\mathrm{d}t = \frac{1}{2}\dot{\boldsymbol{r}}^{\mathrm{T}}\dot{\boldsymbol{r}} + \text{constant}$.

Problem B.5 Use the scalar formulation (B.2) of the product rule of differentiation, in combination with the definitions of scalar and cross product, to prove (B.3) and (B.4).

B.3 Chain Rule

The chain rule is used when the expression is the nested composition of two or more functions. To calculate the time derivative of a single nested function $h(g(t))$, the chain rule yields

$$\dot{h} = \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(h(g(t))\right) = \frac{\mathrm{d}h}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}t}. \tag{B.5}$$

Problem B.6 Calculate the time derivative $\dot{h}$ of $h = \sin(\theta)$ for which you know that θ is time-dependent, using the chain rule.

Problem B.7 Calculate the time derivative of $h = 3\theta\sin(\theta)$ for which you know that θ is time-dependent, using both the product rule and the chain rule.

For a more deeply nested function, the chain rule expands to:

$$\dot{h} = \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(h(g_1(g_2(g_3(t))))\right) = \frac{\mathrm{d}h}{\mathrm{d}g_1}\frac{\mathrm{d}g_1}{\mathrm{d}g_2}\frac{\mathrm{d}g_2}{\mathrm{d}g_3}\frac{\mathrm{d}g_3}{\mathrm{d}t}. \tag{B.6}$$

Example B.1 To calculate $\dot{h} = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\cos\left(\sqrt{x(t)}\right)}\right)$, we decompose h as a function of g_1 first. That is:

1. $h(g_1) = e^{g_1}$. In its turn, g_1 is a function of g_2. That is
2. $g_1(g_2) = \cos(g_2)$. Then, g_2 is a function of g_3, namely
3. $g_2(g_3) = \sqrt{g_3}$. Finally, g_3 is the time-dependent variable:
4. $g_3(t) = x(t)$.

The separate derivatives are:

1. $\frac{\mathrm{d}h(g_1)}{\mathrm{d}g_1} = e^{g_1}$
2. $\frac{\mathrm{d}g_1(g_2)}{\mathrm{d}g_2} = -\sin(g_2)$
3. $\frac{\mathrm{d}g_2(g_3)}{\mathrm{d}g_3} = \frac{1}{2\sqrt{g_3}}$
4. $\frac{\mathrm{d}g_3(t)}{\mathrm{d}t} = \dot{x}$

This way we find

$$\dot{h} = -\frac{\dot{x}e^{\cos\left(\sqrt{x(t)}\right)}\sin\left(\sqrt{x(t)}\right)}{2\sqrt{x(t)}}. \tag{B.7E}$$

Problem B.8 Calculate $\dot{h} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\cos\left(x(t)^2\right)\right)$.

Problem B.9 Show (B.5) from the definition of the time derivative in (B.1).

Problem B.10 Use the Symbolic Toolbox in MATLAB$^{\mathrm{TM}}$ to calculate $\frac{\mathrm{d}h}{\mathrm{d}x}$ from Example B.1 and from Problem B.8 and compare the results with your manual calculations.

B.4 Partial Derivatives

We consider a function f that depends on multiple variables. A *partial derivative* of f is the derivative of this function with respect to a single variable. Taking a partial derivative, one is only interested in the influence of the variation of one of the variables on f, while all the other variables are held constant.

Example B.2 The elevation at a particular location in a mountain landscape can be given as a function of two variables: latitude and longitude. We could be interested in the influence of the variation of latitude, while the longitude is kept constant. To obtain this information, we calculate the partial derivative of elevation with respect to latitude. This provides the slope of the landscape in the requested direction.

So, a partial derivative represents the dependency of an expression on an isolated variable. The partial derivative of a function f with respect to a variable x is denoted by $\frac{\partial f}{\partial x}$.

Example B.3

$$\frac{\partial}{\partial x}\left(ax^2 + 3yx + c\dot{x}^2\right) = 2ax + 3y \tag{B.8E}$$

Example B.4

$$\frac{\partial}{\partial \dot{x}}\left(ax^2 + 3yx + c\dot{x}^2\right) = 2c\dot{x} \tag{B.9E}$$

Example B.5

$$\frac{\partial}{\partial y}\left(ax^2 + 3yx + c\dot{x}^2\right) = 3x \tag{B.10E}$$

We can use partial derivatives to calculate increments of a function f that depends on multiple variables x_i, with $i = 1 \dots N$:

$$\mathrm{d}f = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i. \tag{B.11}$$

Example B.6 Consider a particle moving in three-dimensional space, within a temperature field. The particle's coordinates x, y, and z determine the temperature $f(x, y, z)$. We find the infinitesimal temperature increment $\mathrm{d}f$ that is due to changes in x, in y, and in z as:

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z. \tag{B.12E}$$

Example B.7 In Example B.2 about the mountain landscape, the elevation f is only a function of two variables, namely latitude x_1 and longitude x_2. The relationship (B.11) indicates how much elevation $\mathrm{d}f$ is gained as a function of infinitesimal changes in latitude $\mathrm{d}x_1$ and longitude $\mathrm{d}x_2$.

In case the x_i are all functions of time t, we find the *total derivative* of a function $f(x_1(t), x_2(t) \dots x_N(t))$ with respect to time as:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t}. \tag{B.13}$$

Problem B.11 Consider a function $f(x(t), y(t))$ with

$$f = 3x^2 + 5y, \qquad x = 5t, \qquad y = t^2. \tag{B.14P}$$

Find $\dot{f}$ in two ways: a) by substituting the functions for x and y into f, such that it becomes a function of time only, and then taking its derivative, and b) by using (B.13). You should obtain the same result.

B.5 Gradient

For a function f that depends on multiple variables, for example x, y, and z, the partial derivatives of f in all directions, so in the example respectively in x, y and z direction, can be subsumed in a vector, the *gradient* of f:

$$\nabla f := \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}. \tag{B.15}$$

The symbol ∇ is called the *nabla* operator; it produces the gradient of a scalar function, like here of $f(x, y, z)$. At any specific location, this vector points in the direction of the steepest ascent.

A gradient of a function is always orthogonal to the *level curves* (often also denoted *iso lines*) or *level surfaces* of this function, which denotes the manifolds along which the function does not change its value.

Example B.8 Figure B.1 shows the gradient of a function $f(x, y)$ that specifies elevation for a given location on a hiking map. This gradient gives inclination. Each vector points in direction of the greatest change in height at its specific location. The gradient vectors are always perpendicular to the iso lines, which connect all points that have the same elevation.

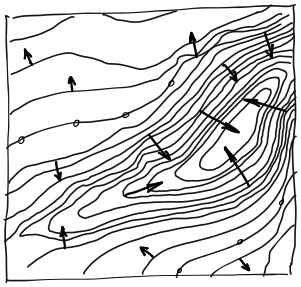


Figure B.1: Visualization of a gradient field. The arrows represent the gradient in elevation on a hiking map with iso lines of elevation.

Problem B.12 We model Earth as a perfect sphere with center E and radius r (see Figure B.2).

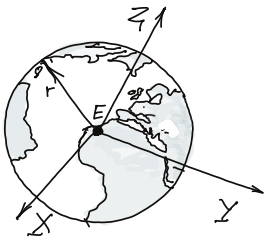


Figure B.2: Earth

a. Describe the location set of points that have equal distance d from the *surface* of the earth in the form $h(X, Y, Z) = d$, where X, Y, Z are the points' coordinates. The unit of the function should be $[h] = \mathrm{m}$.
b. Calculate the gradient of this function using the nabla operator.
c. Draw a few gradient vectors projected on the XY-plane, the YZ-plane, and the XZ-plane. Include at least the gradient at point $(X, Y, Z) = (1.1r, 1.1r, 1.1r)$ and one point on the surface of the Earth.
d. Multiply the gradient by (-1). What might be an interpretation of this (negative) gradient field for the Earth?
e. Check your answer further by plotting the vector field, for example using MATLAB$^{\mathrm{TM}}$. You can use the functions `ndgrid` and `quiver3`. You will also need the `./` notation to divide the elements of a matrix by the elements of another.

Problem B.13 Show that the gradient of a function $f(x, y)$ of two variables x and y at any particular point is orthogonal to the function's level curve through that point. [a]

[a] *Hint*: In order to show this, do the following: Consider an infinitesimal vector that points in a direction that is tangential to the level curve and has the components $(\mathrm{d}x, \mathrm{d}y)^{\mathrm{T}}$. Express the infinitesimal increement $\mathrm{d}f$ of f along the level curve, using (B.11), as a function of the individual increments $\mathrm{d}x$ and $\mathrm{d}y$. This increment should be zero in the direction of the curve. Then, re-write $\mathrm{d}f$ as a scalar product of two vectors. What can you say about these two vectors?

B.6 Integration

For single and multiple integrals, please consult for example Chapter 5 of [70], which explains line, surface, and volume integrals in Sections 16.2, 16.7, and 6.2-6.3, respectively. Pay particular attention to the limits of the integrals, they might be functions of other variables.

C Shorthand Notations

Writing vectors out in components requires additional subscripts. Maintain all subscripts of an original vector can become cumbersome and makes equations less compact. Furthermore, projections require choosing how to typeset several subscripts. Stacked indices are not aesthetic. A comma can be ambiguous, as it sometimes indicates differentiation.

To avoid this choice, one can define new, subscript-free vectors.

Example C.1 A vector ${}^{\mathcal{B}}\boldsymbol{r}_{A/C}$ that describes the relative location of point A with respect to C already has a lengthy subscript. None of these options for the first component are ideal: $r_{A/C_{b_1}}$, $r_{A/Cb1}$, or $r_{A/C,b1}$.

In this specific problem, one could choose to define a new vector $\boldsymbol{a} := \boldsymbol{r}_{A/C}$ and then write components compactly as ${}^{\mathcal{B}}\boldsymbol{a} = (a_{b1}, a_{b2}, a_{b3})^{\mathrm{T}}$.

This advice does not apply to coding software. On the contrary, variables in code need clear names that maintain all identifying subscripts.

Example C.2 The first component of vector ${}^{\mathcal{B}}\boldsymbol{r}_{A/C}$ could for example be coded as `r_AwrtC_b1`. There are many equally good options. It is only advised to explicitly choose a type of definition that maintains the original vector name or meaning, the triad, and the number of the component, and to code consistently. Depending on the choices made, also these could be viable variable names: `rCtoA_b1`, or `B_relativeposition_AwrtC_x`, etc. Since all information encoded via typesetting is lost in plain text, this needs to be replaced by a clear definition of sequence of information.

If elements of a point's position vector are also *Cartesian coordinates*, one can for example choose the axis name as main symbol and the point name as index, or vice versa. However, not all position vector components are also Cartesian coordinates. Components of a position vector in triad $\mathcal{F}$ are only coordinates if the point of reference is the origin of a coordinate system that is aligned with triad $\mathcal{F}$. Therefore, it is important to distinguish between vector components that happen to be also Cartesian coordinates, and components that are only projections onto given axis directions, without further meaning.

◯ **Example C.3** In this book (for example in Figure 4.9 on page 123), we frequently use

- an inertial reference point O,
- a moving reference point Q (for example the center of mass of a system, a geometric center, or the location of a spherical joint),
- a specific point P (for example a mass element within a rigid body),
- an inertial coordinate system XYZ with origin O and triad $\mathcal{N}$,
- a body-fixed coordinate system xyz with origin Q and triad $\mathcal{B}$.

For the mentioned points, we will often (but not strictly) use these shorthand notations for relative and absolute position vectors:

$$\boldsymbol{P} := \boldsymbol{r}_{P/O}, \quad \boldsymbol{Q} := \boldsymbol{r}_{Q/O}, \quad \boldsymbol{p} := \boldsymbol{r}_{P/Q}. \tag{C.1E}$$

For the above vectors, components are also Cartesian coordinates in the cases of ${}^{\mathcal{N}}\boldsymbol{P}, {}^{\mathcal{N}}\boldsymbol{Q}$, and ${}^{\mathcal{B}}\boldsymbol{p}$. The first and the last are both coordinates of point P, just in two different coordinate systems: XYZ and xyz, respectively. Here, there are multiple alternatives to abbreviate components to variables that carry more meaning:

$${}^{\mathcal{N}}\boldsymbol{P} = {}^{\mathcal{N}}\boldsymbol{r}_{P/O} = \begin{pmatrix} P_{n1} \\ P_{n2} \\ P_{n3} \end{pmatrix} =: \begin{pmatrix} X_P \\ Y_P \\ Z_P \end{pmatrix}, \text{ or } {}^{\mathcal{N}}\boldsymbol{P} =: \begin{pmatrix} P_X \\ P_Y \\ P_Z \end{pmatrix}, \tag{C.2E}$$

$${}^{\mathcal{N}}\boldsymbol{Q} = {}^{\mathcal{N}}\boldsymbol{r}_{Q/O} = \begin{pmatrix} Q_{n1} \\ Q_{n2} \\ Q_{n3} \end{pmatrix} =: \begin{pmatrix} X_Q \\ Y_Q \\ Z_Q \end{pmatrix}, \text{ or } {}^{\mathcal{N}}\boldsymbol{Q} =: \begin{pmatrix} Q_X \\ Q_Y \\ Q_Z \end{pmatrix}, \tag{C.3E}$$

$${}^{\mathcal{B}}\boldsymbol{p} = {}^{\mathcal{B}}\boldsymbol{r}_{P/Q} = \begin{pmatrix} p_{b1} \\ p_{b2} \\ p_{b3} \end{pmatrix} =: \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix}, \text{ or } {}^{\mathcal{B}}\boldsymbol{p} =: \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}. \tag{C.4E}$$

It depends on personal preference which alternative to choose in a given problem. The only requirement is to always make all abbreviated definitions explicit. This book mostly uses the first alternative, so X_P rather than P_X.

It is not possible to apply this abbreviated notation in a meaningful way to the three other possible projections, ${}^{\mathcal{B}}\boldsymbol{P}, {}^{\mathcal{B}}\boldsymbol{Q}$, and ${}^{\mathcal{N}}\boldsymbol{p}$, because components are *not* Cartesian coordinates. In these cases, we can only use the explicit subscripts that indicate projections:

$${}^{\mathcal{B}}\boldsymbol{P} = {}^{\mathcal{B}}\boldsymbol{r}_{P/O} = \begin{pmatrix} P_{b1} \\ P_{b2} \\ P_{b3} \end{pmatrix}, {}^{\mathcal{B}}\boldsymbol{Q} = {}^{\mathcal{B}}\boldsymbol{r}_{Q/O} = \begin{pmatrix} Q_{b1} \\ Q_{b2} \\ Q_{b3} \end{pmatrix}, {}^{\mathcal{N}}\boldsymbol{p} = {}^{\mathcal{N}}\boldsymbol{r}_{P/Q} = \begin{pmatrix} p_{n1} \\ p_{n2} \\ p_{n3} \end{pmatrix}. \tag{C.5E}$$

Attempting to write something like "p_X" as shorthand for p_{n1} would be highly misleading, because it gives the impression of an X-coordinate, while the reference point of the position vector is not the origin of XYZ.

D Index Notation with Einstein Summation Convention

Vector, matrix, and tensor equations can be written in a compact and clear way by means of the index notation together with the so-called Einstein summation convention. The symbols are no longer bold faced as opposed to matrix vector notation. For example

$$\boldsymbol{F} = F_i, \qquad \text{with } i = 1 \dots n. \tag{D.1}$$

If in a product two indices are repeated we assume that we have to sum over this index:

$$\boldsymbol{y} = \mathbf{A}\boldsymbol{x} \qquad \Rightarrow \qquad y_i = A_{ij} x_j, \qquad i = 1 \dots n, \quad j = 1 \dots m, \tag{D.2}$$

here the summation is over the index j. This is the so-called Einstein summation convention (as all great scientists, Einstein was lazy and wanted to write as little as possible).

The concept of a transpose of a matrix is somewhat harder to spot in the index notation. Assume we have the following equation $F_i = C_{ji} s_j$. Then if we associate the matrix $\mathbf{C}$ with C_{ij} the above equation in matrix–vector form reads $\boldsymbol{F} = \mathbf{C}^{\mathrm{T}} \boldsymbol{s}$.

Finally, the last shorthand notation is that partial derivatives are denoted by a comma operator followed by the appropriate index, like in

$$\frac{\partial T_i}{\partial q_k} = T_{i,k} \tag{D.3}$$

and

$$\frac{\partial}{\partial q_k} \left(\frac{\partial T_i}{\partial q_j} \right) = T_{i,jk}. \tag{D.4}$$

This last example is unambiguous as opposed to an impossible matrix–vector notation, since we have to deal with three indices here.

E Quaternions, Finite Rotation and Euler Parameters

> Son: *Well, Papa, can you multiply triplets?*
> Father: *No [sadly shaking his head], I can only add and subtract them.*
>
> (William Rowan Hamilton, Conversation with his sons (1843))

A quaternion is a collection of four real parameters, of which the first is considered as a scalar and the other three as a vector in three-dimensional space. In addition, for two quaternions

$$q = \begin{pmatrix} q_0 \\ \boldsymbol{q} \end{pmatrix}, \quad \text{and} \quad p = \begin{pmatrix} p_0 \\ \boldsymbol{p} \end{pmatrix},$$

the sum is defined as

$$q + p = \begin{pmatrix} q_0 + p_0 \\ \boldsymbol{q} + \boldsymbol{p} \end{pmatrix}, \tag{E.1}$$

and their product (non-commutative) as

$$q \circ p = \begin{pmatrix} q_0 p_0 - \boldsymbol{q} \cdot \boldsymbol{p}, \\ q_0 \boldsymbol{p} + p_0 \boldsymbol{q} + (\boldsymbol{q} \times \boldsymbol{p}) \end{pmatrix}. \tag{E.2}$$

The adjoint quaternion of q is defined as $\bar{q} = (q_0, -\boldsymbol{q}^{\mathrm{T}})$ and the length or norm as $|q| = \sqrt{(\bar{q} \circ q)_0} = \sqrt{q_0^2 + \boldsymbol{q} \cdot \boldsymbol{q}}$. From these definitions, we note that $|q \circ p| = |q||p|$.

Problem E.1 Prove that $|q \circ p| = |q||p|$.

There are two special quaternions, the unit element $1 = (1, \mathbf{0}^{\mathrm{T}})^{\mathrm{T}}$ and the zero element $0 = (0, \mathbf{0}^{\mathrm{T}})^{\mathrm{T}}$. The reciprocal of a quaternion $q \neq 0$ is $q^{-1} = \bar{q}/|q|^2$. The quaternion with a norm of one, $|q| = 1$, is a unit quaternion.

If a quaternion is looked upon as a four-dimensional vector, the quaternion product can be described by a matrix-vector product as

$$q \circ p = \begin{pmatrix} q_0 & -\boldsymbol{q}^{\mathrm{T}} \\ \boldsymbol{q} & q_0\mathbf{E}_3 + \tilde{\mathbf{q}} \end{pmatrix} \begin{pmatrix} p_0 \\ \boldsymbol{p} \end{pmatrix} = \mathbf{Q} \begin{pmatrix} p_0 \\ \boldsymbol{p} \end{pmatrix}, \tag{E.3}$$

$$p \circ q = \begin{pmatrix} q_0 & -\boldsymbol{q}^{\mathrm{T}} \\ \boldsymbol{q} & q_0\mathbf{E}_3 - \tilde{\mathbf{q}} \end{pmatrix} \begin{pmatrix} p_0 \\ \boldsymbol{p} \end{pmatrix} = \bar{\mathbf{Q}} \begin{pmatrix} p_0 \\ \boldsymbol{p} \end{pmatrix}. \tag{E.4}$$

Here, we have used the tilde notation for the antisymmetric matrix $\tilde{\mathbf{q}}$ from the vector $\boldsymbol{q}$, which is defined by the matrix-vector notation for the vector cross product $\boldsymbol{q} \times \boldsymbol{x} = \tilde{\mathbf{q}}\boldsymbol{x}$. This skew-symmetric matrix is

$$\tilde{\mathbf{q}} = \begin{pmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{pmatrix} \tag{E.5}$$

The quaternion matrices $\mathbf{Q}$ and $\bar{\mathbf{P}}$ commute, $\mathbf{Q}\bar{\mathbf{P}} = \bar{\mathbf{P}}\mathbf{Q}$. The matrices of the adjoint quaternion $\bar{q}$ are $\mathbf{Q}^{\mathrm{T}}$ and $\bar{\mathbf{Q}}^{\mathrm{T}}$.

If we associate the quaternion $x' = (0,\, \boldsymbol{x}'^{\mathrm{T}})^{\mathrm{T}}$ with the 3-D vector $\boldsymbol{x}'$ and define the operation with the unit quaternion q, as

$$x = q \circ x' \circ q^{-1} = q \circ x' \circ \bar{q}, \tag{E.6}$$

then this transformation, from x' to x, represents a rotation. The resulting quaternion x is a vectorial quaternion with the same length as x'. The case of reflection, the other possibility, can be excluded. The rotation matrix $\mathbf{R}$ in terms of the unit quaternions q can be derived from (E.6) as

$$\boldsymbol{x} = (q_0^2 - \boldsymbol{q}\cdot\boldsymbol{q})\boldsymbol{x}' + 2q_0(\boldsymbol{q}\times\boldsymbol{x}') + 2(\boldsymbol{q}\cdot\boldsymbol{x}')\boldsymbol{q} = \mathbf{R}\boldsymbol{x}', \tag{E.7}$$

with

$$\mathbf{R} = \begin{pmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_2q_1 + q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_3q_1 - q_0q_2) & 2(q_3q_2 + q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{pmatrix} \tag{E.8}$$

This rotation matrix can also be written with the help of the quaternion matrix representation according to

$$\begin{pmatrix} 1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{R} \end{pmatrix} = \mathbf{Q}\bar{\mathbf{Q}}^{\mathrm{T}} = \bar{\mathbf{Q}}^{\mathrm{T}}\mathbf{Q}. \tag{E.9}$$

The quaternion q in rotation matrix $\mathbf{R}$ according to (E.8) is identified as the set of Euler parameters for the description of finite rotation. According to Euler's theorem on finite rotation, a rotation in space can always be described by a rotation along a certain axis by a certain angle. With the unit vector $\hat{\boldsymbol{e}}_\mu$ representing the axis and the angle of rotation μ, right-handed positive, the Euler parameters q can be interpreted as

$$q_0 = \cos(\mu/2) \quad \text{and} \quad \boldsymbol{q} = \sin(\mu/2)\hat{\boldsymbol{e}}_\mu. \tag{E.10}$$

Since the Euler parameters are unit quaternions, the subsidiary condition,

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1, \tag{E.11}$$

must always be satisfied.

The quaternion x' in (E.6) can now be associated with the components of a vector in a body-fixed triad and the quaternion x as the corresponding components expressed in a space-fixed triad.

The Euler parameters for successive rotation are given by the quaternion product of the Euler parameters describing the individual rotations.

Before we derive the rotational equations of motion for a spatial rigid body in terms of Euler parameters, we have to express the angular velocities and accelerations in terms of the Euler parameters and their time derivatives. By differentiation of the rotational transformation (E.6), as in

$$\dot{x} = \dot{q} \circ x' \circ \bar{q} + q \circ x' \circ \dot{\bar{q}}, \tag{E.12}$$

and substitution of the body-fixed coordinates according to $x' = \bar{q} \circ x \circ q$, realizing that $\bar{q} \circ q$ is the unit element $(1,\, \mathbf{0}^{\mathrm{T}})^{\mathrm{T}}$, the velocity reads

$$\dot{x} = \dot{q} \circ \bar{q} \circ x + x \circ q \circ \dot{\bar{q}}. \tag{E.13}$$

The scalar parts of the products $\dot{q} \circ \bar{q}$ and $q \circ \dot{\bar{q}}$ are zero, because q is a unit quaternion, and the vector parts are opposite. Therefore, we may write: $\dot{q} \circ \bar{q} = (0,\, \boldsymbol{w}^{\mathrm{T}})^{\mathrm{T}}$ and $q \circ \dot{\bar{q}} = (0,\, -\boldsymbol{w}^{\mathrm{T}})^{\mathrm{T}}$.

The velocity $\dot{x}$ now has a zero scalar part, as expected, and a vectorial part, $\dot{\boldsymbol{x}} = 2\boldsymbol{w} \times \boldsymbol{x}$, so $\boldsymbol{\omega} = 2\boldsymbol{w}$. We conclude that the angular velocity $\boldsymbol{\omega}$, expressed in the space-fixed reference in terms of the Euler parameters q and their time derivatives, is given by

$$\omega = 2\dot{q} \circ \bar{q} \quad \text{or} \quad \begin{pmatrix} 0 \\ \boldsymbol{\omega} \end{pmatrix} = 2\bar{\mathbf{Q}}^{\mathrm{T}} \begin{pmatrix} \dot{q}_0 \\ \dot{\boldsymbol{q}} \end{pmatrix}. \tag{E.14}$$

The inverse, the time derivatives $\dot{q}$ of the Euler parameters for given q and ω, can be found as

$$\dot{q} = \frac{1}{2}\omega \circ q \quad \text{or} \quad \begin{pmatrix} \dot{q}_0 \\ \dot{\boldsymbol{q}} \end{pmatrix} = \frac{1}{2}\bar{\mathbf{Q}} \begin{pmatrix} 0 \\ \boldsymbol{\omega} \end{pmatrix}. \tag{E.15}$$

These time derivatives are always uniquely defined, in contrast to the classical combination of three parameters for describing spatial rotation as in, for example, Euler angles or Rodrigues parameters.

The angular velocities ${}^{\mathcal{B}}\boldsymbol{\omega}$ expressed in a body-fixed reference triad $\mathcal{B}$ can be derived in the same manner, or by application of the rotational transformation (E.9), as

$$\omega' = 2\bar{q} \circ \dot{q} \quad \text{or} \quad \begin{pmatrix} 0 \\ {}^{\mathcal{B}}\boldsymbol{\omega} \end{pmatrix} = 2\mathbf{Q}^{\mathrm{T}} \begin{pmatrix} \dot{q}_0 \\ \dot{\boldsymbol{q}} \end{pmatrix}, \tag{E.16}$$

and with the inverse

$$\dot{q} = \frac{1}{2} q \circ \omega' \quad \text{or} \quad \begin{pmatrix} \dot{q}_0 \\ \dot{\boldsymbol{q}} \end{pmatrix} = \frac{1}{2}\mathbf{Q} \begin{pmatrix} 0 \\ {}^{\mathcal{B}}\boldsymbol{\omega} \end{pmatrix}. \tag{E.17}$$

The angular accelerations are found by differentiation of the expressions for ω and ω', resulting in

$$\begin{pmatrix} 0 \\ \dot{\boldsymbol{\omega}} \end{pmatrix} = 2\bar{\mathbf{Q}}^{\mathrm{T}} \begin{pmatrix} \ddot{q}_0 \\ \ddot{\boldsymbol{q}} \end{pmatrix} + 2 \begin{pmatrix} |\dot{q}|^2 \\ \mathbf{0} \end{pmatrix}, \tag{E.18}$$

and expressed in the body-fixed reference triad

$$\begin{pmatrix} 0 \\ {}^{\mathcal{B}}\dot{\boldsymbol{\omega}} \end{pmatrix} = 2\mathbf{Q}^{\mathrm{T}} \begin{pmatrix} \ddot{q}_0 \\ \ddot{\boldsymbol{q}} \end{pmatrix} + 2 \begin{pmatrix} |\dot{q}|^2 \\ \mathbf{0} \end{pmatrix}. \tag{E.19}$$

The inverse, the second-order time derivatives $\ddot{q}$ of the Euler parameters in terms of q, $\dot{q}$ and $\dot{\boldsymbol{\omega}}$, can be easily derived.

The equations of motion for the rotation of a rigid body in space, with the components of the inertia tensor as matrix ${}^{\mathcal{B}}\mathbf{I}_C$ and the vector of applied torques ${}^{\mathcal{B}}\boldsymbol{M}_C$, all at the center of mass C expressed in the body-fixed triad, are

$${}^{\mathcal{B}}\mathbf{I}_C\, {}^{\mathcal{B}}\dot{\boldsymbol{\omega}} = {}^{\mathcal{B}}\boldsymbol{M}_C - {}^{\mathcal{B}}\boldsymbol{\omega} \times ({}^{\mathcal{B}}\mathbf{I}_C\, {}^{\mathcal{B}}\boldsymbol{\omega}). \tag{E.20}$$

They can be expressed in terms of Euler parameters and their time derivatives by application of the principle of virtual power and introduction of the Lagrangian multiplier λ for the norm constraint (E.11) written as

$$C_q = q_0^2 + q_1^2 + q_2^2 + q_3^2 - 1 = 0. \tag{E.21}$$

This results in the virtual power equation for a rigid body as

$$({}^{\mathcal{B}}\boldsymbol{M}_C - {}^{\mathcal{B}}\mathbf{I}_C\,{}^{\mathcal{B}}\dot{\boldsymbol{\omega}} - {}^{\mathcal{B}}\boldsymbol{\omega} \times ({}^{\mathcal{B}}\mathbf{I}_C\,{}^{\mathcal{B}}\boldsymbol{\omega}))^{\mathrm{T}}\delta\,{}^{\mathcal{B}}\boldsymbol{\omega} = \lambda\delta\dot{C}_q. \tag{E.22}$$

The virtual constraint rate can be derived from (E.21) as

$$\delta\dot{C}_q = 2q_0\delta\dot{q}_0 + 2\boldsymbol{q}^{\mathrm{T}}\delta\dot{\boldsymbol{q}}. \tag{E.23}$$

Substitution of these expressions and the expressions for the angular velocities (E.16) and the angular accelerations (E.19) in the virtual power equation (E.22) and taking arbitrary virtual Euler parameter velocities yields, after adding the constraints on the accelerations of the Euler parameters as in (E.18) or (E.19), the equations of motion for a rigid body expressed in terms of Euler parameters as

$$\begin{aligned}
&\begin{pmatrix} 4\mathbf{Q}\begin{pmatrix} 0 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & {}^{\mathcal{B}}\mathbf{I}_C \end{pmatrix}\mathbf{Q}^{\mathrm{T}} & 2\begin{pmatrix} q_0 \\ \boldsymbol{q} \end{pmatrix} \\ 2\begin{pmatrix} q_0 & \boldsymbol{q}^{\mathrm{T}} \end{pmatrix} & 0 \end{pmatrix}\begin{pmatrix} \ddot{q}_0 \\ \ddot{\boldsymbol{q}} \\ \lambda \end{pmatrix} \\
&= \begin{pmatrix} 2\mathbf{Q}\begin{pmatrix} 0 \\ {}^{\mathcal{B}}\boldsymbol{M}_C \end{pmatrix} + 8\dot{\mathbf{Q}}\begin{pmatrix} 0 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & {}^{\mathcal{B}}\mathbf{I}_C \end{pmatrix}\dot{\mathbf{Q}}^{\mathrm{T}}\begin{pmatrix} q_0 \\ \boldsymbol{q} \end{pmatrix} \\ -2|\dot{q}|^2 \end{pmatrix}.
\end{aligned} \tag{E.24}$$

The multiplier λ can for this single body be obtained by premultiplying the first four equations by $(q_0,\, \boldsymbol{q}^{\mathrm{T}})^{\mathrm{T}}$, and it is identified as twice the rotational kinetic energy of the body,

$$\lambda = 4\begin{pmatrix} q_0 \\ \boldsymbol{q} \end{pmatrix}^{\mathrm{T}}\dot{\mathbf{Q}}\begin{pmatrix} 0 & \mathbf{0}^{T} \\ \mathbf{0} & {}^{\mathcal{B}}\mathbf{I}_C \end{pmatrix}\dot{\mathbf{Q}}^{\mathrm{T}}\begin{pmatrix} q_0 \\ \boldsymbol{q} \end{pmatrix} = {}^{\mathcal{B}}\boldsymbol{\omega}^{\mathrm{T}}\,{}^{\mathcal{B}}\mathbf{I}_C\,{}^{\mathcal{B}}\boldsymbol{\omega}. \tag{E.25}$$

The transformations of an applied torque, body-fixed ${}^{\mathcal{B}}\boldsymbol{M}_C$ or space-fixed $\boldsymbol{M}$, to the torque parameters $(f_0,\, \boldsymbol{f})$, which are dual to the Euler parameters, are apparently

$$\begin{pmatrix} f_0 \\ \boldsymbol{f} \end{pmatrix} = 2\mathbf{Q}\begin{pmatrix} 0 \\ {}^{\mathcal{B}}\boldsymbol{M}_C \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} f_0 \\ \boldsymbol{f} \end{pmatrix} = 2\bar{\mathbf{Q}}\begin{pmatrix} 0 \\ \boldsymbol{M} \end{pmatrix}. \tag{E.26}$$

F Answers

1.8 ${}^{\mathcal{B}}\boldsymbol{r}_{D/P} = \left(-10x_{C^*},\, 0,\, \frac{l}{2} - \Delta\right)^{\mathrm{T}}$, ${}^{\mathcal{B}}\boldsymbol{r}_{E/P} = \left(0,\, -10y_{C^*},\, \frac{l}{2} - \Delta\right)^{\mathrm{T}}$

1.12 $\ddot{\boldsymbol{r}} = \left(-9\sin\frac{3t}{\mathrm{s}} \quad -12\cos\frac{2t}{\mathrm{s}} \quad 8\cos\frac{4t}{\mathrm{s}}\right)^{\mathrm{T}}\ \mathrm{m/s^2}$

1.14 $\{X_R, Y_L, Y_R, \dot{X}_L\} > 0, \{X_A, \dot{X}_A\} = 0, \{X_L, Y_A, \dot{X}_R, \dot{Y}_L, \dot{Y}_R, \dot{Y}_A\} < 0$

1.19 $\boldsymbol{r}_{A/P} = (l/\,(2d))\,\boldsymbol{r}_{C/P}$

1.20 $a = \frac{m_1 L}{m_1+m_2+m_3}$, $\quad b = \frac{(m_2-m_3)L}{4(m_1+m_2+m_3)}$

1.21 E.

1.25 A.

1.27 $\boldsymbol{F}_{A/D} = -\frac{\boldsymbol{r}_{D/A}}{|\boldsymbol{r}_{D/A}|}\ k\ (|\boldsymbol{r}_{D/A}| - l_n)$

1.28 A.

1.29 F.

2.7 No. The muscle forces are internal forces and therefore cannot change angular momentum of the body as a whole.

2.12 The friction forces do no work in this case.

2.19 Gravity and ideal spring forces are conservative forces, sliding friction is not.

2.26 F.

3.1 The same: $|{}^{\mathcal{N}}\boldsymbol{r}| = |{}^{\mathcal{B}}\boldsymbol{r}|$

3.3 ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_1 = (\frac{1}{2}\sqrt{3},\, \frac{1}{2},\, 0)^{\mathrm{T}}$, ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_2 = (-\frac{1}{2},\, \frac{1}{2}\sqrt{3},\, 0)^{\mathrm{T}}$, ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_3 = (0,\, 0,\, 1)^{\mathrm{T}}$

3.4 1

3.5 0

3.9 The identity matrix

3.11 ${}^{\mathcal{N}}\boldsymbol{r}_{P/O} = {}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}}^{\mathrm{T}}\,{}^{\mathcal{F}}\mathbf{C}_{\mathcal{B}}\,{}^{\mathcal{B}}\boldsymbol{r}_{P/O}$

3.16 ${}^{\mathcal{N}}\hat{\boldsymbol{e}}_a = (0,\, 0,\, 1)^{\mathrm{T}}$, $\varphi_a = \frac{\pi}{6}$

3.17 2

3.26 A.,C.

3.27 ${}^{\mathcal{C}}\boldsymbol{r}_{D/A} = {}^{\mathcal{C}}\boldsymbol{r}_{S/A} + {}^{\mathcal{K}}\mathbf{C}_{\mathcal{C}}^{\mathrm{T}} \begin{pmatrix} 0 \\ -R \\ 0 \end{pmatrix}$

3.28 B., C., D.

3.29 ${}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}} = \begin{bmatrix} \cos\psi & -\sin\psi\cos\gamma & \sin\psi\sin\gamma \\ \sin\psi & \cos\psi\cos\gamma & -\cos\psi\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$

3.30 A., B.

3.36 ${}^{\mathcal{T}}\mathbf{C}_{\mathcal{N}} = {}^{\mathcal{L}}\mathbf{C}_{\mathcal{T}}^{\mathrm{T}}\,{}^{\mathcal{L}}\mathbf{C}_{\mathcal{N}}$

3.37 ${}^{\mathcal{N}}\boldsymbol{D} = \begin{pmatrix} 0 & -R & 0 \end{pmatrix}^{\mathrm{T}} + {}^{\mathcal{L}}\mathbf{C}_{\mathcal{N}}^{\mathrm{T}} \begin{pmatrix} 0 & 0 & r \end{pmatrix}^{\mathrm{T}}$

4.6 $\boldsymbol{\omega}_{\mathcal{N}/\mathcal{F}} = -\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}}$

4.9 Yes, no.

4.15

$$\begin{aligned} \boldsymbol{a} &= (\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta)\hat{\boldsymbol{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta)\hat{\boldsymbol{e}}_\theta \\ &\quad + (r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta)\hat{\boldsymbol{e}}_\varphi. \end{aligned}$$

4.16 ${}^{\mathcal{N}}\hat{\boldsymbol{e}}_{\mathrm{t}} = \left({}^{\mathcal{N}}\dot{\boldsymbol{D}}\right) / \left|{}^{\mathcal{N}}\dot{\boldsymbol{D}}\right|$, ${}^{\mathcal{N}}\hat{\boldsymbol{e}}_{\mathrm{b}} = \left({}^{\mathcal{N}}\dot{\boldsymbol{D}} \times {}^{\mathcal{N}}\ddot{\boldsymbol{D}}\right) / \left|{}^{\mathcal{N}}\dot{\boldsymbol{D}} \times {}^{\mathcal{N}}\ddot{\boldsymbol{D}}\right|$

4.21 1. Centripetal acceleration, 2. Coriolis acceleration

4.22 ${}^{\mathcal{S}}\boldsymbol{p} = (0,\, d,\, 3d)^{\mathrm{T}}$, ${}^{\mathcal{S}}(\dot{\boldsymbol{p}})_{\mathcal{S}} = (0,\, 0,\, 0)^{\mathrm{T}}$, ${}^{\mathcal{S}}(\ddot{\boldsymbol{p}})_{\mathcal{S}} = (0,\, 0,\, 0)^{\mathrm{T}}$

4.31 A.

4.32 A.

4.35 All (A, B, C, D, E) are false.

4.36 The components have the signs

	$\hat{\boldsymbol{f}}_1$	$\hat{\boldsymbol{f}}_2$	$\hat{\boldsymbol{f}}_3$
1.	0	0	0
2.	0	0	−
3.	0	−	0
4.	0	0	+
5.	0	+	0

4.38 ${}^{\mathcal{N}}\boldsymbol{\omega} = \frac{v_C}{r(R-r\cos\theta)}\left(-R\sin\theta, 0, r - R\cos\theta\right)^{\mathrm{T}}$.

5.3 Result: See Example 5.2. Intermediate bodies: A ring and a thin disk.

5.12 ${}^{\mathcal{T}}\mathbf{I}_E = {}^{\mathcal{N}}\mathbf{I}_E = \frac{2}{5}mr^2\mathbf{E}$

5.15 ${}^{\mathcal{K}}\mathbf{I}_S = \frac{2}{5}mR^2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - m_Q\begin{bmatrix}d^2+k^2&0&0\\0&k^2&-dk\\0&-dk&d^2\end{bmatrix}$

5.16 $I_{xx} = I_{yy} = \left(\frac{1}{2}m_{\mathrm{k}} + \frac{1}{6}m_{\mathrm{s}}\right)l^2 + m_{\mathrm{s}}\Delta^2$, $I_{zz} = \left(m_{\mathrm{k}} + \frac{1}{6}m_{\mathrm{s}}\right)l^2$, $I_{xy} = I_{yx} = I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$

5.29 yes, yes

5.38 None

5.39 A.

5.40 C., D.

5.41 No.

5.42 A., E.

5.44 A.

5.45 A.,E.

5.47 correct: A, C, D, G. false: B, E, F

5.48 Correct answers (and some possible alternatives in gray)

$$^{\mathcal{G}}\dot{\boldsymbol{r}}_A = \begin{pmatrix} R\dot{\alpha} \\ 0 \\ 0 \end{pmatrix}, \text{ or } ^{\mathcal{N}}\dot{\boldsymbol{r}}_A = \begin{pmatrix} \dot{C_X} - \frac{d}{2}\dot{\theta}\cos\theta \\ \dot{C_Y} - \frac{d}{2}\dot{\theta}\sin\theta \\ 0 \end{pmatrix},$$

$$\text{or } ^{\mathcal{G}}\dot{\boldsymbol{r}}_A = \begin{pmatrix} \dot{C_X}\cos\theta + \dot{C_Y}\sin\theta - \frac{d}{2}\dot{\theta} \\ -\dot{C_X}\sin\theta + \dot{C_Y}\cos\theta \\ 0 \end{pmatrix}$$

$$^{\mathcal{G}}\dot{\boldsymbol{r}}_B = \begin{pmatrix} R\dot{\beta} \\ 0 \\ 0 \end{pmatrix}, \text{ or } ^{\mathcal{N}}\dot{\boldsymbol{r}}_B = \begin{pmatrix} \dot{C_X} + \frac{d}{2}\dot{\theta}\cos\theta \\ \dot{C_Y} + \frac{d}{2}\dot{\theta}\sin\theta \\ 0 \end{pmatrix},$$

$$\text{or } ^{\mathcal{G}}\dot{\boldsymbol{r}}_B = \begin{pmatrix} \dot{C_X}\cos\theta + \dot{C_Y}\sin\theta + \frac{d}{2}\dot{\theta} \\ -\dot{C_X}\sin\theta + \dot{C_Y}\cos\theta \\ 0 \end{pmatrix}$$

$$^{\mathcal{G}}\dot{\boldsymbol{r}}_S = \begin{pmatrix} \dot{C_X}\cos\theta + \dot{C_Y}\sin\theta + \dot{\gamma}h\cos\gamma \\ -\dot{C_X}\sin\theta + \dot{C_Y}\cos\theta + \dot{\theta}h\sin\gamma \\ -h\dot{\gamma}\sin\gamma \end{pmatrix},$$

$$\text{or } ^{\mathcal{N}}\dot{\boldsymbol{r}}_S = \begin{pmatrix} \dot{C_X} + \dot{\gamma}h\cos\gamma\cos\theta - \dot{\theta}h\sin\gamma\sin\theta \\ \dot{C_Y} + \dot{\gamma}h\cos\gamma\sin\theta + \dot{\theta}h\sin\gamma\cos\theta \\ -h\dot{\gamma}\sin\gamma \end{pmatrix}$$

$$^{\mathcal{G}}\boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ \dot{\alpha} \\ \dot{\theta} \end{pmatrix}, \text{ or } ^{\mathcal{N}}\boldsymbol{\omega}_1 = \begin{pmatrix} -\dot{\alpha}\sin\theta \\ \dot{\alpha}\cos\theta \\ \dot{\theta} \end{pmatrix}, \text{ or } ^{\mathcal{B}}\boldsymbol{\omega}_1 = \begin{pmatrix} -\dot{\theta}\sin\gamma \\ \dot{\alpha} \\ \dot{\theta}\cos\gamma \end{pmatrix}$$

$$^{\mathcal{G}}\boldsymbol{\omega}_2 = \begin{pmatrix} 0 \\ \dot{\beta} \\ \dot{\theta} \end{pmatrix}, \text{ or } ^{\mathcal{N}}\boldsymbol{\omega}_2 = \begin{pmatrix} -\dot{\beta}\sin\theta \\ \dot{\beta}\cos\theta \\ \dot{\theta} \end{pmatrix}, \text{ or } ^{\mathcal{B}}\boldsymbol{\omega}_2 = \begin{pmatrix} -\dot{\theta}\sin\gamma \\ \dot{\beta} \\ \dot{\theta}\cos\gamma \end{pmatrix}$$

$$^{\mathcal{G}}\boldsymbol{\omega}_3 = \begin{pmatrix} 0 \\ \dot{\gamma} \\ \dot{\theta} \end{pmatrix}, \text{ note that this is not needed if } ^{\mathcal{G}}\mathbf{I}_S \text{ is a zero matrix.}$$

$$^{\mathcal{G}}\mathbf{I}_w = \begin{pmatrix} \frac{1}{2}m_wR^2 & 0 & 0 \\ 0 & m_wR^2 & 0 \\ 0 & 0 & \frac{1}{2}m_wR^2 \end{pmatrix} = {}^{\mathcal{B}}\mathbf{I}_w$$

$$^{\mathcal{G}}\mathbf{I}_S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

5.49 E.

5.50 incorrect: A, B, C, D, E, G, I.

5.51 worse: i, ii, iii, v, vi, vii. better: iv.

6.1 The acceleration does not change.

6.17 ${}^{\mathcal{N}}\boldsymbol{p} = m_{\mathrm{p}}({}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}})^{\mathrm{T}}\,{}^{\mathcal{F}}\boldsymbol{v}_P$,
${}^{\mathcal{B}}\boldsymbol{H}_P = {}^{\mathcal{B}}\mathbf{I}_P[{}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}\,{}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} + {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{F}}]$

6.18 B.,D.

6.19 D.

6.20 F.

6.21 C.

6.34 ${}^{\mathcal{T}}\boldsymbol{a}_E = \left({}^{\mathcal{T}}\boldsymbol{F}_{\mathrm{c}} + {}^{\mathcal{T}}\boldsymbol{F}_g\right)/m$,
${}^{\mathcal{T}}(\dot{\boldsymbol{\omega}})_{\mathcal{T}/\mathcal{N}} = {}^{\mathcal{T}}\mathbf{I}_E^{-1}\left({}^{\mathcal{T}}\boldsymbol{r}_{D/E} \times {}^{\mathcal{T}}\boldsymbol{F}_{\mathrm{c}} - {}^{\mathcal{T}}\boldsymbol{\omega}_{\mathcal{T}/\mathcal{N}} \times \left({}^{\mathcal{T}}\mathbf{I}_E\,{}^{\mathcal{T}}\boldsymbol{\omega}_{\mathcal{T}/\mathcal{N}}\right)\right)$

6.35 ${}^{\mathcal{N}}\boldsymbol{F}_P = m\left({}^{\mathcal{N}}\ddot{\boldsymbol{r}}_C + (0,\,0,\,g)^{\mathrm{T}}\right)$

6.36 ${}^{\mathcal{B}}\dot{\boldsymbol{\omega}} = {}^{\mathcal{B}}\mathbf{I}_C^{-1} \cdot \left({}^{\mathcal{B}}\boldsymbol{r}_{P/C} \times \left({}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}\,{}^{\mathcal{N}}\boldsymbol{F}_P\right) - {}^{\mathcal{B}}\boldsymbol{\omega} \times \left({}^{\mathcal{B}}\mathbf{I}_C\,{}^{\mathcal{B}}\boldsymbol{\omega}\right)\right)$

7.5 $\Phi = (X_D - X_G)^2 + (Y_D - Y_G)^2 + (Z_D - Z_G)^2 - l^2 = 0$

7.6 $a_1 = \sin\theta, a_2 = -\cos\theta, a_3 = \frac{1}{2}l$

7.9 The minimal number n of generalized coordinates for this physical system is $n = 5$. The number of scalar constraint equations m that exist between this minimal set of generalized coordinates is $m = 2$.

7.14 $\boldsymbol{Q}_g = (0,\,(5m_{\mathrm{k}} + 3m_{\mathrm{s}})\,dg\sin\theta,\,0,\,0,\,0)^{\mathrm{T}}$

7.27 No.

7.28 $\dot{\theta} = \frac{R}{d}(\dot{\beta} - \dot{\alpha})$

7.29 $\dot{\psi} = -\frac{2v}{l}\tan\delta$

8.4 $C_2 = m\left(\ddot{Z}_D + l\ddot{\alpha}\sin\alpha + l\dot{\alpha}^2\cos\alpha + \frac{1}{2}h\ddot{\beta}\sin\beta + \frac{1}{2}h\dot{\beta}^2\cos\beta + g\right) - Q_2$

8.8 $\hat{\boldsymbol{a}}_X = (1,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0)^{\mathrm{T}}$, $\hat{\boldsymbol{a}}_Y = (0,\,1,\,0,\,0,\,0,\,0,\,0,\,0,\,0)^{\mathrm{T}}$, $\hat{\boldsymbol{a}}_Z = (0,\,0,\,1,\,0,\,0,\,0,\,0,\,0,\,0)^{\mathrm{T}}$

8.22 A., B.

8.23 The contribution to Q_γ is $m_{\mathrm{p}}gh\sin\gamma$

8.29 $\ddot{\psi} + (\dot{B}/B)\dot{\psi} = 0$

9.2 $\boldsymbol{y}(t_c) = \left(1 - \frac{1}{2}k\,\mathrm{s}\right)^2 \boldsymbol{y}_a$

9.10 $\boldsymbol{x}(t = 0.2\,\mathrm{s}) = (0.5\,\mathrm{m},\ -0.081\,\mathrm{m/s})^{\mathrm{T}}$

10.3 $\ddot{\varphi}_1 = -13.5310\,\mathrm{rad/s^2}$

10.4 $\ddot{\varphi}_1 = 2.6671\,\mathrm{rad/s^2}$

11.8 $\dot{x}_C = 1.2821\,\mathrm{m/s}$

11.9 The loss in kinetic energy is $1.988\,32\,\mathrm{N\,m}$

13.4 $\ddot{\varphi}_1 = 11.7\,\mathrm{rad/s^2}$

13.5 $\dot{x}_2 = -0.319\,149\,\mathrm{m/s}$

16.1 $\ddot{\varphi}_1 = 3.441\,54\,\mathrm{rad/s^2}$

16.2 $\ddot{x} = 2.151\,639\,\mathrm{m/s^2}$

18.6 $\dot{\varphi}_1 = 2.3660\,\mathrm{m/s^2}$

18.7 $\ddot{\beta} = 22.3194\,\mathrm{rad/s}$

18.8 $\ddot{\beta} = 28.6537\,\mathrm{rad/s^2}$

18.9 $\dot{\gamma} = 1.690\,41\,\mathrm{rad/s}$

18.10 $\dot{\gamma} = 33.0000\,\mathrm{rad/s}$

A.3 $\boldsymbol{a} + b\hat{\boldsymbol{e}}_b = c\hat{\boldsymbol{e}}_c$

A.29

$$\mathbf{Q} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{F.1}$$

G Solutions

1.8 In general, the position vector of the center of mass of a composite body is calculated by (1.11). In this case, with P taking the role of the reference point A, two added pieces of cork give the expression

$$ {}^{\mathcal{B}}\boldsymbol{r}_{C/P} = \frac{\frac{1}{10}M^* {}^{\mathcal{B}}\boldsymbol{r}_{D/P} + \frac{1}{10}M^* {}^{\mathcal{B}}\boldsymbol{r}_{E/P} + M^* {}^{\mathcal{B}}\boldsymbol{r}_{C^*/P}}{\frac{6}{5}M^*}. \tag{G.1} $$

To make the top spin better, the center of mass needs to be on the z-axis. Therefore, we desire

$$ {}^{\mathcal{B}}\boldsymbol{r}_{C/P} = \begin{pmatrix} 0 & 0 & z_C \end{pmatrix}^{\mathrm{T}}, \tag{G.2} $$

with arbitrary z_C (although it can be shown that a smaller z_C also makes the top spin better).

Furthermore, we must place one piece of cork on skewer 2 and one piece of cork on skewer 3 to correct for both unbalance in the x-direction and unbalance in the y-direction. In that case one of the coordinates of the pieces is always zero and its z-coordinate is always on the set height of the cork at Q. We choose to put cork D on skewer 2 and E on skewer 3:

$$ {}^{\mathcal{B}}\boldsymbol{r}_{D/P} = \begin{pmatrix} x_D \\ 0 \\ \frac{l}{2} - \Delta \end{pmatrix}, \qquad {}^{\mathcal{B}}\boldsymbol{r}_{E/P} = \begin{pmatrix} 0 \\ y_E \\ \frac{l}{2} - \Delta \end{pmatrix} \tag{G.3} $$

By substituting (G.2) and (G.3) in (G.1) a set of equations is obtained:

$$ \begin{pmatrix} \frac{1}{10}M^*x_D + 0 + M^*x_{C^*} \\ 0 + \frac{1}{10}M^*y_E + M^*y_{C^*} \\ 2\frac{1}{10}M^*\left(\frac{l}{2} - \Delta\right) + M^*r_{C^*z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{6}{5}M^*z_C \end{pmatrix} \tag{G.4} $$

This set can be solved to find the missing cork coordinates:

$$ x_D = -10x_{C^*}, \qquad y_E = -10y_{C^*}. \tag{G.5} $$

The position vectors of the two pieces of cork are:

$$ {}^{\mathcal{B}}\boldsymbol{r}_{D/P} = \begin{pmatrix} -10x_{C^*} \\ 0 \\ \frac{l}{2} - \Delta \end{pmatrix}, \qquad {}^{\mathcal{B}}\boldsymbol{r}_{E/P} = \begin{pmatrix} 0 \\ -10y_{C^*} \\ \frac{l}{2} - \Delta \end{pmatrix}. \tag{G.6} $$

1.9 Figure G.1 shows a possible FBD. As there is only point contact, no rotations can be prevented by the ground, so no moments are drawn.

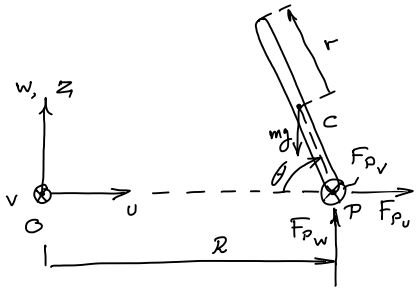


Figure G.1: Disk free-body diagram.

1.13 The kinematics of all animals are described in Figure G.2.

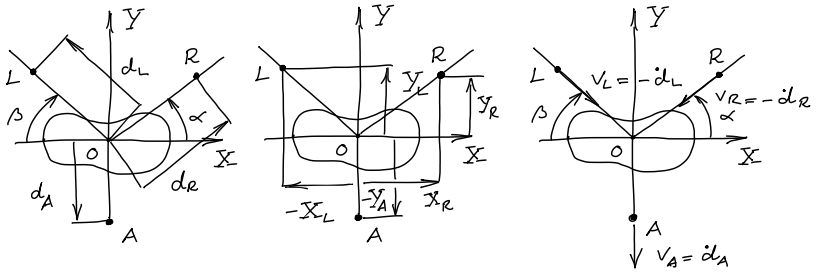


Figure G.2: Describing kinematics of all animals. Left: Location in terms of distance and angle. Center: Cartesian coordinates. Right: Velocity

The correct signs for the rhino's position vector are:

$$\boldsymbol{r}_R = \begin{pmatrix} +X_R \\ +Y_R \\ +Z_R \end{pmatrix} = \begin{pmatrix} +d_R \cos\alpha \\ +d_R \sin\alpha \\ 0 \end{pmatrix} \tag{G.7}$$

The correct signs for its velocity vector are:

$$\boldsymbol{v}_R = \begin{pmatrix} -v_R \cos\alpha \\ -v_R \sin\alpha \\ 0 \end{pmatrix} = +\dot{\boldsymbol{r}}_R = \begin{pmatrix} +\dot{X}_R \\ +\dot{Y}_R \\ +\dot{Z}_R \end{pmatrix} = \begin{pmatrix} \dot{d}_R \cos\alpha - d_R \overset{0}{\dot{\alpha}} \sin\alpha \\ \dot{d}_R \sin\alpha + d_R \overset{0}{\dot{\alpha}} \cos\alpha \\ 0 \end{pmatrix} \tag{G.8}$$

The correct signs for the antelope's position vector are:

$$\boldsymbol{r}_A = \begin{pmatrix} +X_A \\ +Y_A \\ +Z_A \end{pmatrix} = \begin{pmatrix} 0 \\ -d_A \\ 0 \end{pmatrix} \tag{G.9}$$

The correct signs for its velocity vector are:

$$\boldsymbol{v}_A = \begin{pmatrix} 0 \\ -v_A \\ 0 \end{pmatrix} = +\dot{\boldsymbol{r}}_A = \begin{pmatrix} +\dot{X}_A \\ +\dot{Y}_A \\ +\dot{Z}_A \end{pmatrix} = \begin{pmatrix} 0 \\ -\dot{d}_A \\ 0 \end{pmatrix} \tag{G.10}$$

1.22 Using the origin of xyz as reference point in (1.11), the center of mass $\boldsymbol{r}_C$ of the combined body is calculated from:

$$\boldsymbol{r}_C = \frac{m\boldsymbol{r}_\mathrm{D} + m_a\boldsymbol{r}_A}{m + m_a}, \tag{G.11}$$

with $\boldsymbol{r}_A = (r_x,\, r_y,\, r_z)^\mathrm{T}$ the position vector of the mass at A, with components

$$r_x = d, \qquad r_y = R\cos\varphi, \qquad r_z = R\sin\varphi. \tag{G.12}$$

Substituting these values in (G.11) gives

$$\boldsymbol{r}_C = \frac{1}{m + m_a}\left(mR\begin{pmatrix}\frac{1}{40}\\ \frac{1}{50}\\ \frac{1}{50}\end{pmatrix} + m_a\begin{pmatrix}d\\ R\cos\varphi\\ R\sin\varphi\end{pmatrix}\right). \tag{G.13}$$

For static balance, the center of mass must be on the x-axis:

$$\boldsymbol{r}_C \stackrel{!}{=} \begin{pmatrix}r_{Cx} & 0 & 0\end{pmatrix}^\mathrm{T}, \tag{G.14}$$

with arbitrary r_{Cx}. Combination of (G.13) and (G.14) gives the conditions:

$$\frac{mR}{40} + m_a d \stackrel{!}{=} r_{Cx}\,, \tag{G.15}$$

$$\frac{mR}{50} + m_a R\cos\varphi \stackrel{!}{=} 0\,, \tag{G.16}$$

$$\frac{mR}{50} + m_a R\sin\varphi \stackrel{!}{=} 0\,, \tag{G.17}$$

which means that d is arbitrary, and $\sin\varphi = \cos\varphi$. A possible choice is (avoiding negative mass):

$$\varphi = \frac{5}{4}\pi, \qquad d = 0, \qquad m_a = \frac{\sqrt{2}}{50}m. \tag{G.18}$$

1.24 The FBD of the skateboarder is shown in Figure G.3. *Remark:*Note that this FBD would not yet be sufficient to determine the equations of motion and the unknown forces and moments. For example, some forces cannot uniquely be determined, and one needs to allow local slip at the wheels for realistic movement.

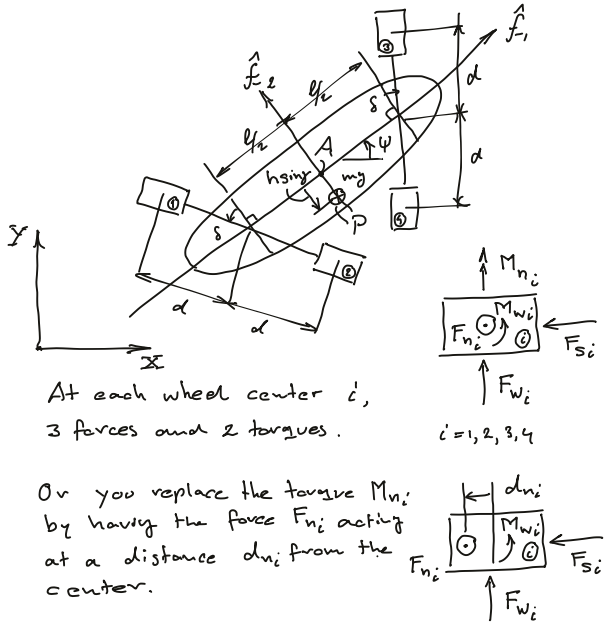


Figure G.3: Free-body diagram of the skateboarder.

1.26 The FBD for the sphere and the cage are shown in Figure G.4.

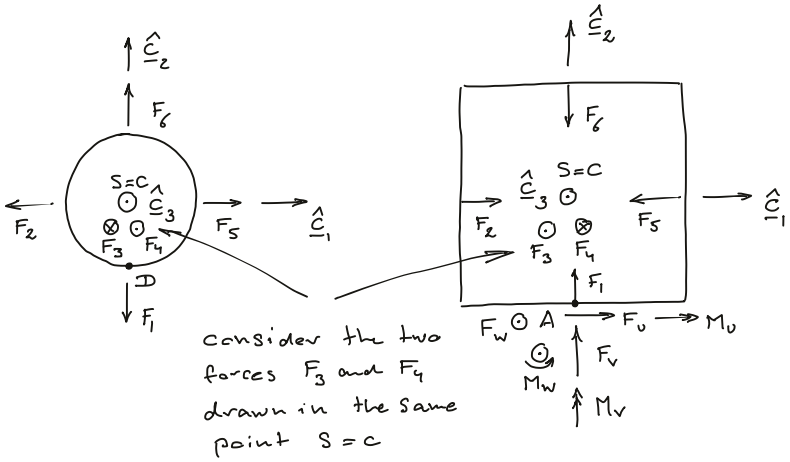


Figure G.4: Free-body diagrams for the accelerometer.

1.31 Possible FBD's of the scooter's footboard and rear wheel are drawn in Figure G.5.

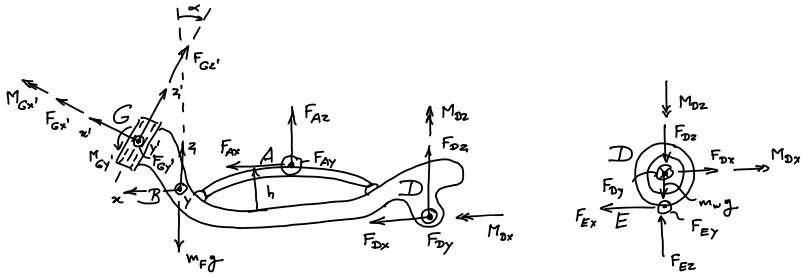


Figure G.5: FBDs of the scooter's footboard and rear wheel.

2.3 From vector addition, we know the relationship

$$\boldsymbol{r}_{i/C} = \boldsymbol{r}_i - \boldsymbol{r}_C, \tag{G.19}$$

which allows us to re-write the left version of (2.18),

$$\boldsymbol{H}_C = \sum_{i=1}^{N} (\boldsymbol{r}_{i/C} \times (m_i \dot{\boldsymbol{r}}_{i/C})), \tag{G.20}$$

to

$$\begin{aligned} \boldsymbol{H}_C &= \sum_{i=1}^{N} (\boldsymbol{r}_{i/C} \times (m_i (\dot{\boldsymbol{r}}_i - \dot{\boldsymbol{r}}_C))) && \text{(G.21)} \\ &= \sum_{i=1}^{N} (\boldsymbol{r}_{i/C} \times (m_i \dot{\boldsymbol{r}}_i)) - \left(\sum_{i=1}^{N} m_i \boldsymbol{r}_{i/C} \right) \times \dot{\boldsymbol{r}}_C && \text{(G.22)} \end{aligned}$$

From the definition of the center of mass, we know that

$$\sum_{i=1}^{N} m_i \boldsymbol{r}_i = \sum_{i=1}^{N} m_i \boldsymbol{r}_C, \tag{G.23}$$

which implies that

$$\sum_{i=1}^{N} m_i \boldsymbol{r}_{i/C} = \sum_{i=1}^{N} m_i (\boldsymbol{r}_i - \boldsymbol{r}_C) = \boldsymbol{0}, \tag{G.24}$$

so (G.22) simplifies to

$$\boldsymbol{H}_C = \sum_{i=1}^{N} \boldsymbol{r}_{i/C} \times (m_i \dot{\boldsymbol{r}}_i). \tag{G.25}$$

2.5 For a (sub)system that has its center of mass in point C, we find angular momentum with respect to another point Q using summation over each particle i in the (sub)system, following (2.9) and (2.10):

$$\boldsymbol{H}_Q = \sum_{i=1}^{N} \left(\boldsymbol{r}_{i/Q} \times \left(m_i \dot{\boldsymbol{r}}_{i/Q} \right) \right) . \tag{G.26}$$

Making use of the vector addition

$$\boldsymbol{r}_{i/Q} = \boldsymbol{r}_{i/C} + \boldsymbol{r}_{C/Q} , \quad \dot{\boldsymbol{r}}_{i/Q} = \dot{\boldsymbol{r}}_{i/C} + \dot{\boldsymbol{r}}_{C/Q} \tag{G.27}$$

we can expand (G.26) to

$$\begin{aligned} \boldsymbol{H}_Q \;=\; & \sum_{i=1}^{N} \left(\boldsymbol{r}_{i/C} \times \left(m_i \dot{\boldsymbol{r}}_{i/C} \right) \right) + \sum_{i=1}^{N} \left(\boldsymbol{r}_{C/Q} \times \left(m_i \dot{\boldsymbol{r}}_{i/C} \right) \right) \\ & + \sum_{i=1}^{N} \left(\boldsymbol{r}_{i/C} \times \left(m_i \dot{\boldsymbol{r}}_{C/Q} \right) \right) + \sum_{i=1}^{N} \left(\boldsymbol{r}_{C/Q} \times \left(m_i \dot{\boldsymbol{r}}_{C/Q} \right) \right) \end{aligned} \tag{G.28}$$

We recognize that the first term is nothing else but the (sub)-system's angular momentum with respect to its center of mass. The second and third term can be simplified by recognizing that $\boldsymbol{r}_{C/Q}$ and $\dot{\boldsymbol{r}}_{C/Q}$ do not depend on i, so they can be factored out of the summation. Furthermore, by virtue of the definition of the center of mass, we have $\sum_{i=1}^{N} m_i \boldsymbol{r}_{i/C} = \boldsymbol{0}$ and also $\sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_{i/C} = \boldsymbol{0}$. See a similar and more explicit reasoning for this in the solution of Problem 2.3.

Finally, we retain only the first and fourth summand of (G.28), which can be written as the expected relationship:

$$\boldsymbol{H}_Q = \boldsymbol{H}_C + \boldsymbol{r}_{C/Q} \times \left(m \dot{\boldsymbol{r}}_{C/Q} \right) . \tag{G.29}$$

2.16 One example is drawn in Figure G.6. We can "follow" the force field along a closed, circular path, so net work is being done by the force on that path.

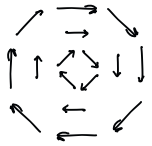


Figure G.6: A non-conservative force field.

2.17 A sliding friction force is directed against the movement direction, so it is tangent to the path. A specific counterexample showing that such a force is not conservative: A particle moves from point A to point B via two different paths: Path 1 is straight from A to B. Path 2 is longer: straight from A to

B, then straight back to A, finally straight back to B. The velocity in this example always has constant magnitude, and friction properties are the same in both directions. The work done by the sliding friction force in the second case is three times as large as in the first case, although the two paths' start and end points were the same. Therefore, the work depends on the path taken, and the force is not conservative.

2.19 (Note that this is also treated as an example in Example 12.7, where illustrations of the conservative cases are shown.)

Gravitational force: For the gravitational force, suppose an object is at a positive height y above an arbitrarily defined level of zero potential (which can for example be the ground), and gravity with acceleration g acts in negative y-direction (see also Figure 12.11). The object's weight mg is the force acting on the object in negative y-direction such that (2.55) becomes

$$F_y = -mg = -\frac{\partial V}{\partial y}, \tag{G.30}$$

which can be integrated to give

$$V(y) = mgy + V_0. \tag{G.31}$$

Only in the special case where V is defined to be zero at $y = 0$, the integration constant is found as $V_0 = 0$.

Spring force: For the spring force, we consider an object that is attached to a fixed point by means of a linear spring with stiffness k (see also Figure 12.12). When there is no force acting, the spring has length x_0. The object can only move in the x-direction, such that (2.55) becomes

$$F_x = -k(x - x_0) = -\frac{\partial V}{\partial x}, \tag{G.32}$$

which can be integrated to give

$$V(x) = \frac{1}{2}k(x - x_0)^2 + V_0. \tag{G.33}$$

In the special case where V is defined to be zero at $x = x_0$, the integration constant is found as $V_0 = 0$.

Friction force during sliding: For the friction force, we consider an example where an object is supported by a flat surface and can only slide straight along the x-axis with friction. Both Coulomb and viscous friction act on it. The Coulomb friction term is proportional to the normal force of magnitude N that is acting on the object from the surface. With Coulomb friction coefficient μ and viscous friction coefficient d, the force F_x acting on the object in positive x-direction is:

$$F_x = -\mu N \mathrm{sgn}(\dot{x}) - d\dot{x}. \tag{G.34}$$

We cannot obtain a potential function $V(x)$ for either of the two damping terms, because the force does not depend on position, but on velocity. So, (2.55) cannot be solved for V. This counterexample shows that sliding friction is not a conservative force. Also note that regardless of the direction of motion, the force will always dissipate energy.

2.25 The system *does not conserve all linear momenta*, because gravity, as an external force, changes the linear momentum component in the vertical direction (so the velocity of the system's center of mass is not constant). The system *does not conserve energy* either, since the internal forces between body and hands do work. However, the system *does conserve angular momenta* about the center of mass, because the only external force (gravity) generates no moment about the system's center of mass.

3.6 We know from (3.17) that the columns form a triad of orthogonal unit vectors. Therefore, one can calculate the third column by taking the cross product of the first with the second column. Be aware that this order is important, such that the triad is right-handed:

$$^{\mathcal{B}}\hat{\boldsymbol{n}}_3 = {}^{\mathcal{B}}\hat{\boldsymbol{n}}_1 \times {}^{\mathcal{B}}\hat{\boldsymbol{n}}_2. \tag{G.35}$$

3.7 A rotation of triads about the Z-axis is depicted in Figure G.7.

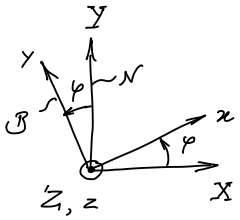


Figure G.7: Rotation of a coordinate triad about the Z-axis.

From the drawing, we can write down the components of units vectors of the rotated x and of the y-axes ($Z = z$ is unchanged), and we find that these are the rows of the rotation matrix.

3.8 A rotation of triads about the Y-axis is depicted in Figure G.8.

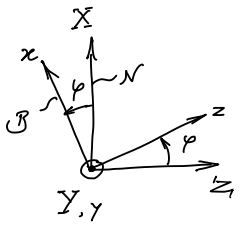


Figure G.8: Rotation of a coordinate triad about the Y-axis.

From the drawing, we can write down the components of units vectors of the x and of the z-axes, and we thereby populate the rows of the rotation matrix ($Y = y$ is unchanged):

$$\mathbf{C}_Y = \begin{pmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix}. \tag{G.36}$$

3.13 The different views and point A are shown in Figure G.9.

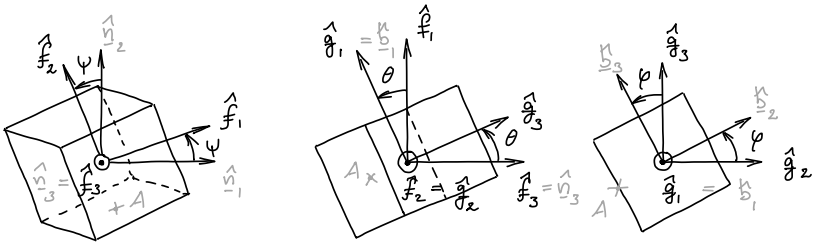


Figure G.9: Cartoon showing three different views of the cube.

3.14 As intermediate triads, we choose $\mathcal{F}$ and $\mathcal{G}$. The transformation steps are as drawn in Figure G.10.

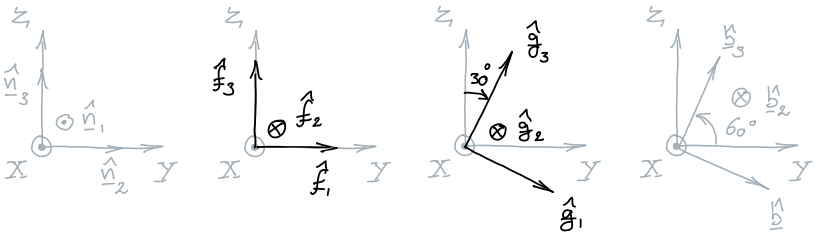


Figure G.10: Transformation steps of the coordinate directions.

The three Euler angles are:

$$\psi = \frac{\pi}{2}, \qquad \theta = \frac{\pi}{6}, \qquad \varphi = 0. \tag{G.37}$$

3.17 According to Euler, when a fixed point exists (in this case the ball's center), the succession of all movements the ball underwent in the first half time can be described by one single equivalent rotation about one axis. This principal axis of rotation passes through the fixed point and thus intersects the ball's surface in two points. These two points must therefore be at the same location as they were before. In Euler's words: "Quomodocunque sphaera circa centrum suum convertatur, semper assignari potest diameter, cuius directio in situ translato conveniat cum situ initiali" [15].

3.25 The transformation steps are as drawn in Figure G.11.

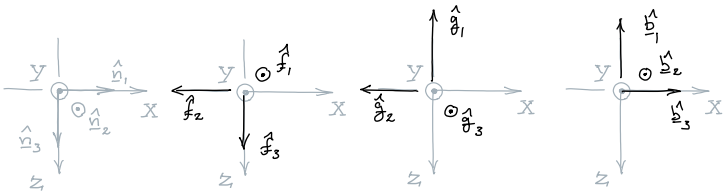


Figure G.11: Transformation steps of the coordinate axes via Euler angles of the model plane.

3.31 The Euler angle θ is 120°. A unit vector for the principal axis in the specified special case would be

$$^{\mathcal{B}}\hat{\boldsymbol{a}} = {}^{\mathcal{B}}\hat{\boldsymbol{f}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{G.38}$$

3.32 We consider an intermediate coordinate system $X'Y'Z'$ with an associated intermediate triad $\mathcal{F}$. This coordinate system and triad only perform the rotation about the Z-axis, but not about the new x-axis, such that $X' = x$ and $Z' = Z$. The total rotation matrix is

$$^{\mathcal{N}}\mathbf{C}_{\mathcal{B}} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}^{\mathrm{T}}, \tag{G.39}$$

where the individual rotation matrices are combined as

$$^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}\,{}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}}. \tag{G.40}$$

The rotation from $\mathcal{N}$ to $\mathcal{F}$ by angle α is depicted in Figure G.12,

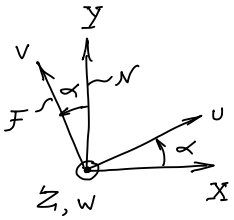


Figure G.12: Rotation of the 3-D pendulum about the Z-axis.

and we obtain

$$^{\mathcal{F}}\mathbf{C}_{\mathcal{N}} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{G.41}$$

The rotation from $\mathcal{F}$ to $\mathcal{B}$ by angle β is depicted in Figure G.13.

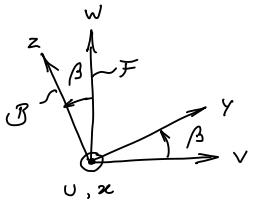


Figure G.13: Rotation of the 3-D pendulum about the x-axis.

From the figure, we see that

$$ {}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix}. \tag{G.42} $$

Applying the successive rotations results in the complete rotation matrix:

$$ {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha\cos\beta & \cos\alpha\cos\beta & \sin\beta \\ \sin\alpha\sin\beta & -\cos\alpha\sin\beta & \cos\beta \end{pmatrix}. \tag{G.43} $$

Its inverse/transpose is the required matrix:

$$ {}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}} = \begin{pmatrix} \cos\alpha & -\sin\alpha\cos\beta & \sin\alpha\sin\beta \\ \sin\alpha & \cos\alpha\cos\beta & -\cos\alpha\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix}. \tag{G.44} $$

3.33 Successive rotation:

$$ {}^{\mathcal{F}}\mathbf{C}_{\mathcal{G}} = {}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}} \cdot {}^{\mathcal{N}}\mathbf{C}_{\mathcal{G}}, \tag{G.45} $$

with

$$ {}^{\mathcal{N}}\mathbf{C}_{\mathcal{G}} = \begin{pmatrix} {}^{\mathcal{N}}\hat{\boldsymbol{g}}_1 & {}^{\mathcal{N}}\hat{\boldsymbol{g}}_2 & {}^{\mathcal{N}}\hat{\boldsymbol{g}}_3 \end{pmatrix}, \tag{G.46} $$

$$ {}^{\mathcal{N}}\hat{\boldsymbol{g}}_2 = {}^{\mathcal{N}}\hat{\boldsymbol{g}}_3 \times {}^{\mathcal{N}}\hat{\boldsymbol{g}}_1, \tag{G.47} $$

and (for rotation about the z-axis)

$$ {}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{G.48} $$

Filling in values gives:

$$ {}^{\mathcal{N}}\hat{\boldsymbol{g}}_2 \overset{\text{(G.47)}}{=} \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \times \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{4} \\ -\frac{1}{4} \end{pmatrix} $$

$$\Rightarrow \quad {}^{\mathcal{N}}\mathbf{C}_{\mathcal{G}} \overset{(G.46)}{=} \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2} \\ \frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{2} \end{pmatrix}, \qquad {}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}} \overset{(G.48)}{=} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overset{(G.45)}{\Rightarrow} \qquad {}^{\mathcal{F}}\mathbf{C}_{\mathcal{G}} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \\ \frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

3.34

$$ {}^{\mathcal{N}}\boldsymbol{F}_s = \underbrace{k\,(|\boldsymbol{l}_s| - l_0)}_{\text{magnitude}} \cdot \underbrace{\frac{\boldsymbol{l}_s}{|\boldsymbol{l}_s|}}_{\text{direction}} \tag{G.49}$$

where $\boldsymbol{l}_s$ is the vector from point P to point Q.

$$ {}^{\mathcal{N}}\boldsymbol{l}_s = -{}^{\mathcal{N}}\boldsymbol{l}_1 + {}^{\mathcal{N}}\boldsymbol{l}_{12} + {}^{\mathcal{N}}\boldsymbol{l}_2 \tag{G.50}$$

$$ {}^{\mathcal{N}}\boldsymbol{l}_1 = {}^{\mathcal{F}}\boldsymbol{l}_1 \qquad \text{(not affected by rotation)} \tag{G.51}$$

$$ {}^{\mathcal{N}}\boldsymbol{l}_2 = \hat{\boldsymbol{g}}_1 \cdot 12\,\mathrm{m} + \hat{\boldsymbol{g}}_3 \cdot 4\,\mathrm{m} \tag{G.52}$$

$$\Rightarrow \qquad {}^{\mathcal{N}}\boldsymbol{l}_2 \overset{(G.52)}{=} \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix} \cdot 12\,\mathrm{m} + \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \cdot 4\,\mathrm{m} = \begin{pmatrix} -6 \\ 7 \\ 5\sqrt{3} \end{pmatrix} \mathrm{m}$$

$$\overset{(G.50)}{\Rightarrow} \qquad {}^{\mathcal{N}}\boldsymbol{l}_s = \begin{pmatrix} 40 \\ 80 \\ 10 \end{pmatrix} \mathrm{m} \qquad \Rightarrow \qquad |\boldsymbol{l}_s| = 90\,\mathrm{m}$$

$$\overset{(G.49)}{\Rightarrow} \qquad {}^{\mathcal{N}}\boldsymbol{F}_s = 9000\,\mathrm{N/m} \cdot (90-89)\,\mathrm{m} \cdot \frac{1}{90\,\mathrm{m}} \cdot \begin{pmatrix} 40 \\ 80 \\ 10 \end{pmatrix} \mathrm{m} = \begin{pmatrix} 4000 \\ 8000 \\ 1000 \end{pmatrix} \mathrm{N} \tag{G.53}$$

Alternative solution

The vector ${}^{\mathcal{N}}\boldsymbol{l}_2$ can also be found by using a rotation matrix:

$$ {}^{\mathcal{N}}\boldsymbol{l}_2 = {}^{\mathcal{N}}\mathbf{C}_{\mathcal{G}} \cdot {}^{\mathcal{G}}\boldsymbol{l}_2 \tag{G.54}$$

3.35 Roll: rotation about r-axis by angle γ is described by ${}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}$. Then,

$$ {}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}}\,{}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}} = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} \tag{G.55}$$

$$\Rightarrow \qquad {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{G.56}$$

$$ {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2}\sqrt{2}\cos\gamma & \frac{1}{2}\sqrt{2}\cos\gamma & \sin\gamma \\ \frac{1}{2}\sqrt{2}\sin\gamma & -\frac{1}{2}\sqrt{2}\sin\gamma & \cos\gamma \end{pmatrix} \tag{G.57}$$

3.37 The position vector of Delft is:

$$^{\mathcal{N}}\boldsymbol{D} = {}^{\mathcal{N}}\boldsymbol{r}_{D/S} = {}^{\mathcal{N}}\boldsymbol{r}_{E/S} + {}^{\mathcal{N}}\boldsymbol{r}_{D/E} \tag{G.58}$$

$$= -R\,{}^{\mathcal{N}}\hat{\boldsymbol{n}}_2 + r\,{}^{\mathcal{N}}\hat{\boldsymbol{l}}_3 = -R\,{}^{\mathcal{N}}\hat{\boldsymbol{n}}_2 + r\,{}^{\mathcal{L}}\mathbf{C}_{\mathcal{N}}^{\mathrm{T}}\,{}^{\mathcal{L}}\hat{\boldsymbol{l}}_3 \tag{G.59}$$

$$^{\mathcal{N}}\boldsymbol{D} = \begin{pmatrix} 0 \\ -R \\ 0 \end{pmatrix} + {}^{\mathcal{L}}\mathbf{C}_{\mathcal{N}}^{\mathrm{T}} \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \tag{G.60}$$

4.4 A unit vector that points in the direction of the instantaneous axis of rotation is given by the angular velocity vector divided by its magnitude:

$$^{\mathcal{B}}\hat{\boldsymbol{u}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \left(= \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} \right) \tag{G.61}$$

4.5 The angular velocity vectors may be added:

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{D}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{F}} + \boldsymbol{\omega}_{\mathcal{F}/\mathcal{D}}. \tag{G.62}$$

4.6 Seen from the opposite triad, this is the inverse rotation. This means the instantaneous axis of rotation has opposite direction, but the magnitude of angular velocity is the same, so:

$$\boldsymbol{\omega}_{\mathcal{N}/\mathcal{F}} = -\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \tag{G.63}$$

4.7 The velocity vector is found by the cross product:

$$^{\mathcal{D}}\boldsymbol{v}_W = {}^{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \times {}^{\mathcal{D}}\boldsymbol{r}_{W/A} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} \times \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} -b\dot{\psi} \\ 0 \\ 0 \end{pmatrix} \tag{G.64}$$

$$\Rightarrow {}^{\mathcal{D}}\boldsymbol{v}_W = -b\dot{\psi}\,{}^{\mathcal{D}}\hat{\boldsymbol{d}}_1 \tag{G.65}$$

4.13 Applying the transport theorem to velocity $\boldsymbol{v}$ gives the acceleration $\boldsymbol{a}$:

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = (\dot{\boldsymbol{v}})_{\mathcal{N}} = (\dot{\boldsymbol{v}})_{\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{v}. \tag{G.66}$$

We can substitute the expression for velocity from (4.44) and the angular velocity from (4.41) and reach:

$$\boldsymbol{a} = \ddot{\rho}\hat{\boldsymbol{e}}_\rho + \ddot{\varphi}\rho\hat{\boldsymbol{e}}_\varphi + \dot{\varphi}\dot{\rho}\hat{\boldsymbol{e}}_\varphi + \ddot{Z}_P\hat{\boldsymbol{n}}_3 + (\dot{\varphi}\hat{\boldsymbol{n}}_3) \times (\dot{\rho}\hat{\boldsymbol{e}}_\rho + \dot{\varphi}\rho\hat{\boldsymbol{e}}_\varphi + \dot{Z}_P\hat{\boldsymbol{n}}_3). \tag{G.67}$$

The cross products of the unit vectors can be determined quickly using the right-hand rule:

$$\hat{\boldsymbol{n}}_3 \times \hat{\boldsymbol{e}}_\rho = \hat{\boldsymbol{e}}_\varphi, \quad \hat{\boldsymbol{n}}_3 \times \hat{\boldsymbol{e}}_\varphi = -\hat{\boldsymbol{e}}_\rho, \quad \hat{\boldsymbol{n}}_3 \times \hat{\boldsymbol{n}}_3 = \mathbf{0}. \tag{G.68}$$

Using this and some simplification, the final result is:

$$\boldsymbol{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\hat{\boldsymbol{e}}_\rho + (\ddot{\varphi}\rho + 2\dot{\varphi}\dot{\rho})\hat{\boldsymbol{e}}_\varphi + \ddot{Z}_P\hat{\boldsymbol{n}}_3. \tag{G.69}$$

4.16 The tangential direction is the direction of the velocity vector, so the unit direction vector is:

$$ {}^{\mathcal{N}}\hat{\boldsymbol{e}}_{\mathrm{t}} = \frac{{}^{\mathcal{N}}\dot{\boldsymbol{D}}}{\left|{}^{\mathcal{N}}\dot{\boldsymbol{D}}\right|} \tag{G.70} $$

The acceleration ${}^{\mathcal{N}}\boldsymbol{a} = {}^{\mathcal{N}}\ddot{\boldsymbol{D}}$ in natural coordinates can only have components in tangential and normal direction, which span the osculating plane (Figure G.14).

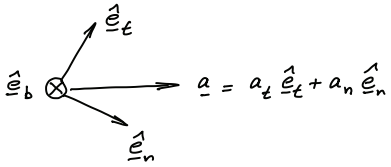


Figure G.14: Earth natural coordinates direction vectors and the acceleration.

The binormal direction is orthogonal to this plane, so also to velocity and acceleration vectors. The right-hand rule dictates its direction (because the normal acceleration component is always positive):

$$ {}^{\mathcal{N}}\hat{\boldsymbol{e}}_{\mathrm{b}} = \frac{{}^{\mathcal{N}}\dot{\boldsymbol{D}} \times {}^{\mathcal{N}}\ddot{\boldsymbol{D}}}{\left|{}^{\mathcal{N}}\dot{\boldsymbol{D}} \times {}^{\mathcal{N}}\ddot{\boldsymbol{D}}\right|}. \tag{G.71} $$

Alternative solution
We find the normal direction by subtracting the acceleration component in tangential direction from the overall acceleration vector, and then normalizing:

$$ {}^{\mathcal{N}}\hat{\boldsymbol{e}}_{\mathrm{n}} = \frac{{}^{\mathcal{N}}\ddot{\boldsymbol{D}} - \left({}^{\mathcal{N}}\ddot{\boldsymbol{D}} \cdot {}^{\mathcal{N}}\hat{\boldsymbol{e}}_{\mathrm{t}}\right) \cdot {}^{\mathcal{N}}\hat{\boldsymbol{e}}_{\mathrm{t}}}{\left|{}^{\mathcal{N}}\ddot{\boldsymbol{D}} - \left({}^{\mathcal{N}}\ddot{\boldsymbol{D}} \cdot {}^{\mathcal{N}}\hat{\boldsymbol{e}}_{\mathrm{t}}\right) \cdot {}^{\mathcal{N}}\hat{\boldsymbol{e}}_{\mathrm{t}}\right|} \tag{G.72} $$

The binormal direction is then found from the cross product of the tangential and normal direction vector, which are orthogonal to each other:

$$ {}^{\mathcal{N}}\hat{\boldsymbol{e}}_{\mathrm{b}} = {}^{\mathcal{N}}\hat{\boldsymbol{e}}_{\mathrm{t}} \times {}^{\mathcal{N}}\hat{\boldsymbol{e}}_{\mathrm{n}}. \tag{G.73} $$

Substituting (G.70) and (G.72) leads to the final answer.

4.25 The Coriolis acceleration $\boldsymbol{a}_C$ is

$$ {}^{\mathcal{F}}\boldsymbol{a}_C = 2\,{}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times {}^{\mathcal{F}}(\dot{\boldsymbol{p}})_{\mathcal{F}}, \tag{G.74} $$

with $\boldsymbol{p}$ the relative position of C with respect to O.

$$ {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}, \; {}^{\mathcal{F}}\boldsymbol{p} = \begin{pmatrix} 0 \\ h\sin(\gamma) - \frac{D}{2} \\ h\cos(\gamma) \end{pmatrix}, \; {}^{\mathcal{F}}(\dot{\boldsymbol{p}})_{\mathcal{F}} = h \begin{pmatrix} 0 \\ \dot{\gamma}\cos(\gamma) \\ -\dot{\gamma}\sin(\gamma) \end{pmatrix}. \tag{G.75} $$

This gives

$$ {}^{\mathcal{F}}\boldsymbol{a}_C = \begin{pmatrix} -2\dot{\psi}\dot{\gamma}h\cos(\gamma) \\ 0 \\ 0 \end{pmatrix}. \tag{G.76} $$

4.33 The instantaneous axis of rotation passes through *points A and P*, which are *points with zero velocity*. For this axis to be vertical and thereby being parallel to the Z-axis, the distance from the Z-axis to P needs to be zero:

$$ R = 0. \tag{G.77} $$

Alternative solution

The result in (4.68E) can be re-used in the solution for this question. The instantaneous axis of rotation is defined by the direction of the vector

$$ \boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{F}} N + \cancel{\boldsymbol{\omega}_{\mathcal{F}} G}^{\,0} + \boldsymbol{\omega}_{\mathcal{B}} G = \dot{\varphi}\hat{\boldsymbol{n}}_3 + \dot{\psi}\hat{\boldsymbol{g}}_3 \tag{G.78} $$

as shown in Figure G.15.

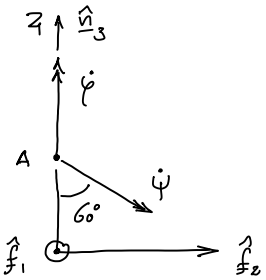


Figure G.15: Angular rotation vectors of the thin disk on an arm.

To be vertical, this vector needs to be equal to $\omega\hat{\boldsymbol{n}}_3$, with unknown scalar value ω:

$$ \boldsymbol{\omega} = \dot{\varphi}\hat{\boldsymbol{n}}_3 + \dot{\psi}\hat{\boldsymbol{g}}_3 \stackrel{!}{=} \omega\hat{\boldsymbol{n}}_3. \tag{G.79} $$

As $\hat{\boldsymbol{g}}_3$ is not aligned with $\hat{\boldsymbol{n}}_3$, $\dot{\psi}$ must be zero. We can re-use (4.68E), to see that R must be zero for any nonzero $\dot{\varphi}$.

4.37 A free-body diagram could look as in Figure G.16. The cuts expose the forces acting on the particle, namely those exerted by the spring and by the tube (but only in x'-direction, see problem statement), and gravity.

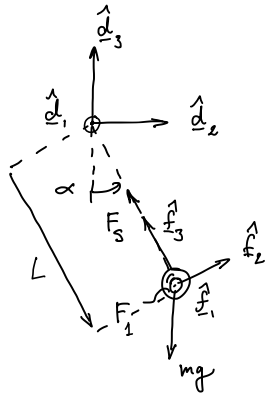


Figure G.16: FBD of the particle in a tube.

Newton's second law with the sum of the forces on the particle $\sum \boldsymbol{F}$ and the acceleration of the particle $\boldsymbol{a}$ is:

$$\sum \boldsymbol{F} = m\boldsymbol{a}, \text{ with } \sum \boldsymbol{F} = (F_s - mg\cos\alpha)\hat{\boldsymbol{f}}_3 - (mg\sin\alpha)\hat{\boldsymbol{f}}_2. \tag{G.80}$$

Acceleration is calculated using the rotating triad $\mathcal{F}$:

$$\boldsymbol{a} = \ddot{\boldsymbol{Q}} + (\ddot{\boldsymbol{p}})_{\mathcal{F}} + \dot{\boldsymbol{\omega}}_{\mathcal{F}/\mathcal{N}} \times \boldsymbol{p} + \boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times \big(\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times \boldsymbol{p}\big) + 2\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times (\dot{\boldsymbol{p}})_{\mathcal{F}}, \tag{G.81}$$

with the choices:

Theory (4.69)	**This example**
point P	location of particle
origin of XYZ	point O
point Q	point O
$\mathcal{F}$	$\mathcal{F}$
$\mathcal{N}$	$\mathcal{N}$

The values for relative position and for angular velocity are:

$${}^{\mathcal{F}}\boldsymbol{p} = \begin{pmatrix} 0 \\ 0 \\ -L \end{pmatrix}, {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} = \begin{pmatrix} \dot{\alpha} \\ \dot{\beta}\sin\alpha \\ \dot{\beta}\cos\alpha \end{pmatrix} \tag{G.82}$$

The needed angular acceleration can be expressed as a function of the time derivatives of the angles and the current configuration:

$${}^{\mathcal{F}}\dot{\boldsymbol{\omega}}_{\mathcal{F}/\mathcal{N}} = {}^{\mathcal{F}}(\dot{\boldsymbol{\omega}}_{\mathcal{F}/\mathcal{N}})_{\mathcal{F}} = \begin{pmatrix} \ddot{\alpha} \\ \ddot{\beta}\sin\alpha + \dot{\beta}\dot{\alpha}\cos\alpha \\ \ddot{\beta}\cos\alpha - \dot{\beta}\dot{\alpha}\sin\alpha \end{pmatrix} \tag{G.83}$$

Substituting all necessary terms for acceleration in (G.81) provides:

$$
{}^{\mathcal{F}}\boldsymbol{a} = \begin{pmatrix} -2L\dot{\alpha}\dot{\beta}\cos\alpha - L\ddot{\beta}\sin\alpha \\ -L\dot{\beta}^2\cos\alpha\sin\alpha + L\ddot{\alpha} \\ L\dot{\alpha}^2 + L\dot{\beta}^2\sin^2\alpha \end{pmatrix}. \tag{G.84}
$$

Finally, Newton's second law in y''-direction can be established from (G.80), and simplification for the case $\ddot{\beta} = \ddot{\alpha} = 0$ delivers:

$$
\Rightarrow \qquad \underbrace{-mg\sin\alpha}_{\sum \boldsymbol{F}\cdot\hat{\boldsymbol{f}}_2} = \underbrace{-mL\dot{\beta}^2\cos\alpha\sin\alpha}_{m\boldsymbol{a}\cdot\hat{\boldsymbol{f}}_2}, \tag{G.85}
$$

which in the case of $\alpha \neq 0$ leads to the expected relationship

$$
\cos\alpha = \frac{g}{L\dot{\beta}^2} \tag{G.86}
$$

4.38 From the disk's kinematic diagram in Figure G.17, we obtain the rotation matrix, (angular) velocity vectors, and position vectors:

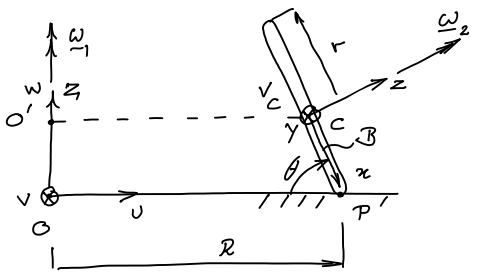


Figure G.17: Steady rolling disk kinematic diagram.

$$
{}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \tag{G.87}
$$

$$
{}^{\mathcal{N}}\boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ 0 \\ \omega_1 \end{pmatrix}, \qquad {}^{\mathcal{B}}\boldsymbol{\omega}_2 = \begin{pmatrix} 0 \\ 0 \\ \omega_2 \end{pmatrix}, \qquad {}^{\mathcal{N}}\boldsymbol{v}_C = \begin{pmatrix} 0 \\ v_C \\ 0 \end{pmatrix}, \tag{G.88}
$$

$$
{}^{\mathcal{B}}\boldsymbol{r}_{C/P} = \begin{pmatrix} -r \\ 0 \\ 0 \end{pmatrix}, \qquad {}^{\mathcal{N}}\boldsymbol{r}_{C/O'} = \begin{pmatrix} R - r\cos\theta \\ 0 \\ 0 \end{pmatrix}. \tag{G.89}
$$

The velocity vector $\boldsymbol{v}_C$ of C can be expressed in two ways. First, by a pure rotation around the Z-axis:

$$
{}^{\mathcal{N}}\boldsymbol{v} = {}^{\mathcal{N}}\boldsymbol{\omega}_1 \times {}^{\mathcal{N}}\boldsymbol{r}_{C/O'} = \begin{pmatrix} 0 \\ \omega_1\,(R - r\cos\theta) \\ 0 \end{pmatrix}. \tag{G.90}
$$

This shows

$$\omega_1 = \frac{v_C}{R - r\cos\theta}. \tag{G.91}$$

Second, by using the fact that the contact point P has zero velocity, which means it must be a point on the disk's instantaneous axis of rotation, so:

$$^{\mathcal{N}}\boldsymbol{v}_C = \left(^{\mathcal{N}}\boldsymbol{\omega}_1 + {}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}}\,{}^{\mathcal{B}}\boldsymbol{\omega}_2\right) \times ({}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}}\,{}^{\mathcal{B}}\boldsymbol{r}_{C/P}), \tag{G.92}$$

$$= \begin{pmatrix} 0 \\ -r\left(\omega_1\cos\theta + \omega_2\right) \\ 0 \end{pmatrix}. \tag{G.93}$$

This shows

$$\omega_2 = -\omega_1\cos\theta - \frac{v_C}{r}, \tag{G.94}$$

and with (G.91) we obtain

$$\omega_2 = -\frac{R}{r}\omega_1 = -\frac{Rv_C}{r\left(R - r\cos\theta\right)}. \tag{G.95}$$

Finally, we are able to populate the complete angular velocity vector as a function of v_C:

$$^{\mathcal{N}}\boldsymbol{\omega} = {}^{\mathcal{N}}\boldsymbol{\omega}_1 + {}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}}\,{}^{\mathcal{B}}\boldsymbol{\omega}_2 \tag{G.96}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \frac{v_C}{R - r\cos\theta} \end{pmatrix} + \begin{pmatrix} \frac{Rv_C}{r(r\cos\theta - R)}\sin\theta \\ 0 \\ \frac{Rv_C}{r(r\cos\theta - R)}\cos\theta \end{pmatrix}, \tag{G.97}$$

$$\Rightarrow {}^{\mathcal{N}}\boldsymbol{\omega} = \frac{v_C}{r(R - r\cos\theta)} \begin{pmatrix} -R\sin\theta \\ 0 \\ r - R\cos\theta \end{pmatrix}. \tag{G.98}$$

4.41 In this case, we do not need to make use of the rotating coordinate system in the FBD (see Figure G.18).

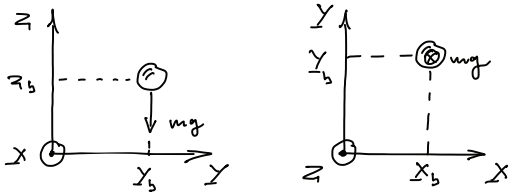


Figure G.18: Projected FBDs of the ball during flight after release by the person.

$$\boldsymbol{F} = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} = m\ddot{\boldsymbol{r}}_{\mathrm{b}} \qquad \Rightarrow \qquad \ddot{X}_{\mathrm{b}} = 0,\ \ddot{Y}_{\mathrm{b}} = 0,\ \ddot{Z}_{\mathrm{b}} = -g \tag{G.99}$$

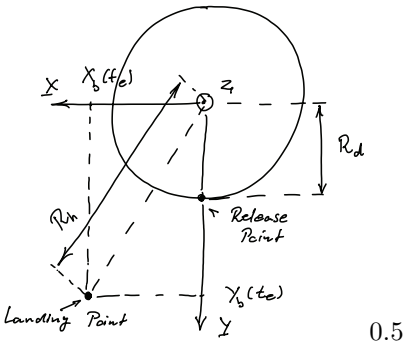
Integrating twice over time t gives the Z-coordinate:

$$Z_\mathrm{b} = -\frac{1}{2}gt^2 + v_{Z0}t + h. \tag{G.100}$$

With that and the given $v_{Z0} = \frac{1}{2}\sqrt{hg}$, we can solve for the time t_e when the ball hits the ground, which implies that $Z_\mathrm{b}(t_\mathrm{e}) = 0$:

$$t_\mathrm{e} = \frac{v_{Z0} \pm \sqrt{v_{Z0}^2 + 2gh}}{g} = 2\sqrt{\frac{h}{g}}. \tag{G.101}$$

We recognize that we have a situation as depicted in Figure G.19,



0.5

Figure G.19: The release and landing point of the ball of the person on a disk.

and that the coordinates in the XY-plane are

$$X_\mathrm{b}(t_\mathrm{e}) = v_{X0}t_\mathrm{e}, \qquad Y_\mathrm{b}(t_\mathrm{e}) = R_\mathrm{d} + v_{Y0}t_\mathrm{e}. \tag{G.102}$$

Since we know from the Pythagorean theorem that

$$R_\mathrm{h}^2 = \left(X_\mathrm{b}(t_\mathrm{e})\right)^2 + \left(Y_\mathrm{b}(t_\mathrm{e})\right)^2 = R_\mathrm{d}^2 + \left(v_{X0}^2 + v_{Y0}^2\right)t_\mathrm{e}^2 + 2v_{Y0}t_\mathrm{e}R_\mathrm{d}, \tag{G.103}$$

we obtain

$$R_\mathrm{h} = \sqrt{R_\mathrm{d}^2 + 19h^2 + 2hR_\mathrm{d}\sqrt{3}}. \tag{G.104}$$

5.11 The quadcopters's skeleton can be decomposed into two bodies, as shown in Figure G.20.

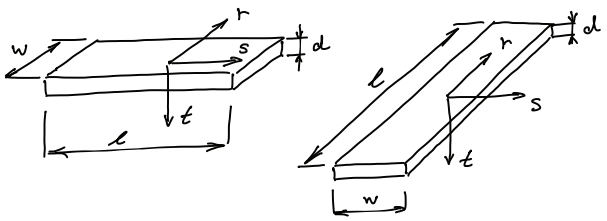


Figure G.20: Two skeleton bodies of the quadcopter. Left body 1 and right body 2.

Both bodies are symmetric to all of the coordinate planes, so all products of inertia are zero. The moments of inertia can be added:

$$\mathbf{I}_C = \begin{pmatrix} I_{rr} & 0 & 0 \\ 0 & I_{ss} & 0 \\ 0 & 0 & I_{tt} \end{pmatrix} \tag{G.105}$$

$$I_{tt} = 2\frac{M}{12}\left(W^2 + L^2\right) \tag{G.106}$$

$$I_{ss} = \underbrace{\frac{M}{12}\left(d^2 + W^2\right)}_{\text{body 1}} + \underbrace{\frac{M}{12}\left(d^2 + L^2\right)}_{\text{body 2}} \overset{d \approxeq 0}{=} \frac{M}{12}\left(W^2 + L^2\right) \tag{G.107}$$

$$I_{rr} = I_{ss} \tag{G.108}$$

$$\Rightarrow \qquad \mathbf{I}_C = \frac{M}{12}\left(W^2 + L^2\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{G.109}$$

5.12 A solid homogeneous sphere is symmetrical about infinitely many planes, so its inertia tensor with respect to the center of mass is diagonal and has the same inertia about all the axes irrespective of the triad:

$${}^{\mathcal{T}}\mathbf{I}_E = {}^{\mathcal{N}}\mathbf{I}_E = \frac{2}{5}mr^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{G.110}$$

5.16 The tensor's only nonzero elements are the diagonal terms, because the body is symmetric to two different coordinate planes:

$${}^{\mathcal{B}}\mathbf{I}_Q = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}, \tag{G.111}$$

where filling in the moment of inertia for a thin rod, and application of the parallel-axis theorem for all elements leads to

$$I_{xx} = I_{yy} = 2m_{\mathrm{k}}\left(\tfrac{l}{2}\right)^2 + 2\tfrac{1}{12}m_{\mathrm{s}}l^2 + m_{\mathrm{s}}\Delta^2, \tag{G.112}$$

$$I_{zz} = 4m_{\mathrm{k}}\left(\tfrac{l}{2}\right)^2 + 2\tfrac{1}{12}m_{\mathrm{s}}l^2, \tag{G.113}$$

which resolves to the simplified expressions:

$$I_{xx} = I_{yy} = \left(\tfrac{1}{2}m_{\mathrm{k}} + \tfrac{1}{6}m_{\mathrm{s}}\right)l^2 + m_{\mathrm{s}}\Delta^2, \tag{G.114}$$

$$I_{zz} = \left(m_{\mathrm{k}} + \tfrac{1}{6}m_{\mathrm{s}}\right)l^2. \tag{G.115}$$

5.25 In 2-D, a body's angular velocity vector $\boldsymbol{\omega}$ is always perpendicular to the 2-D-plane, so there is only planar movement, which can be rotational and translational. So, equations simplify by using (5.14) with (5.77). The angular velocity vector can in fact be reduced to one scalar value, its component orthogonal to the plane.

5.26 The body's rotational kinetic energy T_{rot} can alternatively be calculated as:

$$T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\mathbf{I}_C\boldsymbol{\omega}, \tag{G.116}$$

where $\mathbf{I}_C$ is the body's inertia tensor,

or by

$$T_{\text{rot}} = \frac{1}{2}I_{\text{isa}}|\boldsymbol{\omega}|^2, \tag{G.117}$$

where I_{isa} is the scalar moment of inertia about the body's instantaneous axis of rotation.

The inertia tensor is

$${}^{\mathcal{B}}\mathbf{I}_C = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}, \tag{G.118}$$

so we obtain

$$T_{\text{rot}} = \frac{1}{2}I_{xx}\omega_x^2 + \frac{1}{2}I_{yy}\omega_y^2 + \frac{1}{2}I_{zz}\omega_z^2 = \frac{1}{2}I_{\text{isa}}|\boldsymbol{\omega}|^2, \tag{G.119}$$

which resolves to:

$$I_{\text{isa}} = \frac{I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2}{\omega_x^2 + \omega_y^2 + \omega_z^2}. \tag{G.120}$$

Alternative solution:

Given an inertia tensor $\mathbf{I}$, the moment of inertia in the direction of a unit vector $\hat{\boldsymbol{u}}$ can be calculated using

$$I_{\text{isa}} = \hat{\boldsymbol{u}}^{\mathrm{T}}\mathbf{I}_C\hat{\boldsymbol{u}}. \tag{G.121}$$

The direction vector here is the one pointing in direction of the instantaneous axis of rotation, so

$${}^{\mathcal{B}}\hat{\boldsymbol{u}} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \frac{1}{\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}}. \tag{G.122}$$

Using again (G.118) leads to the same final result for I_{isa} as in (G.120).

5.29 There is only a spring and gravitational forces acting on the plane, so it is a conservative system. The kinetic energy then is the same when returning to an exactly identical configuration. However, angular velocity can be nonzero, at the cost of translational speed. The conservation of energy principle does not state anything about the type of kinetic energy.

5.38 None, because only a point that is fixed with respect to the body *and* with respect to a fixed coordinate system is allowed as reference for this equation.

5.41 No, the principal axis of inertia is an intrinsic property of the body, and the instantaneous axis of rotation describes the body's movement in a given situation. For example, for a rotor that is constrained to only rotate about a certain axis, that axis is the instantaneous axis of rotation (which is constant in this example), and it does not need to coincide with the principal axis of inertia.

5.43 A range of possible checks are (not all independent):

- *Prediction and Reasoning:* The vector ${}^{\mathcal{F}}\hat{\boldsymbol{b}}_1$ should be an eigenvector of ${}^{\mathcal{F}}\mathbf{I}_C$, because x is a principal axis of inertia, so ${}^{\mathcal{F}}\mathbf{I}_C\,{}^{\mathcal{F}}\hat{\boldsymbol{b}}_1 \stackrel{?}{=} I_{xx}\,{}^{\mathcal{F}}\hat{\boldsymbol{b}}_1$, with scalar I_{xx}.
  Execution and Conclusion: ${}^{\mathcal{F}}\mathbf{I}_C\,{}^{\mathcal{F}}\hat{\boldsymbol{b}}_1 = \begin{pmatrix} \frac{3}{5}I_1 \\ \frac{4}{5}(I_2+2I_3) \\ \frac{4}{5}(I_2-I_3) \end{pmatrix} \neq I_{xx}\begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}$.
  This is not plausible.
- *Prediction and Reasoning:* $\mathbf{I}_C$ should be symmetric, which means that $I_{xy} = I_{yx}, I_{xz} = I_{zx}$ and $I_{yz} = I_{zy}$.
  Execution and Conclusion: $I_{xy} = I_2 - I_3 = I_{yx}$, and also $I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$. This is plausible.
- *Prediction and Reasoning:* The invariants of $\mathbf{I}_C$ must be invariant, so the trace should be $\text{trace}({}^{\mathcal{F}}\mathbf{I}_C) \stackrel{?}{=} \text{trace}({}^{\mathcal{B}}\mathbf{I}_C) = I_1 + I_2 + I_3$ and the determinant should be $\det({}^{\mathcal{F}}\mathbf{I}_C) \stackrel{?}{=} \det({}^{\mathcal{B}}\mathbf{I}_C) = I_1 I_2 I_3$.
  Execution and Conclusion: $\text{trace}({}^{\mathcal{F}}\mathbf{I}_C) = I_1 + 3I_2 + 3I_3 \neq I_1 + I_2 + I_3$ and $\det({}^{\mathcal{F}}\mathbf{I}_C) = I_1(I_2^2 + 3I_2I_3 + I_3^2) \neq I_1 I_2 I_3$. This is not plausible.
- *Prediction and Reasoning:* For the special case of $I_1 = I_2 = I_3$, the body's equivalent ellipsoid is a sphere. In that case, the off-diagonal terms should always be zero.
  Execution and Conclusion: In that case, the only off-diagonal elements resolve to $I_2 - I_3 = 0$. This is plausible.
- *Prediction and Reasoning:* For the special case of $I_1 = I_2 = I_3$, the body's equivalent ellipsoid is a sphere. In that case, the diagonal elements should be the same (I_1) in any triad.
  Execution and Conclusion: In that case, the second diagonal element is $3I_1 \neq I_1$. This is not plausible.

- *Prediction and Reasoning:* I_3 is the minimal moment of inertia I_{isa}, so diagonal terms should not be smaller than I_3.
  Execution and Conclusion: $I_1, I_2 + 2I_3, 2I_2 + I_3$ are all $> I_3$. This is plausible.

5.46 The inertia tensor is calculated as

$$ {}^{\mathcal{B}}\mathbf{I}_S^* = {}^{\mathcal{B}}\mathbf{I}_S + \begin{pmatrix} m({z_P}^2 + {y_P}^2) & -mx_P y_P & -mx_P z_P \\ -mx_P y_P & m({x_P}^2 + {z_P}^2) & -my_P z_P \\ -mx_P z_P & -my_P z_P & m({x_P}^2 + {y_P}^2) \end{pmatrix}, \quad \text{(G.123)} $$

with $x_P = r$, $y_P = 0$, and $z_P = \frac{d}{2}$.

$$ {}^{\mathcal{B}}\mathbf{I}_S^* = {}^{\mathcal{B}}\mathbf{I}_S + m \begin{pmatrix} \frac{d^2}{4} & 0 & -\frac{1}{2}rd \\ 0 & r^2 + \frac{d^2}{4} & 0 \\ -\frac{1}{2}rd & 0 & r^2 \end{pmatrix} \quad \text{(G.124)} $$

The answer to whether x, y, z are principal axes of inertia:

Only the y-axis is a principal axis of inertia, because $I_{yx} = I_{yz} = 0$. The x and z-axes are not, because the other off-diagonal elements in the inertia tensor are nonzero.

6.2 In mid-air without air friction, only gravity acts on the phone, as shown in Figure G.21.

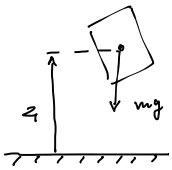


Figure G.21: Free-body diagram of the free-falling smartphone.

We can calculate the phone's acceleration from Newton's second law:

$$ \sum F_Z = m\ddot{Z} = -mg. \quad \text{(G.125)} $$

Defining $t = 0$ at the moment the phone is dropped and introducing $v_0 := v(t = 0)$ and $Z_0 := Z(t = 0)$ results in:

$$ \ddot{Z}(t) = -g \quad \text{(G.126)} $$

$$ \dot{Z}(t) = v_0 - gt \quad \text{(G.127)} $$

$$ Z(t) = Z_0 + v_0 t - \tfrac{1}{2}gt^2 \quad \text{(G.128)} $$

We know the begin conditions and the height at the moment the phone hits the ground flat face:

$$ v_0 = 0, \qquad Z_0 = H, \qquad Z(\Delta t) = \tfrac{d}{2}. \quad \text{(G.129)} $$

Substituting these and rearranging gives us the time Δt:

$$\Rightarrow \qquad \tfrac{d}{2} = H - \tfrac{1}{2} g \Delta t^2 \qquad \Rightarrow \qquad \Delta t = \sqrt{\frac{2H-d}{g}}. \tag{G.130}$$

6.11 For each equation, the reference point for moment and angular momentum needs to be the same, so

$$A = B = D = E, \tag{G.131}$$

and the point is required to be either a fixed point (with respect to the body and to the world) or the center of mass of the body.

The triads can be subdivided in a set of triads that rotate with the same angular velocity (these may or may not be body-fixed):

$$\mathcal{G}_1 = \mathcal{G}_2 \,(= \mathcal{F})\,, \tag{G.132}$$

some fixed (inertial) triads,

$$\mathcal{G}_3 = \mathcal{G}_5 = \mathcal{G}_7 = \mathcal{G}_9 = \mathcal{G}_{11} \,(= \mathcal{N})\,, \tag{G.133}$$

and some triads that must be body-fixed,

$$\mathcal{G}_4 = \mathcal{G}_6 = \mathcal{G}_8 = \mathcal{G}_{10} \,(= \mathcal{B})\,. \tag{G.134}$$

6.22 Drawing a FBD reveals that there are no external moments $\boldsymbol{M}_C$ acting on the rigid body. Therefore, Euler's second law simplifies to

$${}^{\mathcal{N}}(\dot{\boldsymbol{H}}_C)_{\mathcal{N}} = \boldsymbol{0},$$

which means that the vector ${}^{\mathcal{N}}\boldsymbol{H}_C$ is constant, so also its length $|{}^{\mathcal{N}}\boldsymbol{H}_C|$ is constant.

The vector ${}^{\mathcal{B}}\boldsymbol{H}_C$ is calculated from

$${}^{\mathcal{B}}\boldsymbol{H}_C = {}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}}\, {}^{\mathcal{N}}\boldsymbol{H}_C.$$

This vector is *not* constant, because the rotation matrix is not constant. However, rotation of a vector does not change its length. Therefore, the vector ${}^{\mathcal{B}}\boldsymbol{H}_C$ has the same length as ${}^{\mathcal{N}}\boldsymbol{H}_C$,

$$|{}^{\mathcal{B}}\boldsymbol{H}_C| = |{}^{\mathcal{N}}\boldsymbol{H}_C|,$$

which is constant.

Alternative solution:

The squared length of a vector can be calculated using:

$$|{}^{\mathcal{B}}\boldsymbol{H}_C|^2 = {}^{\mathcal{B}}\boldsymbol{H}_C^{\mathrm{T}}\, {}^{\mathcal{B}}\boldsymbol{H}_C$$

Therefore, we can calculate the time derivative using the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|{}^{\mathcal{B}}\boldsymbol{H}_C|^2\right) = 2\, {}^{\mathcal{B}}\boldsymbol{H}_C^{\mathrm{T}} \cdot {}^{\mathcal{B}}(\dot{\boldsymbol{H}}_C)_{\mathcal{N}} = 2\, {}^{\mathcal{B}}\boldsymbol{H}_C^{\mathrm{T}} \cdot {}^{\mathcal{B}}\boldsymbol{M}_C = \boldsymbol{0},$$

because $\boldsymbol{M}_C = \boldsymbol{0}$ as explained above.

6.23 A possible FBD of the wheel is shown in Figure G.22.

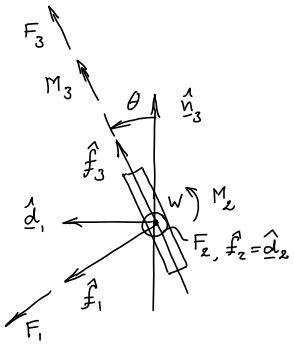


Figure G.22: Free-body diagram of the wheel held by a person.

6.24 The point W is neither fixed in space, nor the center of mass of the system. So, one needs to apply Euler's Second Law in the generic form of (2.27). Only in special cases would the term $\boldsymbol{r}_{C/P} \times (m\ddot{\boldsymbol{r}}_P)$ resolve to zero, for example if the center of mass C of the system is at height h and on a line between the Z-axis and point W, while the chair rotates at constant angular rate $\dot{\psi}$, meaning that the acceleration of point W will point towards the Z-axis. In that case, $\boldsymbol{r}_{C/W}$ and $\ddot{\boldsymbol{r}}_W$ are aligned, and their cross product is the zero vector. Except for such special cases, however, the equation is incorrect.

6.25 We notice that the term

$$^{\mathcal{F}}(\dot{\mathbf{I}}_{\mathrm{w}})_{\mathcal{F}} \tag{G.135}$$

resolves to zero in this special case, because the wheel is rotationally symmetric, so it is not included in (6.80P).

For (6.80P) to be correct, $^{\mathcal{F}}\boldsymbol{\omega}_1$ must be the angular velocity of the rigid body (the wheel), and $^{\mathcal{F}}\boldsymbol{\omega}_2$ must be the angular velocity of triad $\mathcal{F}$, because that is the triad used for the relative derivative (bottom right index of $^{\mathcal{F}}(\dot{\boldsymbol{\omega}}_1)_{\mathcal{F}}$):

$$^{\mathcal{F}}\boldsymbol{\omega}_1 = {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}, \quad {}^{\mathcal{F}}\boldsymbol{\omega}_2 = {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \tag{G.136}$$

We can use vector addition to calculate these angular velocities:

$$^{\mathcal{F}}\boldsymbol{\omega}_2 = {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} + {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{D}} = {}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} = \begin{pmatrix} -\dot{\psi}\sin\theta \\ \dot{\theta} \\ \dot{\psi}\cos\theta \end{pmatrix} \tag{G.137}$$

$$^{\mathcal{F}}\boldsymbol{\omega}_1 = {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} + {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{F}} = {}^{\mathcal{F}}\boldsymbol{\omega}_2 + \begin{pmatrix} \Omega \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\dot{\psi}\sin\theta + \Omega \\ \dot{\theta} \\ \dot{\psi}\cos\theta \end{pmatrix} \tag{G.138}$$

The relative derivative is easy to calculate, because the components of ${}^{\mathcal{F}}\boldsymbol{\omega}_1$ are already expressed in $\mathcal{F}$. So, we may take the derivatives of the components:

$$ {}^{\mathcal{F}}(\dot{\boldsymbol{\omega}}_1)_{\mathcal{F}} = \begin{pmatrix} -\dot{\psi}\dot{\theta}\cos\theta - \ddot{\psi}\sin\theta + \dot{\Omega} \\ \ddot{\theta} \\ -\dot{\psi}\dot{\theta}\sin\theta + \ddot{\psi}\cos\theta \end{pmatrix} \tag{G.139} $$

6.27 The reaction wheel (or fly-wheel-) effect is then canceled out, such that the rotors exert no unwanted resultant yaw moment on the skeleton during simultaneously accelerating or decelerating of all rotors.

Also, this cancels the gyroscopic effect, such that it prevents the rotors from exerting net moments on the skeleton when rotating the quadcopter about the r- or s-axis.

6.30 Euler's second law about center of mass A gives:

$$ \sum {}^{\mathcal{B}}\boldsymbol{M}_A = {}^{\mathcal{B}}\dot{\boldsymbol{H}}_A \tag{G.140} $$

with

$\sum {}^{\mathcal{B}}\boldsymbol{M}_A$: Sum of external moments about center of mass A.

${}^{\mathcal{B}}\dot{\boldsymbol{H}}_A$: Rate of change of the phone's angular momentum about center of mass A.

$$ {}^{\mathcal{B}}\boldsymbol{H}_A = {}^{\mathcal{B}}\boldsymbol{I}_A\, {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathrm{p}} \tag{G.141} $$

with

$\mathbf{I}_A$: Inertia tensor of the phone with respect to its center of mass A.

The triad $\mathcal{B}$ is body-fixed:

$$ {}^{\mathcal{B}}\dot{\boldsymbol{H}}_A = {}^{\mathcal{B}}\mathbf{I}_A\, {}^{\mathcal{B}}\dot{\boldsymbol{\omega}}_{\mathrm{p}} + {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathrm{p}} \times \left({}^{\mathcal{B}}\mathbf{I}_A\, {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathrm{p}} \right) \tag{G.142} $$

$$ \sum {}^{\mathcal{B}}\boldsymbol{M}_A = {}^{\mathcal{B}}\boldsymbol{r}_{C/A} \times {}^{\mathcal{B}}\boldsymbol{F}_C \tag{G.143} $$

$$ \Rightarrow \qquad {}^{\mathcal{B}}\dot{\boldsymbol{\omega}}_{\mathrm{p}} = {}^{\mathcal{B}}\mathbf{I}_A^{-1} \left(\underbrace{{}^{\mathcal{B}}\boldsymbol{r}_{C/A} \times {}^{\mathcal{B}}\boldsymbol{F}_C}_{\boldsymbol{M}_A} - {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathrm{p}} \times \underbrace{\left({}^{\mathcal{B}}\boldsymbol{I}_A\, {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathrm{p}} \right)}_{\boldsymbol{H}_A} \right) \tag{G.144} $$

Fill in:

$$ {}^{\mathcal{B}}\mathbf{I}_A = \frac{m}{12} \begin{pmatrix} b^2 + h^2 & 0 & 0 \\ 0 & d^2 + h^2 & 0 \\ 0 & 0 & d^2 + b^2 \end{pmatrix} $$

$$
{}^{\mathcal{B}}\boldsymbol{r}_{C/A} = \frac{1}{2}\begin{pmatrix} d \\ -b \\ -h \end{pmatrix} \qquad \Rightarrow \qquad {}^{\mathcal{B}}\boldsymbol{M}_A \overset{\text{(G.143)}}{=} \frac{1}{2}\begin{pmatrix} -bF_C \\ -dF_C \\ 0 \end{pmatrix}
$$

$$
{}^{\mathcal{B}}\mathbf{I}_A^{-1} = \frac{12}{m}\begin{pmatrix} \frac{1}{b^2+h^2} & 0 & 0 \\ 0 & \frac{1}{d^2+h^2} & 0 \\ 0 & 0 & \frac{1}{d^2+b^2} \end{pmatrix}
$$

$$
{}^{\mathcal{B}}\boldsymbol{H}_A \overset{\text{(G.141)}}{=} \frac{\omega_{\text{p}}\sqrt{2}m}{24}\begin{pmatrix} -b^2-h^2 \\ d^2+h^2 \\ 0 \end{pmatrix}
$$

$$
\overset{\text{(G.144)}}{\Rightarrow} \qquad {}^{\mathcal{B}}\dot{\boldsymbol{\omega}}_{\text{p}} = \begin{pmatrix} -\frac{6bF_C}{m(b^2+h^2)} \\ -\frac{6dF_C}{m(d^2+h^2)} \\ \frac{\omega_{\text{p}}^2(d^2-b^2)}{2(d^2+b^2)} \end{pmatrix}
$$

6.31 To calculate acceleration $\boldsymbol{a}_C$ of C, we use (4.69) with the associations

(4.69)	This problem
Q	O
P	C
$\mathcal{F}$	$\mathcal{D}$

The definitions lead to

$$
\begin{gathered}
{}^{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}, \quad {}^{\mathcal{D}}(\dot{\boldsymbol{\omega}})_{\mathcal{D}/\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \\ \ddot{\psi} \end{pmatrix}, \\
{}^{\mathcal{D}}\ddot{\boldsymbol{Q}} = {}^{\mathcal{D}}\ddot{\boldsymbol{r}}_{C/O} = \mathbf{0}, \quad {}^{\mathcal{D}}\boldsymbol{p} = {}^{\mathcal{D}}\boldsymbol{r}_{C/O} = \frac{(H-c)}{\sqrt{R^2+H^2}}\begin{pmatrix} 0 \\ H \\ R \end{pmatrix}
\end{gathered} \tag{G.145}
$$

Since ${}^{\mathcal{D}}\boldsymbol{p}$ is a constant vector in this problem, $(\dot{\boldsymbol{p}})_{\mathcal{D}}$ and $(\ddot{\boldsymbol{p}})_{\mathcal{D}}$ are both zero vectors. The resultant force acting on the cone is

$$
\begin{aligned}
\sum \boldsymbol{F} &= m\boldsymbol{a}_C \\
&= m\Big(\ddot{\boldsymbol{Q}} + (\ddot{\boldsymbol{p}})_{\mathcal{D}} + \boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \times \big(\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \times \boldsymbol{p}\big) \\
&\qquad\qquad + 2\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \times (\dot{\boldsymbol{p}})_{\mathcal{D}} + \dot{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{N}} \times \boldsymbol{p}\Big)
\end{aligned} \tag{G.146}
$$

$$
\Rightarrow \qquad \sum {}^{\mathcal{D}}\boldsymbol{F} = -\frac{H(H-c)m}{\sqrt{R^2+H^2}}\begin{pmatrix} \ddot{\psi} \\ \dot{\psi}^2 \\ 0 \end{pmatrix} \tag{G.147}
$$

A quicker alternative is to calculate $\boldsymbol{a}_C$ directly from the transport theorem. We define the shorthand $\boldsymbol{C} := \boldsymbol{r}_{C/O}$ and obtain

$$ {}^{\mathcal{D}}\boldsymbol{v} := {}^{\mathcal{D}}\dot{\boldsymbol{C}} = {}^{\mathcal{D}}(\dot{\boldsymbol{C}})_{\mathcal{D}} + {}^{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \times {}^{\mathcal{D}}\boldsymbol{C} = -\frac{H(H-c)}{\sqrt{R^2+H^2}}\dot{\psi}\,{}^{\mathcal{D}}\hat{\boldsymbol{d}}_1 \tag{G.148} $$

, because ${}^{\mathcal{D}}(\dot{\boldsymbol{C}})_{\mathcal{D}}$ is a zero vector. We proceed to:

$$ {}^{\mathcal{D}}\boldsymbol{a}_C = {}^{\mathcal{D}}\ddot{\boldsymbol{C}} = {}^{\mathcal{D}}(\dot{\boldsymbol{v}})_{\mathcal{D}} + {}^{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{D}/\mathcal{N}} \times {}^{\mathcal{D}}\boldsymbol{v} = -\frac{H(H-c)}{\sqrt{R^2+H^2}}\ddot{\psi}\,{}^{\mathcal{D}}\hat{\boldsymbol{d}}_1, \tag{G.149} $$

because ${}^{\mathcal{D}}(\dot{\boldsymbol{v}})_{\mathcal{D}}$ is also a zero vector.

6.32

$$ \sum {}^{\mathcal{F}}\boldsymbol{M}_O = {}^{\mathcal{F}}\dot{\boldsymbol{H}}_O = {}^{\mathcal{F}}(\dot{\boldsymbol{H}}_O)_{\mathcal{F}} + {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times {}^{\mathcal{F}}\boldsymbol{H}_O \tag{G.150} $$

$$ = {}^{\mathcal{F}}\mathbf{I}_O\,{}^{\mathcal{F}}(\dot{\boldsymbol{\omega}}_{\mathrm{b}})_{\mathcal{F}} + {}^{\mathcal{F}}(\dot{\mathbf{I}}_O)_{\mathcal{F}}\boldsymbol{\omega}_{\mathrm{b}} + \boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times \left({}^{\mathcal{F}}\mathbf{I}_O\,{}^{\mathcal{F}}\boldsymbol{\omega}_{\mathrm{b}}\right) \tag{G.151} $$

with,

$\boldsymbol{\omega}_{\mathrm{b}}$: Angular velocity of the body (cone).

$\mathbf{I}_O$: Inertia tensor of the body with respect to O.

$$ {}^{\mathcal{F}}\mathbf{I}_O = \begin{pmatrix} I_{tC}+md^2 & 0 & 0 \\ 0 & I_a & 0 \\ 0 & 0 & I_{tC}+md^2 \end{pmatrix} \tag{G.152} $$

with $d = H - c = \frac{3\sqrt{3}}{4}R$ the distance between O and C.

$$ {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} = \begin{pmatrix} 0 \\ \frac{1}{2}\dot{\psi} \\ \frac{1}{2}\sqrt{3}\dot{\psi} \end{pmatrix}, \quad {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathrm{b}} = \begin{pmatrix} 0 \\ \frac{1}{2}\dot{\psi}+\dot{\gamma} \\ \frac{1}{2}\sqrt{3}\dot{\psi} \end{pmatrix}, \quad {}^{\mathcal{F}}(\dot{\boldsymbol{\omega}}_{\mathrm{b}})_{\mathcal{F}} = \begin{pmatrix} 0 \\ \frac{1}{2}\ddot{\psi}+\ddot{\gamma} \\ \frac{1}{2}\sqrt{3}\ddot{\psi} \end{pmatrix} $$

$$ \Rightarrow {}^{\mathcal{F}}\boldsymbol{H}_O = \begin{pmatrix} 0 \\ I_a\left(\frac{1}{2}\dot{\psi}+\dot{\gamma}\right) \\ \left(I_{tC}+m\frac{27}{16}R^2\right)\frac{1}{2}\sqrt{3}\dot{\psi} \end{pmatrix} $$

$$ \Rightarrow {}^{\mathcal{F}}(\dot{\boldsymbol{H}}_O)_{\mathcal{F}} = \begin{pmatrix} 0 \\ I_a\left(\frac{1}{2}\ddot{\psi}+\ddot{\gamma}\right) \\ \left(I_{tC}+m\frac{27}{16}R^2\right)\frac{1}{2}\sqrt{3}\ddot{\psi} \end{pmatrix} \tag{G.153} $$

$$ \sum \boldsymbol{M}_O = \begin{pmatrix} \frac{1}{4}\sqrt{3}\dot{\psi}^2\left(I_{tC}+m\frac{27}{16}R^2\right) - \frac{1}{2}\sqrt{3}\dot{\psi}I_a\left(\frac{1}{2}\dot{\psi}+\dot{\gamma}\right) \\ I_a\left(\frac{1}{2}\ddot{\psi}+\ddot{\gamma}\right) \\ \left(I_{tC}+m\frac{27}{16}R^2\right)\frac{1}{2}\sqrt{3}\ddot{\psi} \end{pmatrix} \tag{G.154} $$

6.33 Combined body to compensate for moment unbalance:

$$\mathbf{I}_C = \mathbf{I}_\mathrm{D} + \mathbf{I}_a \qquad \Rightarrow \qquad \mathbf{I}_a \stackrel{!}{=} \mathbf{I}_C - \mathbf{I}_\mathrm{D} \tag{G.155}$$

with $\mathbf{I}_a$ the inertia tensor of the additional mass about O. Off-diagonal elements of $\mathbf{I}_a$:

$$I_{xy} = -\int_m xy \,\mathrm{d}m = -m_a r_x r_y \tag{G.156}$$

$$I_{xz} = -m_a r_x r_z \tag{G.157}$$

$$I_{yz} = -m_a r_y r_z \tag{G.158}$$

with $\boldsymbol{r}_\mathrm{A} = (r_x,\, r_y,\, r_z)^\mathrm{T}$ the position vector of the mass at A.

$$r_x = d, \qquad r_y = R\cos\varphi, \qquad r_z = R\sin\varphi \tag{G.159}$$

Fill in:

$$\stackrel{\text{(G.155), (G.156)}}{\Rightarrow} \qquad -m_a dR\cos\varphi \stackrel{!}{=} -\frac{1}{200}mR^2 \tag{G.160}$$

$$\stackrel{\text{(G.155), (G.157)}}{\Rightarrow} \qquad -m_a dR\sin\varphi \stackrel{!}{=} -\frac{\sqrt{3}}{200}mR^2 \tag{G.161}$$

$$\stackrel{\text{(G.155), (G.158)}}{\Rightarrow} \qquad -m_a R^2\sin\varphi\cos\varphi \stackrel{!}{=} -\frac{1}{1000}mR^2 \tag{G.162}$$

From (G.160) and (G.161), one can find φ:

$$\Rightarrow \qquad \varphi = \frac{\pi}{3} \Rightarrow \qquad \sin\varphi = \frac{1}{2}\sqrt{3}, \qquad \cos\varphi = \frac{1}{2} \tag{G.163}$$

$$\stackrel{\text{(G.162)}}{\Rightarrow} \qquad m_a = \frac{1}{750}\sqrt{3}m, \qquad \stackrel{\text{(G.160)}}{\Rightarrow} \qquad d = \frac{Rm}{100m_a} = \frac{5}{2}\sqrt{3}R \tag{G.164}$$

6.34 Newton's second law relates $\sum \boldsymbol{F}_E$, the sum of all external forces acting on the planet, to the acceleration of its center of mass E:

$$\sum \boldsymbol{F}_E = m\boldsymbol{a}_E. \tag{G.165}$$

For this specific case, there are two forces:

$$^{\mathcal{T}}\boldsymbol{F}_\mathrm{c} + {}^{\mathcal{T}}\boldsymbol{F}_g = m\,{}^{\mathcal{T}}\boldsymbol{a}_E, \tag{G.166}$$

so

$$^{\mathcal{T}}\boldsymbol{a}_E = \frac{{}^{\mathcal{T}}\boldsymbol{F}_\mathrm{c} + {}^{\mathcal{T}}\boldsymbol{F}_g}{m}. \tag{G.167}$$

Euler's second law relates $\sum \boldsymbol{M}_E$, the sum of all external moments acting on the planet with respect to its center of mass E, to the rate of change of the corresponding angular momentum $\boldsymbol{H}_E$:

$$\sum \boldsymbol{M}_E = \dot{\boldsymbol{H}}_E. \tag{G.168}$$

In this specific case, only the impact force creates a moment, and $\mathcal{T}$ is a body-fixed triad, so we can use the special form:

$$ {}^{\mathcal{T}}\boldsymbol{r}_{D/E} \times {}^{\mathcal{T}}\boldsymbol{F}_{\mathrm{c}} = {}^{\mathcal{T}}\mathbf{I}_E \, {}^{\mathcal{T}}(\dot{\boldsymbol{\omega}})_{\mathcal{T}/\mathcal{N}} + {}^{\mathcal{T}}\boldsymbol{\omega}_{\mathcal{T}/\mathcal{N}} \times \left({}^{\mathcal{T}}\mathbf{I}_E \, {}^{\mathcal{T}}\boldsymbol{\omega}_{\mathcal{T}/\mathcal{N}} \right). \tag{G.169} $$

Solving (G.169) for the requested angular acceleration gives:

$$ {}^{\mathcal{T}}(\dot{\boldsymbol{\omega}})_{\mathcal{T}/\mathcal{N}} = {}^{\mathcal{T}}\mathbf{I}_E^{-1} \left({}^{\mathcal{T}}\boldsymbol{r}_{D/E} \times {}^{\mathcal{T}}\boldsymbol{F}_{\mathrm{c}} - {}^{\mathcal{T}}\boldsymbol{\omega}_{\mathcal{T}/\mathcal{N}} \times \left({}^{\mathcal{T}}\mathbf{I}_E \, {}^{\mathcal{T}}\boldsymbol{\omega}_{\mathcal{T}/\mathcal{N}} \right) \right). \tag{G.170} $$

6.35 The sum of forces in Newton's second law

$$ \sum \boldsymbol{F} = m\boldsymbol{a} \tag{G.171} $$

in this case includes the ground reaction force and gravity:

$$ {}^{\mathcal{N}}\boldsymbol{F}_P + \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} = m \, {}^{\mathcal{N}}\ddot{\boldsymbol{r}}_C. \tag{G.172} $$

This gives

$$ {}^{\mathcal{N}}\boldsymbol{F}_P = m \left({}^{\mathcal{N}}\ddot{\boldsymbol{r}}_C + \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix} \right). \tag{G.173} $$

6.36 In Euler's second law

$$ \sum \boldsymbol{M}_C = (\dot{\boldsymbol{H}}_C)_{\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{H}_C, \tag{G.174} $$

the only moment about the center of mass is caused by the ground reaction force, so

$$ {}^{\mathcal{B}}\boldsymbol{r}_{P/C} \times \left({}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} \, {}^{\mathcal{N}}\boldsymbol{F}_P \right) = {}^{\mathcal{B}}\mathbf{I}_C \, {}^{\mathcal{B}}\dot{\boldsymbol{\omega}} + {}^{\mathcal{B}}\boldsymbol{\omega} \times \left({}^{\mathcal{B}}\mathbf{I}_C \, {}^{\mathcal{B}}\boldsymbol{\omega} \right), \tag{G.175} $$

which can be solved for the angular acceleration:

$$ {}^{\mathcal{B}}\dot{\boldsymbol{\omega}} = {}^{\mathcal{B}}\mathbf{I}_C^{-1} \cdot \left({}^{\mathcal{B}}\boldsymbol{r}_{P/C} \times \left({}^{\mathcal{B}}\mathbf{C}_{\mathcal{N}} \, {}^{\mathcal{N}}\boldsymbol{F}_P \right) - {}^{\mathcal{B}}\boldsymbol{\omega} \times \left({}^{\mathcal{B}}\mathbf{I}_C \, {}^{\mathcal{B}}\boldsymbol{\omega} \right) \right). \tag{G.176} $$

6.37 Some plausibility checks:

- Special cases: if the wheel is upright, $\theta = \frac{\pi}{2}$, then the equation becomes $0 + (0 - 12R)\, v_C^2 / \left(R - 0\right)^2 = 0$, so $v_C^2/R = 0$.
  That means either the radius would have to be infinitely large (meaning the wheel goes straight) or the velocity would be zero, so the wheel just stands still. This is plausible, because an upright wheel cannot roll in circles, it needs to lean.

- Check whether units are consistent:

$$[8g\cos\theta] = \mathrm{m/s^2}, \tag{G.177}$$

$$\left[(5r\sin(2\theta) - 12R\sin\theta)\,\frac{v_C^2}{(R-\cos\theta)^2}\right] = \mathrm{m}\cdot\frac{\mathrm{m^2/s^2}}{(\mathrm{m}-[1])^2}. \tag{G.178}$$

  The term in the denominator has inconsistent units, so the result is not plausible.
  Remark: If derived correctly we would find the denominator of (G.178) should include radius r: $(R - r\cos\theta)^2$.

Alternatively, also without deriving the equations, one could check that the units are correct for the expected physical quantity, one could linearize the system, etc.

7.5 The distance from point D to point G is the constant length l. This can be written as

$$\Phi = (X_D - X_G)^2 + (Y_D - Y_G)^2 + (Z_D - Z_G)^2 - l^2 = 0. \tag{G.179}$$

7.10 See Table G.1.

Table G.1: Classification of constraints for the scooter and person system.

constraint imposed because of	holonomic	scleronomic
hinge connecting rear wheel and footboard	yes	yes
wheel rolling on the ground without slip in circumferential direction	no	yes
wheel not slipping laterally on the ground	no	yes
ground surface not deforming in normal direction	yes	yes

7.14 The virtual work principle gives

$$\delta W = \boldsymbol{F}_g \cdot \delta\boldsymbol{r} = \boldsymbol{Q}_g \cdot \delta\boldsymbol{q}, \tag{G.180}$$

with the gravitational forces $\boldsymbol{F}_g$ and virtual displacements $\delta\boldsymbol{r}$ in Cartesian coordinates, and the generalized gravitational forces $\boldsymbol{Q}_g$ and virtual displacements $\delta\boldsymbol{q}$. The generalized force components follow from the Cartesian forces as

$$Q_{gj} = \boldsymbol{F}_g \cdot \frac{\partial\boldsymbol{r}}{\partial q_j}. \tag{G.181}$$

The force considered here is the gravitational force acting on the top's center of mass in the global negative Z-direction:

$${}^{\mathcal{N}}\boldsymbol{F}_g = \begin{pmatrix} 0 \\ 0 \\ -(5m_\mathrm{k} + 3m_\mathrm{s})\,g \end{pmatrix}. \tag{G.182}$$

As position vector $\boldsymbol{r}$ we generally require the absolute position of the point of application of the force. In this special case, knowing that the gravitational force does no work during pure displacement in the XY-plane, we may use the relative position vector of the force application point C with respect to P (instead of the position vector from the inertial origin to C). This vector expressed in the inertial triad is

$$ {}^{\mathcal{N}}\boldsymbol{r}_{C/P} = {}^{\mathcal{N}}\mathbf{C}_{\mathcal{F}}\,{}^{\mathcal{F}}\boldsymbol{r}_{C/P} = {}^{\mathcal{N}}\mathbf{C}_{\mathcal{F}}\begin{pmatrix}0\\0\\d\end{pmatrix} = \begin{pmatrix} d\sin\varphi\sin\theta \\ -d\cos\varphi\sin\theta \\ d\cos\theta \end{pmatrix}. \tag{G.183} $$

There is one generalized force component for each of the five generalized coordinates: $\boldsymbol{Q}_g = (Q_\varphi,\, Q_\theta,\, Q_\psi,\, Q_{X_P},\, Q_{Y_P})^{\mathrm{T}}$. According to (G.181) we take the partial derivatives of the position vector with respect to all generalized coordinates to obtain these generalized force components:

$$ \frac{\partial\,{}^{\mathcal{N}}\boldsymbol{r}_{C/P}}{\partial\varphi} = \begin{pmatrix} d\cos\varphi\sin\theta \\ d\sin\varphi\sin\theta \\ 0 \end{pmatrix}, \quad \frac{\partial\,{}^{\mathcal{N}}\boldsymbol{r}_{C/P}}{\partial\theta} = \begin{pmatrix} d\sin\varphi\cos\theta \\ -d\cos\varphi\cos\theta \\ -d\sin\theta \end{pmatrix}, $$

$$ \frac{\partial\,{}^{\mathcal{N}}\boldsymbol{r}_{C/P}}{\partial\psi} = \mathbf{0}, \quad \frac{\partial\,{}^{\mathcal{N}}\boldsymbol{r}_{C/P}}{\partial X_P} = \mathbf{0}, \quad \frac{\partial\,{}^{\mathcal{N}}\boldsymbol{r}_{C/P}}{\partial Y_P} = \mathbf{0}. $$

Taking the scalar product of the Cartesian forces with these partial derivatives then provides the generalized force components, so the generalized gravitational force vector is:

$$ \boldsymbol{Q}_g = \begin{pmatrix} 0 & (5m_{\mathrm{k}} + 3m_{\mathrm{s}})\, dg\sin\theta & 0 & 0 & 0 \end{pmatrix}. \tag{G.184} $$

7.15 Ideal constraint forces do no (virtual) work, and all interaction forces are ideal constraint forces in this system. This gives

$$ Q_\theta = 0\,\mathrm{N\,m} \tag{G.185} $$

Alternative solution

We name $\boldsymbol{F}_C$ the ground reaction force vector and $\delta\boldsymbol{r}_C$ the virtual displacement of point C. The contact point is momentarily at rest, which means that $\delta\boldsymbol{r}_C = \mathbf{0}$. Therefore:

$$ \delta W = \boldsymbol{F}_C \cdot \delta\boldsymbol{r}_C = Q_i\delta q_i = 0 \qquad \Rightarrow \qquad Q_\theta = 0\,\mathrm{N\,m}. \tag{G.186} $$

Alternative solution

With $\boldsymbol{F}_C$ the ground reaction force vector and $\delta\boldsymbol{r}_C$ the virtual displacement of point C we see when we realize that $\frac{\partial\boldsymbol{r}_C}{\partial\theta} = \mathbf{0}$,

$$ Q_\theta = \boldsymbol{F}_C \cdot \frac{\partial\boldsymbol{r}_C}{\partial\theta} = 0\,\mathrm{N\,m}. \tag{G.187} $$

Alternative solution

We can also look at a free-body diagram with the virtual displacement included as in Figure G.23.

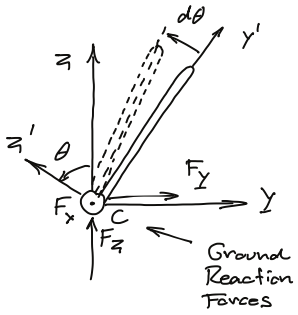


Figure G.23: FBD of the free rolling disk with the virtual displacement.

We see that no forces do work during the virtual displacement $\delta\theta$,

$$\delta W_\theta = Q_\theta \delta\theta = 0 \qquad \Rightarrow \qquad Q_\theta = 0\,\mathrm{N\,m}. \tag{G.188}$$

7.16 Figure G.24 shows a possible FBD for the point mass. Note that the force F_r is drawn because the massless rod has been cut away, which can only transmit a force in its direction.

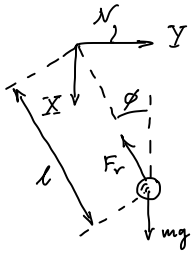


Figure G.24: Free-body diagram of the pendulum with a point mass on a rod.

7.20 The principles of virtual work and d'Alembert for the left mass 1 combine to:

$$\delta W_1 = (\boldsymbol{F}_\mathrm{g} - m\ddot{\boldsymbol{r}}_1) \cdot \delta\boldsymbol{r}_1 = 0 \tag{G.189}$$

Here, the only applied force is the gravitational force:

$${}^{\mathcal{N}}\boldsymbol{F}_\mathrm{g} = \begin{pmatrix} 0 & 0 & -mg \end{pmatrix}^\mathrm{T} \tag{G.190}$$

To find the absolute acceleration vector, first the position vector with components in the inertial triad $\mathcal{N}$, as defined in Example 2.2, is required:

$${}^{\mathcal{N}}\boldsymbol{r}_1 = \begin{pmatrix} 2L\sin\theta\sin\psi \\ -2L\sin\theta\cos\psi \\ 2L\,(1-\cos\theta) \end{pmatrix} \tag{G.191}$$

From this vector, the velocity and acceleration vectors can be derived. For the accelerations, here we only need to determine the nonzero contribution, thus only the contribution of $\dot{\psi}$:

$$ {}^{\mathcal{N}}\dot{\boldsymbol{r}}_1 = \begin{pmatrix} 2L\cos\theta\sin\psi \\ -2L\cos\theta\cos\psi \\ 2L\sin\theta \end{pmatrix} \dot{\theta} + \begin{pmatrix} 2L\sin\theta\cos\psi \\ 2L\sin\theta\sin\psi \\ 0 \end{pmatrix} \dot{\psi} \tag{G.192} $$

$$ {}^{\mathcal{N}}\ddot{\boldsymbol{r}}_1(\dot{\theta}=0,\,\ddot{\theta}=0,\,\ddot{\psi}=0) = \begin{pmatrix} -2L\sin\theta\sin\psi \\ 2L\sin\theta\cos\psi \\ 0 \end{pmatrix} \dot{\psi}^2 \tag{G.193} $$

The variation vector is derived from the position vector by taking the partial derivatives or by looking at the geometrical relationship of the variations (see Figure G.25).

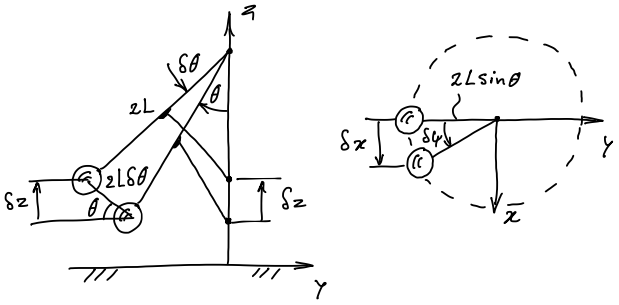


Figure G.25: Variations in coordinates for the governor.

Using the geometric approach, the variations of the position vector, expressed in the $\mathcal{F}$-triad, can be read from Figure G.25.

$$ {}^{\mathcal{F}}\delta\boldsymbol{r}_1 = \begin{pmatrix} 0 \\ -2L\cos\theta \\ 2L\sin\theta \end{pmatrix} \delta\theta + \begin{pmatrix} 2L\sin\theta \\ 0 \\ 0 \end{pmatrix} \delta\psi. \tag{G.194} $$

To map them to the $\mathcal{N}$-triad, we use the rotation matrix

$$ {}^{\mathcal{N}}\mathbf{C}_{\mathcal{F}} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{G.195} $$

and the relationship

$$ {}^{\mathcal{N}}\delta\boldsymbol{r}_1 = {}^{\mathcal{N}}\mathbf{C}_{\mathcal{F}} \cdot {}^{\mathcal{F}}\delta\boldsymbol{r}_1, \tag{G.196} $$

which leads to

$$ {}^{\mathcal{N}}\delta\boldsymbol{r}_1 = \begin{pmatrix} 2L\cos\theta\sin\psi \\ -2L\cos\theta\cos\psi \\ 2L\sin\theta \end{pmatrix} \delta\theta + \begin{pmatrix} 2L\sin\theta\cos\psi \\ 2L\sin\theta\sin\psi \\ 0 \end{pmatrix} \delta\psi. \tag{G.197} $$

The same result can be found using partial derivatives.

Substituting (G.197) and (G.193) in the expression for virtual work (G.189) with $\dot{\psi} = \omega$ gives

$$\begin{aligned}\delta W_1 = &\left(\begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} - m \begin{pmatrix} -2L\sin\theta\sin\psi \\ 2L\sin\theta\cos\psi \\ 0 \end{pmatrix} \omega^2 \right) \\ &\cdot \left(\begin{pmatrix} 2L\cos\theta\sin\psi \\ -2L\cos\theta\sin\psi \\ 2L\sin\theta \end{pmatrix} \delta\theta + \begin{pmatrix} 2L\sin\theta\cos\psi \\ 2L\sin\theta\sin\psi \\ 0 \end{pmatrix} \delta\psi \right) \qquad \text{(G.198)} \\ = &\left(4mL^2\sin\theta\cos\theta\left(\sin^2\psi + \cos^2\psi\right)\omega^2 - 2mgL\sin\theta\right)\delta\theta \\ &+ \left(4mL^2\sin^2\theta\left(\sin\psi\cos\psi - \cos\psi\sin\psi\right) + 0\right)\delta\psi = 0 \\ &\forall\, \delta\theta \neq 0,\, \delta\psi \neq 0. \qquad \text{(G.199)}\end{aligned}$$

The coefficient of the variation in $\delta\psi$ is zero, while the coefficient of the variation in $\delta\theta$ leads to:

$$\sin\theta\left(2L\omega^2\cos\theta - g\right) = 0. \qquad \text{(G.200)}$$

From this, we see that the angle for "equilibrium" can have two values:

$$\theta = 0 \text{ , or } \theta = \arccos\frac{g}{2L\omega^2}. \qquad \text{(G.201)}$$

Alternative solution

The calculation is considerably shorter in the $\mathcal{F}$-triad. However, it is important not to forget any terms that occur due to the rotation of this triad. The gravitational force vector in the $\mathcal{F}$-triad is the same as in the $\mathcal{N}$-triad:

$${}^{\mathcal{F}}\boldsymbol{F}_{\mathrm{g}} = \begin{pmatrix} 0 & 0 & -mg \end{pmatrix}^{\mathrm{T}}. \qquad \text{(G.202)}$$

Acceleration of particle 1,expressed in $\mathcal{F}$, for $\dot{\theta} = 0$, $\ddot{\theta} = 0$, and $\ddot{\psi} = 0$ is:

$${}^{\mathcal{F}}\ddot{\boldsymbol{r}}_1 = {}^{\mathcal{F}}(\ddot{\boldsymbol{r}}_1)_{\mathcal{F}} + {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times \left({}^{\mathcal{F}}(\dot{\boldsymbol{r}}_1)_{\mathcal{F}} + {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{F}/\mathcal{N}} \times {}^{\mathcal{F}}\boldsymbol{r}_1 \right) \qquad \text{(G.203)}$$

$$= \boldsymbol{0} + \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \begin{pmatrix} 2L\omega\sin\theta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2L\omega^2\sin\theta \\ 0 \end{pmatrix}. \qquad \text{(G.204)}$$

Its virtual displacement in the direction of the angle θ only in the $\mathcal{F}$-triad is (see Figure G.25, left):

$${}^{\mathcal{F}}\delta\boldsymbol{r}_{1\theta} = \begin{pmatrix} 0 \\ -2L\cos\theta \\ 2L\sin\theta \end{pmatrix} \delta\theta. \qquad \text{(G.205)}$$

Substituting these in virtual work for particle 1 gives

$$\delta W_{1\theta} = \left({}^{\mathcal{F}}\boldsymbol{F}_{\mathrm{g}} - m\, {}^{\mathcal{F}}\ddot{\boldsymbol{r}}_1 \right) \cdot^{\mathcal{F}} \delta \boldsymbol{r}_{1\theta} = 0 \tag{G.206}$$

$$= \left(4mL^2\omega^2 \sin\theta\cos\theta - 2mgL\sin\theta \right) \delta\theta = 0, \tag{G.207}$$

and simplification delivers the expression that should be zero for arbitrary $\delta\theta$, as above:

$$\sin\theta \left(2L\omega^2 \cos\theta - g \right) = 0. \tag{G.208}$$

From here, we find the same "equilibrium" values as in the first solution.

7.25 See Table G.2 for the force classification.

Table G.2: Classification of forces for the scooter and person system.

force	conservative	ideal constraint	external	d'Alembert
force in leaf spring	yes	no	no	no
weight of the person's left arm	yes	no	yes	no
muscle force of the person's biceps	no	no	no	no
shear force in point B in y-direction	-	yes	no	no
normal force exerted by ground on front wheel	-	yes	yes	no

7.26 A range of possible checks are (not all independent):

- *Prediction and Reasoning:* If $\theta = 0$, the system's heading is in direction of the X-axis. Then it must hold that $\dot{C}_Y = 0$, and $\dot{C}_X$ is not constrained.
  Execution and Conclusion: Substituting $\theta = 0$ in the constraint equation gives $\dot{C}_X \cdot 1 - \dot{C}_Y \cdot = 0 \Rightarrow \dot{C}_X = 0$.It should be the other way around. So, this is not plausible.
- *Prediction and Reasoning:* The system has three constraint equations, two preventing forward slip of the individual wheels, one preventing lateral slip. The former two must involve $\dot{\alpha}$ or $\dot{\beta}$, so the given equation can only be for lateral slip. The lateral-slip constraint should result from stating that the velocity of point C is always perpendicular to the axle, so $\dot{\boldsymbol{r}}_C \cdot \hat{\boldsymbol{g}}_2 = 0$. Writing the vectors out in the $\mathcal{N}$-triad gives: $(\dot{C}_X, \dot{C}_Y, 0)(-\sin\theta, \cos\theta, 0)^{\mathrm{T}} = -\dot{C}_X \sin\theta + \dot{C}_Y \cos\theta = 0$. This should give the same equation as the one stated.
  Execution and Conclusion: In the two equations, the coefficients with sin and cos are exchanged. If we add the two equations, we obtain:

$(\dot{C}_X + \dot{C}_Y)(\cos\theta - \sin\theta) = 0$. This would be true only for very specific values of θ, such as $\pi/4$, or for very specific velocities where $\dot{C}_X = -\dot{C}_Y$. So, it is not generally true, and the equation is not plausible.

- *Prediction and Reasoning:* It must hold that $\sum_{j=1}^{n} a_j \dot{q}_j = 0$ for a movement that complies with the constraint. The given equation states that $a_1 = \cos(\theta)$, and $a_2 = -\sin(\theta)$.
  For movement that complies with the constraint, the velocity vector can be written as a function of speed v as

$$^{\mathcal{N}}\boldsymbol{v}_C = (v\ \cos(\theta), v\ \sin(\theta), 0)^T,$$

  so we are parameterizing $\dot{q_1} = v\ \cos(\theta)$, and $\dot{q_2} = v\ \sin(\theta)$.
  Execution and Conclusion: Filling in the parameterized velocity in the constraint equation provides the following:

$$\sum_{j=1}^{n} a_j \dot{q}_j = a_1 \dot{q_1} + a_2 \dot{q_2} = v\ (\cos^2\theta - \sin^2\theta) \neq 0.$$

  This is not plausible.
- *Prediction and Reasoning:* If θ is constant, then (increase in) $\dot{C_X}$ must be proportional to (increase in) $\dot{C_Y}$, so that the board maintains the same velocity direction vector.
  Execution and Conclusion: $\dot{C_Y} = \dot{C_X} \cdot \frac{\cos(\theta)}{\sin(\theta)} \propto \dot{C_X}$ if θ is constant. This is plausible.
- *Prediction and Reasoning:* If $\theta = \frac{\pi}{2}$, the system's heading is in direction of the Y-axis. Then, it must hold that $\dot{C_X} = 0$, and $\dot{C_Y}$ is not constrained.
  Execution and Conclusion: Substituting $\theta = \frac{\pi}{2}$ in the constraint equation gives $\dot{C_X} \cdot 0 - \dot{C_Y} \cdot 1 = 0 \Rightarrow \dot{C_Y} = 0$ This should be the other way around, so the equation is not plausible.
- *Prediction and Reasoning:* Units of all terms that are added should match: $[\dot{C_X}\cos(\theta)] = [\dot{C_Y}\sin(\theta)]$.
  Execution and Conclusion: $[\dot{C_X}\cos(\theta)] = [\dot{C_Y}\sin(\theta)] =$ m/s. This is plausible.
- *Prediction and Reasoning:* If $\theta = \frac{\pi}{4}$, then the equation must resolve to $\dot{C_X} = \dot{C_Y}$ (moving diagonally).
  Execution and Conclusion: Substituting $\theta = \frac{\pi}{4}$ in the constraint equation gives $\dot{C_X}\frac{\sqrt{2}}{2} = \dot{C_Y}\frac{\sqrt{2}}{2} \Rightarrow \dot{C_X} = \dot{C_Y}$. This is plausible.
- *Prediction and Reasoning:* There are three constraint forces: two preventing the wheels from slipping in the $\hat{\boldsymbol{g}}_1$-direction, so linking $\dot{\beta}$ and $\dot{\alpha}$ to $\dot{C_X}, \dot{C_Y}$ and $\dot{\theta}$, and a third preventing sideways slip. The equation contains no $\dot{\beta}$ nor $\dot{\alpha}$, so it can only be the third. With the point of application $^{\mathcal{N}}\boldsymbol{r} = (C_X\ C_Y\ 0)^T$, we can calculate the expected generalized constraint force associated with C_X:

$C_{C_X} = \boldsymbol{F} \cdot \frac{\partial \boldsymbol{r}}{\partial C_X} = F \cdot {}^{\mathcal{N}}\hat{\boldsymbol{g}}_{\boldsymbol{2}} \cdot (1\ 0\ 0)^T = F \cdot (-\sin\theta)$. The same for C_Y: $C_{C_Y} = \boldsymbol{F} \cdot \frac{\partial \boldsymbol{r}}{\partial C_Y} = F \cdot {}^{\mathcal{N}}\hat{\boldsymbol{g}}_{\boldsymbol{2}} \cdot (0\ 1\ 0)^T = F \cdot \cos\theta$. This must be proportional to λa_{C_X} and to λa_{C_Y}, respectively, with the Lagrange multipler λ and the constraint equation coefficients a_{C_X} and a_{C_Y}.
Execution and Conclusion: Following the constraint equation, $\boldsymbol{a} = (\cos(\theta)\ -\sin(\theta)\ 0\ 0\ 0\ 0)^T$, so $\lambda a_{C_X} = \lambda \cos(\theta)$. This is not proportional to $\sin\theta$ as contained in C_{C_X}. This is not plausible.

7.30 The virtual work of this force vector $\boldsymbol{F}_A$ in Cartesian space and of the generalized force vector $\boldsymbol{Q}$ in configuration space must be equal:

$$\delta W = \boldsymbol{F}_A \cdot \delta \boldsymbol{r}_A = \boldsymbol{Q} \cdot \delta \boldsymbol{q}, \qquad \text{with } \boldsymbol{q} = \begin{pmatrix} \psi \\ \theta \\ \alpha \end{pmatrix}, \tag{G.209}$$

where $\boldsymbol{q}$ are the generalized coordinates. Therefore, the generalized force components can be found from

$$Q_j = \boldsymbol{F}_A \cdot \frac{\partial \boldsymbol{r}_A}{\partial q_j}, \qquad \text{with } {}^{\mathcal{N}}\boldsymbol{F}_A = \begin{pmatrix} 0 \\ 0 \\ F_A \end{pmatrix} \text{ and } {}^{\mathcal{B}}\boldsymbol{r}_A = \begin{pmatrix} 0 \\ -\frac{L}{2} \\ 0 \end{pmatrix}. \tag{G.210}$$

The position vector of the point of application in the $\mathcal{N}$-triad is found from

$${}^{\mathcal{N}}\boldsymbol{r}_A = {}^{\mathcal{N}}\mathbf{C}_{\mathcal{B}}\, {}^{\mathcal{B}}\boldsymbol{r}_A = \frac{L}{2} \begin{pmatrix} \cos\psi \sin\alpha + \sin\psi \cos\theta \cos\alpha \\ \sin\psi \sin\alpha - \cos\psi \cos\theta \cos\alpha \\ -\sin\theta \cos\alpha \end{pmatrix}. \tag{G.211}$$

To obtain the generalized force associated with ψ, we calculate:

$$\frac{\partial {}^{\mathcal{N}}\boldsymbol{r}_A}{\partial q_1} = \frac{\partial {}^{\mathcal{N}}\boldsymbol{r}_A}{\partial \psi} = \begin{pmatrix} \cdot \\ \cdot \\ 0 \end{pmatrix} \quad \Rightarrow \quad Q_1 = Q_\psi = F_A \cdot 0$$
$$\Rightarrow \quad Q_\psi = 0 \tag{G.212}$$

and for the generalized force associated with θ, we calculate:

$$\frac{\partial {}^{\mathcal{N}}\boldsymbol{r}_A}{\partial q_2} = \frac{\partial {}^{\mathcal{N}}\boldsymbol{r}_A}{\partial \theta} = \begin{pmatrix} \cdot \\ \cdot \\ -\frac{L}{2} \cos\theta \cos\alpha \end{pmatrix}$$
$$\Rightarrow \quad Q_\theta = -\tfrac{L}{2} F_A \cos\theta \cos\alpha. \tag{G.213}$$

8.1

$$V = -mgL\cos\alpha + \frac{1}{2}k\,(L - L_0)^2$$

V: potential energy defined zero for $L = L_0$ and $\alpha = \frac{\pi}{2}$.
L_0: unstretched length.

$$\overset{L_0=0}{\Rightarrow} \qquad V = -mgL\cos\alpha + \frac{1}{2}kL^2$$

8.3 Yes. It is important to remember that also some internal forces can do (virtual) work on a system. As an example, consider two particles connected by a linear motor that changes the distance between particles, or muscle forces on a human joint, generating power to flex or extend limbs.

8.4 Lagrange's equation for Z_D:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{Z}_D}\right) - \frac{\partial \mathcal{L}}{\partial Z_D} = Q_2 + C_2, \tag{G.214}$$

with

$$\frac{\partial \mathcal{L}}{\partial \dot{Z}_D} = \tfrac{1}{2}m\left(2\dot{Z}_D + 2l\dot{\alpha}\sin\alpha + h\dot{\beta}\sin\beta\right), \tag{G.215}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{Z}_D}\right) = \tfrac{1}{2}m\left(2\ddot{Z}_D + 2l\ddot{\alpha}\sin\alpha + 2l\dot{\alpha}^2\cos\alpha + h\ddot{\beta}\sin\beta + h\dot{\beta}^2\cos\beta\right), \tag{G.216}$$

$$\frac{\partial \mathcal{L}}{\partial Z_D} = \tfrac{1}{2}m\cdot(-2g). \tag{G.217}$$

Resolved for C_2, this gives

$$C_2 = m\left(\ddot{Z}_D + l\ddot{\alpha}\sin\alpha + l\dot{\alpha}^2\cos\alpha + \tfrac{1}{2}h\ddot{\beta}\sin\beta + \tfrac{1}{2}h\dot{\beta}^2\cos\beta + g\right) - Q_2. \tag{G.218}$$

8.5 Lagrangian:

$$\mathcal{L} = T - V \tag{G.219}$$

with kinetic energy for body b, and hand 1 and 2,

$$T = T_\mathrm{b} + T_1 + T_2 \tag{G.220}$$

and potential energy,

$$V = V_\mathrm{b} + V_1 + V_2 \tag{G.221}$$

We define $V = 0$ when the ballerina stands on the ground, such that we can write (G.221) as

$$V = (m_\mathrm{b} + 2m_\mathrm{h})gd \tag{G.222}$$

For the body kinetic energy, we can use that I_b is the moment of inertia about the instantaneous axis of rotation, such that the rotational kinetic energy term with body inertia tensor $\mathbf{I}$ simplifies to its scalar version:

$$T_\mathrm{b} = \tfrac{1}{2}m_\mathrm{b}\dot{d}^2 + \tfrac{1}{2}\boldsymbol{\omega}^\mathrm{T}\mathbf{I}\boldsymbol{\omega} = \tfrac{1}{2}m_\mathrm{b}\dot{d}^2 + \tfrac{1}{2}I_\mathrm{b}|\boldsymbol{\omega}|^2$$

For the hands, we first define some position vectors (Figure G.26).

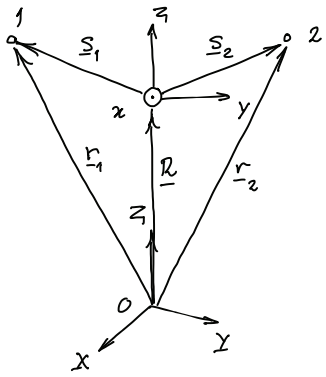


Figure G.26: Position vectors for the ballerina.

The defined vectors are:

$\boldsymbol{R}$: position vector of point S with respect to O; ${}^{\mathcal{B}}\boldsymbol{R} = \begin{pmatrix} 0 & 0 & d+l_1 \end{pmatrix}^{\mathrm{T}}$

$\boldsymbol{s}_i$: position vector of hand i with respect to S; ${}^{\mathcal{B}}\boldsymbol{s}_1 = \begin{pmatrix} 0 & -r & l_2 \end{pmatrix}^{\mathrm{T}}$

$\boldsymbol{r}_i$: position vector of hand i with respect to O; $\boldsymbol{r}_i = \boldsymbol{R} + \boldsymbol{s}_i$

Taking the derivative for hand 1 gives

$$\dot{\boldsymbol{r}}_1 = \dot{\boldsymbol{R}} + (\dot{\boldsymbol{s}}_1)_{\mathcal{B}} + \boldsymbol{\omega} \times \boldsymbol{s}_1$$

Filling this equation in for vectors with components expressed in the $\mathcal{B}$-triad, where ${}^{\mathcal{B}}\boldsymbol{\omega} = (0,\, 0,\, \dot{\theta})^{\mathrm{T}}$, gives

$${}^{\mathcal{B}}\dot{\boldsymbol{r}}_1 = \begin{pmatrix} 0 \\ -\dot{r} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{d} \end{pmatrix} + \begin{pmatrix} r\dot{\theta} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} r\dot{\theta} \\ -\dot{r} \\ \dot{d} \end{pmatrix}$$

We can obtain the kinetic energy of the hands (although they move in different directions, the magnitude of the velocity and thus the kinetic energies of either hand is equal):

$$T_1 = T_2 = \tfrac{1}{2} m_{\mathrm{h}} |\dot{\boldsymbol{r}}_1|^2 = \tfrac{1}{2} m_{\mathrm{h}} (r^2\dot{\theta}^2 + \dot{r}^2 + \dot{d}^2). \tag{G.223}$$

Now, we are able to substitute all kinetic energy terms and obtain

$$\overset{(\mathrm{G.220}),\,(\mathrm{G.223})}{\Rightarrow} \qquad T = \tfrac{1}{2} I_{\mathrm{b}} \dot{\theta}^2 + m_{\mathrm{h}} (r^2\dot{\theta}^2 + \dot{r}^2) + (m_{\mathrm{h}} + \tfrac{1}{2} m_{\mathrm{b}}) \dot{d}^2.$$

Subsequently, the Lagrangian (G.219) can be written out:

$$\overset{(\mathrm{G.219}),\,(\mathrm{G.222})}{\Rightarrow} \qquad \mathcal{L} = \tfrac{1}{2} I_{\mathrm{b}} \dot{\theta}^2 + m_{\mathrm{h}} (r^2\dot{\theta}^2 + \dot{r}^2) + (2m_{\mathrm{h}} + m_{\mathrm{b}}) (\tfrac{1}{2}\dot{d}^2 - gd). \tag{G.224}$$

Next, the generalized force Q_F corresponding to the force F_h exerted by the ballerina on the point masses is sought. Therefore, we look at how the force vectors acting on the hands $\boldsymbol{F}_{\mathrm{h}i}$ look like by noting that the forces on the hands are as depicted in Figure G.27.

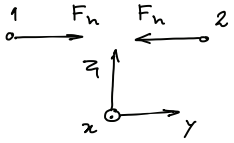


Figure G.27: Forces on the hands of the ballerina.

So, ${}^{\mathcal{B}}\boldsymbol{F}_{\mathrm{h}1} = (0,\, F_\mathrm{h},\, 0)^\mathrm{T}$ and ${}^{\mathcal{B}}\boldsymbol{F}_{\mathrm{h}2} = (0,\, -F_\mathrm{h},\, 0)^\mathrm{T}$, which can be used to write the virtual work of Q_F:

$$\delta W_F = Q_{Fr}\delta r + Q_{F\theta}\delta\theta + Q_{Fd}\delta d \tag{G.225}$$
$$= \boldsymbol{F}_{\mathrm{h}1}\cdot\delta\boldsymbol{r}_1 + \boldsymbol{F}_{\mathrm{h}2}\cdot\delta\boldsymbol{r}_2, \tag{G.226}$$

with

$${}^{\mathcal{B}}\delta\boldsymbol{r}_1 = \begin{pmatrix} r\delta\theta & -\delta r & \delta d \end{pmatrix}^\mathrm{T},$$
$${}^{\mathcal{B}}\delta\boldsymbol{r}_2 = \begin{pmatrix} -r\delta\theta & \delta r & \delta d \end{pmatrix}^\mathrm{T}.$$

The equal and opposite reaction forces acting on the body do no work as there is no displacement of the shoulders in direction of the forces (in fact only the virtual displacement δr is relevant). We can extract the generalized forces exerted by the ballerina as

$$\overset{(\mathrm{G.226})}{\Rightarrow} \qquad \delta W_F = -2F_\mathrm{h}\delta r \qquad \overset{(\mathrm{G.225})}{\Rightarrow} \qquad Q_{Fr} = -2F_\mathrm{h} \tag{G.227}$$

At this point we can fill in Lagrange's equation $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) - \frac{\partial\mathcal{L}}{\partial r} = Q_r + Q_{Fr}$:

$$\overset{(\mathrm{G.224}),(\mathrm{G.227})}{\Rightarrow} \qquad \frac{\mathrm{d}}{\mathrm{d}t}(2m_\mathrm{h}\dot{r}) - 2m_\mathrm{h}\dot{\theta}^2 r = Q_r - 2F_\mathrm{h}$$

Finally, the first equation of motion is obtained:

$$\Rightarrow \qquad \ddot{r} = \frac{-F_\mathrm{h} + \frac{1}{2}Q_r}{m_\mathrm{h}} + r\dot{\theta}^2 \tag{G.228}$$

8.8 Each (ideal) constraint force vector $\boldsymbol{C}$ is orthogonal to $\delta\boldsymbol{q}$, just as its associated vector $\boldsymbol{a}$ to be found from the coefficients of each constraint equation:

$$\boldsymbol{a}^\mathrm{T}\delta\boldsymbol{q} = 0, \tag{G.229}$$

$$\dot{X}_D = 0 \qquad \Rightarrow \qquad \delta X_D = 0, \tag{G.230}$$
$$\dot{Y}_D = 0 \qquad \Rightarrow \qquad \delta Y_D = 0, \tag{G.231}$$
$$\dot{Z}_D = 0 \qquad \Rightarrow \qquad \delta Z_D = 0. \tag{G.232}$$

$$\hat{\boldsymbol{a}}_X = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{\boldsymbol{a}}_Y = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{\boldsymbol{a}}_Z = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{G.233}$$

8.12 When in contact with the ground, the vertical velocity of the toes needs to be zero:

$$d = 0 \qquad \Rightarrow \qquad \dot{d} = 0. \tag{G.234}$$

With the derivatives of the generalized coordinates $\dot{\boldsymbol{q}} = (\dot{r},\, \dot{\theta},\, \dot{d})^{\mathrm{T}}$, the constraint can be written as:

$$\boldsymbol{a}^{\mathrm{T}}\dot{\boldsymbol{q}} = 0 \qquad \Rightarrow \qquad \boldsymbol{a} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}}. \tag{G.235}$$

The generalized constraint forces $\boldsymbol{C}$ give the only nonzero unknown generalized force,

$$\boldsymbol{C} = \lambda \boldsymbol{a} \qquad \Rightarrow C_d = \lambda. \tag{G.236}$$

From (8.45P), we find the multiplier λ:

$$\left.\begin{aligned} &\ddot{d} = \frac{Q_d + \lambda}{2m_{\mathrm{h}} + m_{\mathrm{b}}} - g \\ &d = 0 \quad \text{(from constraint)} \end{aligned}\right\} \Rightarrow \qquad \lambda = (2m_{\mathrm{h}} + m_{\mathrm{b}})g - Q_d. \tag{G.237}$$

The virtual work is obtained by looking at the constraint force vector acting on the toes $\boldsymbol{F}_{\mathrm{c}}$ and virtual displacement δd at the toes as shown in Figure G.28.

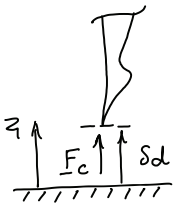


Figure G.28: Force and virtual displacement of the toe of the ballerina.

Then, with the toes' virtual displacement vector $\boldsymbol{r}_{\mathrm{t}}$, the virtual work is:

$$\delta W = C_d \delta d = \boldsymbol{F}_{\mathrm{c}}^{\mathrm{T}} \delta \boldsymbol{r}_{\mathrm{t}} = |\boldsymbol{F}_{\mathrm{c}}| \delta d, \tag{G.238}$$

Comparison of coefficients delivers the magnitude of the constraint force:

$$|\boldsymbol{F}_{\mathrm{c}}| = C_d = \lambda \qquad \Rightarrow \qquad |\boldsymbol{F}_{\mathrm{c}}| = (2m_{\mathrm{h}} + m_{\mathrm{b}})g - Q_d. \tag{G.239}$$

8.20 Choosing C_X:
$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{C_X}}\right) - \frac{\partial \mathcal{L}}{\partial C_X} = 0$, because there are no generalized applied forces.
With $\frac{\partial \mathcal{L}}{\partial \dot{C_X}} = 2D\dot{C_X} + G\dot{\gamma}\cos(\gamma)$ and $\frac{\partial \mathcal{L}}{\partial C_X} = 0$,
we obtain $\frac{\mathrm{d}}{\mathrm{d}t}(2D\dot{C_X} + G\dot{\gamma}\cos(\gamma)) = 0$
$\Rightarrow 2D\ddot{C_X} + G\ddot{\gamma}\cos\gamma - G\dot{\gamma}^2\sin\gamma = 0$
Alternative solution:
Choosing γ:
$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\gamma}}\right) - \frac{\partial \mathcal{L}}{\partial \gamma} = 0$, because there are no generalized applied forces.
With $\frac{\partial \mathcal{L}}{\partial \dot{\gamma}} = 2E\dot{\gamma} + G\dot{C_X}\cos(\gamma)$ and $\frac{\partial \mathcal{L}}{\partial \gamma} = -G\dot{C_X}\dot{\gamma}\sin(\gamma) - H\sin(\gamma)$,
we obtain $\frac{\mathrm{d}}{\mathrm{d}t}(2E\dot{\gamma} + G\dot{C_X}\cos(\gamma)) - (-G\dot{C_X}\dot{\gamma}\sin(\gamma) - H\sin(\gamma)) = 0$
$\Rightarrow 2E\ddot{\gamma} + G\ddot{C_X}\cos\gamma + H\sin\gamma = 0$

Remark: Note that in 2-D, C_X and γ form a minimal set of generalized coordinates. The constraint (7.74P) *(like the rolling constraints) has no further effect.*

8.21 We choose as generalized coordinates: X_A,Y_A,ψ,γ , where X_A and Y_A are defined as the X- and Y-coordinates of point A, respectively.

The Lagrangian is:

$$\mathcal{L} = T - V$$

with potential energy:

$$V = mgh\cos\gamma$$

and kinetic energy:

$$T = \frac{1}{2}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^{\mathrm{T}}\mathbf{I}_P\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \frac{1}{2}m_{\mathrm{p}}|\dot{\boldsymbol{r}}_P|^2.$$

The velocity of point P can be found using vector addition,

$$\dot{\boldsymbol{r}}_P = \dot{\boldsymbol{r}}_{P/O} = \dot{\boldsymbol{r}}_{A/O} + \dot{\boldsymbol{r}}_{P/A}\,,$$

and the transport theorem:

$$\dot{\boldsymbol{r}}_{P/A} = (\dot{\boldsymbol{r}}_{P/A})_{\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{P/A}.$$

The relative derivative $(\dot{\boldsymbol{r}}_{P/A})_{\mathcal{B}}$ is the zero vector in any triad, as the body does not deform. The other variables, expressed in convenient triads, are

$${}^{\mathcal{N}}\dot{\boldsymbol{r}}_{A/O} = \begin{pmatrix} \dot{X}_A \\ \dot{Y}_A \\ 0 \end{pmatrix}, {}^{\mathcal{F}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \begin{pmatrix} \dot{\gamma} \\ 0 \\ \dot{\psi} \end{pmatrix}, {}^{\mathcal{B}}\boldsymbol{r}_{P/A} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}.$$

Using the rotation matrices

$${}^{\mathcal{B}}\mathbf{C}_{\mathcal{F}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{pmatrix}, {}^{\mathcal{F}}\mathbf{C}_{\mathcal{N}} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we map all variables to $\mathcal{B}$ or $\mathcal{F}$, respectively, to obtain ${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ and ${}^{\mathcal{F}}\dot{\boldsymbol{r}}_{P/O}$ as

$$ {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \begin{pmatrix} \dot{\gamma} \\ \dot{\psi}\sin\gamma \\ \dot{\psi}\cos\gamma \end{pmatrix}, {}^{\mathcal{F}}\dot{\boldsymbol{r}}_{P/O} = \begin{pmatrix} \dot{X}_A\cos\psi + \dot{Y}_A\sin\psi + \dot{\psi}h\sin\gamma \\ -\dot{X}_A\sin\psi + \dot{Y}_A\cos\psi - \dot{\gamma}h\cos\gamma \\ -\dot{\gamma}h\sin\gamma \end{pmatrix}. $$

Substitution into the Lagrangian and simplification gives the final result:

$$ \begin{aligned} \mathcal{L} = &- m_{\mathrm{p}}gh\cos\gamma + \frac{1}{2}(\dot{\gamma} \quad \dot{\psi}\sin\gamma \quad \dot{\psi}\cos\gamma)\,{}^{\mathcal{B}}\mathbf{I}_P\,(\dot{\gamma} \quad \dot{\psi}\sin\gamma \quad \dot{\psi}\cos\gamma)^{\mathrm{T}} \\ &+ \frac{1}{2}m_{\mathrm{p}}[\dot{X}_A(\dot{X}_A + 2h\dot{\psi}\sin\gamma\cos\psi + 2h\dot{\gamma}\sin\psi\cos\gamma) \\ &+ \dot{Y}_A(\dot{Y}_A + 2h\dot{\psi}\sin\gamma\sin\psi + 2h\dot{\gamma}\cos\psi\cos\gamma) + h^2\dot{\psi}^2\sin^2\gamma + h^2\dot{\gamma}^2]. \end{aligned} $$

8.24 Lagrangian =: $\mathcal{L} = T - V$, with

$V = mgh\cos\gamma$: potential energy of the body due to gravity. $V := 0$ on the ground. m, g, h, and γ are given in the problem statement.

$T = a\dot{\gamma}^2 + b\dot{\psi}^2(1 - c\sin\gamma)^2$: kinetic energy of the body (given).

The Lagrangian becomes

$$ \mathcal{L} = a\dot{\gamma}^2 + b\dot{\psi}^2(1 - c\sin\gamma)^2 - mgh\cos\gamma. \tag{G.240} $$

Lagrange's equation with multipliers:

$$ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\right) - \frac{\partial\mathcal{L}}{\partial q_j} = Q_j'' + \sum_{i=1}^{m}\lambda_i a_{ij}. \tag{G.241} $$

With generalized coordinates:

$$ \boldsymbol{q} := \begin{pmatrix} \psi \\ \gamma \end{pmatrix}. \tag{G.242} $$

There are no applied forces. Also, there are no constraint forces, because the body has as many DoF as generalized coordinates. So

$$ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\right) - \frac{\partial\mathcal{L}}{\partial q_j} = 0. \tag{G.243} $$

Evaluating the individual terms provides

$$\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = 2b\dot{\psi}(1 - c\sin\gamma)^2, \tag{G.244}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) = 2b\ddot{\psi}(1 - c\sin\gamma)^2 - 4bc\dot{\psi}\dot{\gamma}(1 - c\sin\gamma)\cos\gamma, \tag{G.245}$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = 0, \tag{G.246}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\gamma}} = 2a\dot{\gamma}, \tag{G.247}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\gamma}}\right) = 2a\ddot{\gamma}, \tag{G.248}$$

$$\frac{\partial \mathcal{L}}{\partial \gamma} = -2bc\dot{\psi}^2\,(1 - c\sin\gamma)\cos\gamma + mgh\sin\gamma. \tag{G.249}$$

Substituting these terms in (G.243) leads to the equations of motion:

$$q_1 = \psi: \qquad 2b\ddot{\psi}(1 - c\sin\gamma)^2 - 4bc\dot{\psi}\dot{\gamma}(1 - c\sin\gamma)\cos(\gamma) = 0, \tag{G.250}$$

$$q_2 = \gamma: \qquad 2a\ddot{\gamma} + 2bc\dot{\psi}^2\,(1 - c\sin\gamma)\cos\gamma - mgh\sin\gamma = 0. \tag{G.251}$$

8.25 We choose as generalized coordinates:

$$\boldsymbol{q} = \begin{pmatrix} \varphi \\ l \end{pmatrix} \tag{G.252}$$

The constraint constant length l of the rod gives:

$$\dot{l} = 0 \qquad \Rightarrow \qquad \delta l = 0 \qquad \Rightarrow \qquad \boldsymbol{a}^{\mathrm{T}}\delta\boldsymbol{q} = 0 \tag{G.253}$$

with,

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{G.254}$$

The Lagrangian is $\mathcal{L} = T - V$, so we need to find kinetic energy T and potential energy V.

The kinetic energy is $T = \frac{1}{2}m|\boldsymbol{v}|^2$. To find the required velocity we can express the position vector of the mass $\boldsymbol{r}$ in Cartesian coordinates:

$$\boldsymbol{r} = \begin{pmatrix} l\cos\varphi \\ l\sin\varphi \end{pmatrix} \tag{G.255}$$

and after differentiating we can find the square of the velocity magnitude:

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \begin{pmatrix} \dot{l}\cos\varphi - l\dot{\varphi}\sin\varphi \\ \dot{l}\sin\varphi + l\dot{\varphi}\cos\varphi \end{pmatrix} \qquad \Rightarrow \qquad |\boldsymbol{v}|^2 = \dot{l}^2 + l^2\dot{\varphi}^2 \tag{G.256}$$

The potential energy, with $V = 0$ defined for $\varphi = \frac{\pi}{2}$, is $V = -mgl\cos\varphi$.

So now we have both the kinetic energy and the potential energy and we can assemble the Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{l}^2 + l^2\dot{\varphi}^2\right) + mgl\cos\varphi \tag{G.257}$$

And we get Lagrange's equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} = \alpha_1\lambda = 0\lambda = C_1 \quad \Rightarrow \quad ml^2\ddot{\varphi} + mgl\sin\varphi = 0 = C_1 \tag{G.258}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{l}}\right) - \frac{\partial\mathcal{L}}{\partial l} = \alpha_2\lambda = 1\lambda = C_2 \quad \Rightarrow \quad \overset{0}{\overset{\uparrow}{m\ddot{l}}} - ml\dot{\varphi}^2 - mg\cos\varphi = C_2 \tag{G.259}$$

We can express the virtual work done by the generalized force C_2 as:

$$\delta W_2 = C_2\delta l = -F_r\delta l \tag{G.260}$$

This can be solved to find the force in the rod (with the notion that $\ddot{l} = 0$):

$$F_r = -C_2 = ml\dot{\varphi}^2 + mg\cos\varphi \tag{G.261}$$

And as can be seen in Figure G.24 a positive force pulls the point mass towards the rod, such that it is a tensile force.

Alternative solution

Instead of using Cartesian coordinates to find the Absolute velocity, we can also use polar coordinates. Then we have a triad defined by unit vectors as depicted in Figure G.29.

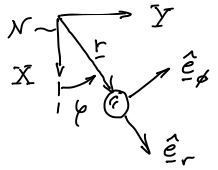


Figure G.29: Polar coordinates for the pendulum with a point mass on a rod.

And the position vector becomes,

$$\boldsymbol{r} = l\hat{\boldsymbol{e}}_r \tag{G.262}$$

After differentiating with use of the transport theorem we find the same magnitude of the velocity squared:

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{l}\hat{\boldsymbol{e}}_r + l\dot{\hat{\boldsymbol{e}}}_r = \dot{l}\hat{\boldsymbol{e}}_r + l\dot{\varphi}\hat{\boldsymbol{e}}_\varphi \qquad \Rightarrow \qquad |\boldsymbol{v}|^2 = \dot{l}^2 + l^2\dot{\varphi}^2 \tag{G.263}$$

This can be used in the kinetic energy and the calculation continues similarly as before.

8.29 Lagrange's equation for ψ is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) - \frac{\partial \mathcal{L}}{\partial \psi} = Q_\psi, \tag{G.264}$$

with the Lagrangian

$$\mathcal{L} = T - V. \tag{G.265}$$

Neither the kinetic energy T nor the potential energy V, with

$$T = \tfrac{1}{2} A \dot{\theta}^2 + \tfrac{1}{2} B \dot{\psi}^2, \tag{G.266}$$

$$V = C\left(2L\left(1-\cos\theta\right)\right)^2 + D\left(2L\left(1-\cos\theta\right)\right) + E, \tag{G.267}$$

are functions of ψ. In addition, the potential energy does not depend on $\dot{\psi}$. Finally, there is no applied force. So, several terms are zero:

$$\frac{\partial \mathcal{L}}{\partial \psi} = 0, \qquad \frac{\partial V}{\partial \dot{\psi}} = 0, \qquad Q_\psi = 0. \tag{G.268}$$

Therefore, (G.264) simplifies to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\psi}}\right) = \dot{B}\dot{\psi} + B\ddot{\psi} = 0. \tag{G.269}$$

This can also be written as:

$$\ddot{\psi} + \frac{\dot{B}}{B}\dot{\psi} = 0. \tag{G.270}$$

Remark: A generalized coordinate like ψ is an *ignorable* (or *cyclic*) coordinate, which means that the Lagrangian does not depend on it, although it may depend on its derivative. As a consequence, the generalized momentum $\frac{\partial \mathcal{L}}{\partial \dot{\psi}}$ associated with this variable is constant.

8.30 The two coefficients are equal:

$$B = Q. \tag{G.271}$$

This is plausible, because in the special case that $\dot{\theta} = 0$, the two particles form a rigid body with zero velocity of its center of mass and with moment of inertia $I_{zz} = B = Q$ about the z-axis. So, kinetic energy can be calculated by

$$T = \tfrac{1}{2} I_{zz} \dot{\psi}^2, \tag{G.272}$$

and angular momentum can be calculated by

$$H_z = I_{zz}\dot{\psi}. \tag{G.273}$$

Remark: One can continue to reason that we know from Lagrange's equation $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\psi}}\right) = 0$ that $\frac{\partial T}{\partial \dot{\psi}}$ is constant. At the same time, we know that $\frac{\partial T}{\partial \dot{\psi}} = B\dot{\psi}$, so

$$B\dot{\psi} = \text{constant}. \tag{G.274}$$

The same equation of motion follows from this reasoning and from Euler's second law about the z-axis:

$$T = \tfrac{1}{2} I_{zz}\dot{\psi}^2 \qquad \Rightarrow \qquad \frac{\partial T}{\partial \dot{\psi}} = I_{zz}\dot{\psi} = \text{constant}, \tag{G.275}$$

$$\sum M_z = \dot{H}_z = 0 \qquad \Rightarrow \qquad H_z = I_{zz}\dot{\psi} = \text{constant}. \tag{G.276}$$

One can also interpret the variables B and Q as "generalized mass/inertia" with respect to changes in $\dot{\psi}$, so it makes sense they are the same.

9.2 We can choose Euler's method for numerical integration, with time t:

$$\boldsymbol{y}(t+h) = \boldsymbol{y}(t) + h\dot{\boldsymbol{y}}(t). \tag{G.277}$$

Because the step size h is half the requested duration, we have to conduct two steps of numerical integration.

Substituting the given values for the first integration step in (G.277), until $t_b = t_a + h$, gives:

$$\boldsymbol{y}(t_b) = \boldsymbol{y}(t_a) + h\dot{\boldsymbol{y}}(t_a) = \boldsymbol{y}(t_a) - hk\boldsymbol{y}(t_a), \tag{G.278}$$

and for the second step:

$$\boldsymbol{y}(t_c) = \boldsymbol{y}(t_b) + h\dot{\boldsymbol{y}}(t_b) = \boldsymbol{y}(t_b) - hk\boldsymbol{y}(t_b). \tag{G.279}$$

The calculated values from the first step (G.278) can be substituted in the second step (G.279), with $\boldsymbol{y}_a = \boldsymbol{y}(t_a)$:

$$\boldsymbol{y}(t_c) = \boldsymbol{y}_a - 2hk\boldsymbol{y}_a + h^2k^2\boldsymbol{y}_a \tag{G.280}$$

$$= (1 - hk)^2\,\boldsymbol{y}_a. \tag{G.281}$$

The numerical value for h is substituted to give the requested function:

$$\boldsymbol{y}(t_c) = \left(1 - \tfrac{1}{2}k\,\mathrm{s}\right)^2 \boldsymbol{y}_a. \tag{G.282}$$

9.10 Euler's method states that given the state vector $\boldsymbol{x}$ at time t_0, the state at time t_1 can be approximated by linear extrapolation:

$$\boldsymbol{x}(t_1) \approx \boldsymbol{x}(t_0) + \Delta t\dot{\boldsymbol{x}}(t_0) \tag{G.283}$$

with $\Delta t = t_1 - t_0$.

Fort the given problem, this can be applied as

$$\boldsymbol{x}(t=0.2\,\mathrm{s}) \approx \begin{pmatrix} Z_K(t=0\,\mathrm{s}) \\ \dot{Z}_K(t=0\,\mathrm{s}) \end{pmatrix} + \Delta t \begin{pmatrix} \dot{Z}_K(t=0\,\mathrm{s}) \\ \ddot{Z}_K(t=0\,\mathrm{s}) \end{pmatrix}, \tag{G.284}$$

with $\Delta t = 0.2\,\mathrm{s}$.

The acceleration at $t = 0\,\mathrm{s}$ is calculated from the equation of motion:

$$\ddot{Z}_K(t=0\,\mathrm{s}) = -\Omega^2 Z_K(t=0\,\mathrm{s}). \tag{G.285}$$

Substituting (G.285) in (G.284), the final result is calculated:

$$\boldsymbol{x}(t=0.2\,\mathrm{s}) = \begin{pmatrix} 0.5\,\mathrm{m} \\ 0\,\mathrm{m/s} \end{pmatrix} + 0.2\,\mathrm{s} \begin{pmatrix} 0\,\mathrm{m/s} \\ -0.9^2\,\mathrm{rad^2/s^2} \cdot 0.5\,\mathrm{m} \end{pmatrix}, \tag{G.286}$$

$$\boldsymbol{x}(t=0.2\,\mathrm{s}) = \begin{pmatrix} 0.5\,\mathrm{m} \\ -0.081\,\mathrm{m/s} \end{pmatrix}. \tag{G.287}$$

9.11 The chosen state vector and its derivative are:

$$\boldsymbol{y} = \begin{pmatrix} \alpha \\ \dot{\alpha} \end{pmatrix}, \qquad \dot{\boldsymbol{y}} = \begin{pmatrix} \dot{\alpha} \\ \ddot{\alpha} \end{pmatrix} = \begin{pmatrix} \dot{\alpha} \\ -\frac{g}{L}\sin(\alpha) \end{pmatrix}. \tag{G.288}$$

The initial conditions are:

$$\boldsymbol{y}(t=0) = \begin{pmatrix} 0 \\ 2\,\mathrm{rad/s} \end{pmatrix}, \qquad \dot{\boldsymbol{y}}(t=0) = \begin{pmatrix} 2\,\mathrm{rad/s} \\ 0 \end{pmatrix}. \tag{G.289}$$

Euler's integration scheme states:

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\dot{\boldsymbol{y}}_n, \Rightarrow \boldsymbol{y}_1 = \boldsymbol{y}_0 + h\dot{\boldsymbol{y}}_0. \tag{G.290}$$

We need two integration steps to reach the end time. The first step gives:

$$\boldsymbol{y}_1 = \begin{pmatrix} 0 \\ 2\,\mathrm{rad/s} \end{pmatrix} + \frac{\pi}{8}\,\mathrm{s} \cdot \begin{pmatrix} 2\,\mathrm{rad/s} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{4}\,\mathrm{rad} \\ 2\,\mathrm{rad/s} \end{pmatrix}. \tag{G.291}$$

The second step gives:

$$\boldsymbol{y}_2 = \begin{pmatrix} \frac{\pi}{4}\,\mathrm{rad} \\ 2\,\mathrm{rad/s} \end{pmatrix} + \frac{\pi}{8}\,\mathrm{s} \cdot \begin{pmatrix} 2\,\mathrm{rad/s} \\ -\frac{g}{L} \cdot \frac{1}{2}\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2}\,\mathrm{rad} \\ 2\,\mathrm{rad/s} - \frac{g\pi\sqrt{2}}{16L}\,\mathrm{s} \end{pmatrix}. \tag{G.292}$$

The requested angular rate is the second component of this vector:

$$\dot{\alpha}\left(t = \tfrac{\pi}{4}\,\mathrm{s}\right) = 2\,\mathrm{rad/s} - \frac{g\pi\sqrt{2}}{16L}\,\mathrm{s}. \tag{G.293}$$

9.12 The chosen state vector and its derivative are:

$$\boldsymbol{y} = \begin{pmatrix} r \\ \theta \\ d \\ \dot{r} \\ \dot{\theta} \\ \dot{d} \end{pmatrix} \qquad \Rightarrow \qquad \dot{\boldsymbol{y}} = \begin{pmatrix} \dot{r} \\ \dot{\theta} \\ \dot{d} \\ \ddot{r} \\ \ddot{\theta} \\ \ddot{d} \end{pmatrix} \tag{G.294}$$

with $\ddot{r}$, $\ddot{\theta}$, and $\ddot{d}$ as given in (G.228), (8.44P), and (8.45P) respectively.

We apply Euler's method for one time step: $\boldsymbol{y}_1 \approx \boldsymbol{y}_0 + \Delta T \cdot \dot{\boldsymbol{y}}_0$.

$$\Rightarrow \qquad \begin{pmatrix} 1\,\mathrm{m} \\ 0\,\mathrm{rad} \\ 0.2\,\mathrm{m} \\ 0\,\mathrm{m/s} \\ 1\,\mathrm{rad/s} \\ 0\,\mathrm{m/s} \end{pmatrix} + 0.1\,\mathrm{s} \begin{pmatrix} 0\,\mathrm{m/s} \\ 1\,\mathrm{rad/s} \\ 0\,\mathrm{m/s} \\ 1\,\mathrm{m/s^2} \\ 0\,\mathrm{rad/s^2} \\ -10\,\mathrm{m/s^2} \end{pmatrix} = \begin{pmatrix} 1\,\mathrm{m} \\ 0.1\,\mathrm{rad} \\ 0.2\,\mathrm{m} \\ 0.1\,\mathrm{m/s} \\ 1\,\mathrm{rad/s} \\ -1\,\mathrm{m/s} \end{pmatrix} \approx \boldsymbol{y}_1 \tag{G.295}$$

The requested values are the first three states:

$$\Rightarrow \qquad r \approx 1\,\mathrm{m}, \qquad \theta \approx 0.1\,\mathrm{rad}, \qquad d \approx 0.2\,\mathrm{m} \tag{G.296}$$

10.2 a. Derivation of the equations of motion. See Example 10.4.

For the numeric calculation of the mass moment of inertia of the bars at the center of mass, you can take the small width ($w \ll l$) into account, $I = \frac{1}{12}m(l^2 + w^2)$, or you can use the approximation for a slender bar and neglect the width, $I = \frac{1}{12}ml^2$. The final results do not differ too much. If you compare the two cases, you will see that the calculated accelerations and forces show some differences in the 4^{rd} digit. Both approaches are correct. Here we have used $I = \frac{1}{12}m(l^2 + w^2)$.

The absolute values of the joint constraint forces $\boldsymbol{F}_\mathrm{c}$ should be correct. However, the signs may differ because of the sign convention used.

b. Both bars vertical up and zero speeds. The CoM accelerations $\ddot{\boldsymbol{x}}$ and joint constraint forces $\boldsymbol{F}_\mathrm{c}$ are,

$$\ddot{\boldsymbol{x}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\varphi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 6.2971\,\mathrm{m/s^2} \\ 0.0000\,\mathrm{m/s^2} \\ -20.9903\,\mathrm{rad/s^2} \\ 10.5097\,\mathrm{m/s^2} \\ 0.0000\,\mathrm{m/s^2} \\ 6.9483\,\mathrm{rad/s^2} \end{pmatrix}, \quad \boldsymbol{F}_\mathrm{c} = \begin{pmatrix} H_A \\ V_A \\ H_B \\ V_B \end{pmatrix} = \begin{pmatrix} -1.9876 \\ 0.0000 \\ 0.4943 \\ 0.0000 \end{pmatrix} \mathrm{N}.$$

Discussion: y-accelerations zero, OK. Bar1 angular acceleration counterclockwise, bar2 clockwise, OK. Vertical joint forces zero, OK. Horizontal joint force in A larger than in B, OK.

c. Both bars horizontal to the right and zero speeds. The CoM accelerations $\ddot{\boldsymbol{x}}$ and joint constraint forces $\boldsymbol{F}_\mathrm{c}$ are,

$$\ddot{\boldsymbol{x}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\varphi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 0\,\mathrm{m/s^2} \\ 0\,\mathrm{m/s^2} \\ 0\,\mathrm{rad/s^2} \\ 0\,\mathrm{m/s^2} \\ 0\,\mathrm{m/s^2} \\ 0\,\mathrm{rad/s^2} \end{pmatrix}, \quad \boldsymbol{F}_\mathrm{c} = \begin{pmatrix} H_A \\ V_A \\ H_B \\ V_B \end{pmatrix} = \begin{pmatrix} -13.8615 \\ 0 \\ -6.9308 \\ 0 \end{pmatrix} \mathrm{N}.$$

Discussion: The system is in static equilibrium, all accelerations are zero, OK. Horizontal joint force in A two times the one in B, mg, OK. Vertical joint force zero, OK.

d. Both bars horizontal to the right and with an initial angular speed of $\omega = 60$ rpm anti-clockwise on both bars. The CoM velocities $\dot{\boldsymbol{x}}$ are,

$$\dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\varphi}_1 \\ \dot{x}_2 \\ \dot{y}_2 \\ \dot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 0\,\mathrm{m/s} \\ 1.8850\,\mathrm{m/s} \\ 6.2832\,\mathrm{rad/s} \\ 0\,\mathrm{m/s} \\ 5.6549\,\mathrm{m/s} \\ 6.2832\,\mathrm{rad/s} \end{pmatrix},$$

and the CoM accelerations $\ddot{\boldsymbol{x}}$, and joint constraint forces $\boldsymbol{F}_\mathrm{c}$ are

$$\ddot{\boldsymbol{x}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\varphi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} -11.8435\,\mathrm{m/s^2} \\ 0\,\mathrm{m/s^2} \\ 0\,\mathrm{rad/s^2} \\ -35.5306\,\mathrm{m/s^2} \\ 0\,\mathrm{m/s^2} \\ 0\,\mathrm{rad/s^2} \end{pmatrix}, \boldsymbol{F}_\mathrm{c} = \begin{pmatrix} H_A \\ V_A \\ H_B \\ V_B \end{pmatrix} = \begin{pmatrix} -47.3313 \\ 0 \\ -32.0331 \\ 0 \end{pmatrix} \mathrm{N}.$$

Discussion: Only non-zero accelerations are the centrifugal accelerations "$\omega^2 r$"in the minus x-direction, OK. On bar 3 times the one on bar1, OK. Zero vertical forces, OK. Horizontal forces are static forces "mg" plus centrifugal forces "$m\omega^2 r$", OK.

11.3 Double pendulum, systematic approach.

a. For the systematic approach see Example 11.5.

b. Both bars vertical up and zero speeds. The CoM accelerations $\ddot{x}_j$ and the Lagrange multipliers λ_k are,

$$\ddot{x}_j = \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\varphi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 6.2971\,\mathrm{m/s^2} \\ 0.0000\,\mathrm{m/s^2} \\ -20.9903\,\mathrm{rad/s^2} \\ 10.5097\,\mathrm{m/s^2} \\ 0.0000\,\mathrm{m/s^2} \\ 6.9483\,\mathrm{rad/s^2} \end{pmatrix}, \quad \lambda_k = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 1.9876 \\ 0.0000 \\ -0.4943 \\ 0.0000 \end{pmatrix} \mathrm{N}.$$

Discussion: The accelerations $\ddot{x}_j$ are identical to the ones found in Problem 10.2b, and the Lagrange multipliers λ_k have the same magnitude as the constraint forces $\boldsymbol{F}_\mathrm{c}$ from Problem 10.2b but have different signs. This is a consequence of how we defined our constraints. Here the Lagrange multipliers λ_k are the tensile forces in the joints. You can check this by investigating the static equilibrium equations $C_{k,j}\lambda_k = f_j$. So if you have defined your constraints in another way the Lagrange multipliers will have a different value and or sign, however the force f_j calculated from the static equilibrium equations $C_{k,j}\lambda_k = f_j$ should always be the same, see for instance Example 11.8.

c. Both bars horizontal to the right and zero speeds. The CoM accelerations $\ddot{x}_j$ and the Lagrange multipliers λ_k are,

$$\ddot{x}_j = \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\varphi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 0\,\mathrm{m/s^2} \\ 0\,\mathrm{m/s^2} \\ 0\,\mathrm{rad/s^2} \\ 0\,\mathrm{m/s^2} \\ 0\,\mathrm{m/s^2} \\ 0\,\mathrm{rad/s^2} \end{pmatrix}, \quad \lambda_k = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 13.8615 \\ 0 \\ 6.9308 \\ 0 \end{pmatrix} \mathrm{N}.$$

Discussion: Compared to Problem 10.2c: identical accelerations $\ddot{x}_j$ and again Lagrange multipliers have opposite signs.

d. Both bars horizontal to the right and with an initial angular speed of $\omega = 60$ rpm anti-clockwise on both bars. The CoM velocities $\dot{x}_i$ are,

$$\dot{x}_i = \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\varphi}_1 \\ \dot{x}_2 \\ \dot{y}_2 \\ \dot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 0\,\mathrm{m/s} \\ 1.8850\,\mathrm{m/s} \\ 6.2832\,\mathrm{rad/s} \\ 0\,\mathrm{m/s} \\ 5.6549\,\mathrm{m/s} \\ 6.2832\,\mathrm{rad/s} \end{pmatrix},$$

and the CoM accelerations $\ddot{x}_j$ and the Lagrange multipliers λ_k are,

$$\ddot{x}_j = \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\varphi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} -11.8435\,\mathrm{m/s^2} \\ 0\,\mathrm{m/s^2} \\ 0\,\mathrm{rad/s^2} \\ -35.5306\,\mathrm{m/s^2} \\ 0\,\mathrm{m/s^2} \\ 0\,\mathrm{rad/s^2} \end{pmatrix}, \lambda_k = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 47.3313 \\ 0 \\ 32.0331 \\ 0 \end{pmatrix} \mathrm{N}.$$

Discussion: You can check if your initial speeds, $\dot{x}_i$, fulfill the constraints, by checking if the time derivatives of the constraints, $\dot{C}_k = C_{k,i}\dot{x}_i$, are all zero. Compared to Problem 10.2d: identical accelerations $\ddot{x}_j$ and again Lagrange multipliers have opposite signs.

11.4 For point C to move over a vertical line through the origin, see Problem 11.1.

a. Both bars vertical up and zero speeds. The CoM accelerations $\ddot{x}_j$ and the Lagrange multipliers λ_k are,

$$\ddot{x}_j = \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\varphi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 7.3447\,\mathrm{m/s^2} \\ 0.0000\,\mathrm{m/s^2} \\ -24.4825\,\mathrm{rad/s^2} \\ 7.3447\,\mathrm{m/s^2} \\ 0.0000\,\mathrm{m/s^2} \\ 24.4825\,\mathrm{rad/s^2} \end{pmatrix}, \quad \lambda_k = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} 1.7417 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 1.7417 \end{pmatrix} \mathrm{N}.$$

Discussion: Zero vertical acceleration, OK. Horizontal acceleration bar 1 and bar 2 are identical, OK. Horizontal accelerations are $\ddot{x}_1 = -\frac{l}{2}\ddot{\varphi}_1$, and $\ddot{x}_2 = \frac{l}{2}\ddot{\varphi}_2$, OK. Angular acceleration bar 1 and bar 2 same magnitude and opposite sign, OK. Only non-zero constraint force are horizontal force in A and C (λ_1 and λ_5), OK.

b. Both bars vertical up and with an initial angular speed of $\omega = 60$ rpm clockwise on bar 1. The CoM velocities $\dot{x}_i$ are,

$$\dot{x}_i = \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\varphi}_1 \\ \dot{x}_2 \\ \dot{y}_2 \\ \dot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 1.8850\,\mathrm{m/s} \\ 0\,\mathrm{m/s} \\ -6.2832\,\mathrm{rad/s} \\ 1.8850\,\mathrm{m/s} \\ 0\,\mathrm{m/s} \\ 6.2832\,\mathrm{rad/s} \end{pmatrix},$$

The CoM accelerations $\ddot{x}_j$ and the Lagrange multipliers λ_k are,

$$\ddot{x}_j = \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\varphi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 7.3447\,\mathrm{m/s^2} \\ -11.8435\,\mathrm{m/s^2} \\ -24.4825\,\mathrm{rad/s^2} \\ 7.3447\,\mathrm{m/s^2} \\ -35.5306\,\mathrm{m/s^2} \\ 24.4825\,\mathrm{rad/s^2} \end{pmatrix}, \quad \lambda_k = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} 1.7417 \\ 33.4698 \\ 0.0000 \\ 25.1024 \\ 1.7417 \end{pmatrix} \mathrm{N}.$$

Discussion: Horizontal acceleration and angular accelerations are identical to item e, but now, due to the initial speed, we also have vertical accelerations, OK. Vertical acceleration bar 2 is 3 times bar 1, OK. Horizontal joint constraint force are identical to those in item e, but now we also have vertical joint constraint forces, OK. Can you explain why the horizontal joint constraint force in B, λ_3, is zero?

c. Both bars horizontal, that is point C is at the origin, with zero speeds, but an additional applied force of $F_y = 100$ N in B, that is at the end of bar 1 in the vertical direction. If your try to find a solution then you will get a system warning, something like this:

```
Warning:  Matrix is close to singular or badly scaled.
Results may be inaccurate.  RCOND = 1.784688e-32.
```

The problem is that the system is in a singular configuration. You would

expect the system to have only one degree of freedom, because you have six coordinates and five constraints, but in this singular configuration the system has two degrees of freedom. Mode 1 is where point C moves up and down on the vertical and mode 2 is where point C stays in the origin and both bars rotate together about the origin. This singularity can easily be checked by looking at the rank of the system matrix $\mathbf{A} = [\mathbf{M}, \mathbf{C}_{,\mathbf{x}}^T; \mathbf{C}_{,\mathbf{x}}, \mathbf{0}]$, this turns out to be 10, whereas you have a 11 equations and 11 unknowns to solve. If you determine the null space of the system matrix, $\mathbf{A}\mathbf{v} = \mathbf{0}$, (MATLAB$^{\text{TM}}$: `null(A)`), you get something like:

$$\mathbf{v} = \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\varphi}_1 \\ \dot{x}_2 \\ \dot{y}_2 \\ \dot{\varphi}_2 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ -0.5774 \\ 0.0000 \\ -0.5774 \\ 0.0000 \\ 0.5774 \end{pmatrix},$$

Here you see that in the absence of external loading there can be joint forces in the system, or prestress. The problem is of course not in the mass matrix, that one is regular. However, in this configuration the five constraint equations on the accelerations are not independent. You can check this by looking at the rank of the Jacobian of the constraint equations, `rank`$(C_{k,i})$, this turns out to be four while you have five equations.

11.5 Double pendulum with a spring added to the system, determine the accelerations and constraint forces.

Method: Add the elongation of the spring as

$$C_l = \sqrt{(x_E - x_D)^2 + (y_E - y_D)^2} - l_0,$$

and then add the spring force σ_l via the Jacobian of the spring elongation, as in $C_{l,i}\sigma_l$, to the righthand side of the equations, see (11.49).

Results: Both bars vertical up and zero speeds. The CoM accelerations $\ddot{x}_j$ and the Lagrange multipliers λ_k are,

$$\ddot{x}_j = \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\varphi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} -10.4661\,\mathrm{m/s^2} \\ 0.0000\,\mathrm{m/s^2} \\ 34.8869\,\mathrm{rad/s^2} \\ 2.0845\,\mathrm{m/s^2} \\ 0.0000\,\mathrm{m/s^2} \\ -76.7222\,\mathrm{rad/s^2} \end{pmatrix}, \quad \lambda_k = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} -1.0092 \\ -27.7231 \\ 5.4581 \\ 0.0000 \end{pmatrix} \mathrm{N}.$$

Discussion: No initial speeds and with the bars along the positive y-axis means $\ddot{y}_1$ and $\ddot{y}_2$ should be zero: ok. No y-loading on bar 2 and no initial speed thus zero constraint force in B, $\lambda_4 = 0$: check ok. Idem y-constraint force in A should be compressive force from the spring, $\lambda_2 = -4/5\sigma_l$: check ok. Finally, compared to Problem 11.3b the accelerations of bar 1 are less because the spring pulls against gravity: check ok.

11.6 Double pendulum driven by a rotary motor in a revolute joint.

Method: Adding a motor constraint at the revolute joint between bar 1 and bar 2 in B, where we assume that the motor has a constant angular speed $\omega(= 30\,\text{rpm})$ clockwise. The motor constraint is then $C_m = (\varphi_2 - \varphi_1) - \omega t$. Since the we assume that the motor has a constant angular speed, this is just a kinematic constraint and the motor torque follows from the calculations as the Lagrange multiplier to this constraint.

Results: Both bars along the y-axis and an angular speed of $-\omega$ on bar 1. Note that the CoM velocities $\dot{x}_i$ which fulfill all the constraints (including the motor constraint!) are,

$$\dot{x}_i = \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\varphi}_1 \\ \dot{x}_2 \\ \dot{y}_2 \\ \dot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 0.9425\,\text{m/s} \\ 0\,\text{m/s} \\ -3.1416\,\text{rad/s} \\ 1.8850\,\text{m/s} \\ 0\,\text{m/s} \\ 0\,\text{rad/s} \end{pmatrix},$$

You can easily check this by calculating $\dot{C}_k = C_{k,i}\dot{x}_i$ and see if they are all zero except of course the motor constraint, see Prescribed Motion Section 11.3. Then the CoM accelerations $\ddot{x}_j$ and the Lagrange multipliers λ_k are,

$$\ddot{x}_j = \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\varphi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 3.6772\,\text{m/s}^2 \\ -2.9609\,\text{m/s}^2 \\ -12.2572\,\text{rad/s}^2 \\ 11.0315\,\text{m/s}^2 \\ -5.9218\,\text{m/s}^2 \\ -12.2572\,\text{rad/s}^2 \end{pmatrix}, \lambda_k = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} 3.4699 \\ 6.2756 \\ -0.8630 \\ 4.1837 \\ 0.5205 \end{pmatrix} \text{N}.$$

The motor torque is $T_m = \lambda_5 = 0.5205\,\text{N m}$ and the power that the motor supplies to the system is $P_m = T_m\omega = 1.6351\,\text{N m/s}$

Discussion: Since the angular speed between bar 1 and bar 2 has to be constant, the angular acceleration of bar 1 and bar 2 should be identical: check ok.

11.7 Double pendulum where end point C of bar 2 hits a flat wall along the y-axis, determine velocities after impact.

Method: Set up the impact equations according to (11.84). For the impact at point C from bar 2 at a wall along the y-axis (wall is where $x < 0$) we

introduce the contact condition $C_5 = x_2 + l/2\cos\varphi_2 - 0$. Contact occurs when $C_5 = 0$. Note that the relative velocity between point C and the wall, $\dot{C}_5$, is in general not zero, this is the impact.
Results: Both bars along the y-axis and with an initial angular speed of $\omega(= 30\,\text{rpm})$ counter-clockwise. The CoM velocities just before impact $\dot{x}_i^-$ are,

$$\dot{x}_i^- = \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\varphi}_1 \\ \dot{x}_2 \\ \dot{y}_2 \\ \dot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} -0.9425\,\text{m/s} \\ 0\,\text{m/s} \\ 3.1416\,\text{rad/s} \\ -2.8274\,\text{m/s} \\ 0\,\text{m/s} \\ 3.1416\,\text{rad/s} \end{pmatrix},$$

Now check the time derivatives of the constraints before impact $\dot{C}_k^- = C_{k,i}\dot{x}_i^-$ and see that they are all zero, except the relative speed before impact at point C, thus $\dot{C}_5^- = -3.7699(= \dot{x}_2 - l/2\dot{\varphi}_2)$.
The CoM velocities after impact $\dot{x}_j^+$ and the Lagrange multipliers ρ_k which are the reaction impulses ($\rho_{1..4}$) and contact impulses (ρ_5) for the three values of the restitution coefficient $e = (0, 0.9, 1)$ are,

$$\dot{x}_j^+ = \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\varphi}_1 \\ \dot{x}_2 \\ \dot{y}_2 \\ \dot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} -1.4113\,\text{m/s} \\ 0.0000\,\text{m/s} \\ 4.7042\,\text{rad/s} \\ -1.4113\,\text{m/s} \\ 0.0000\,\text{m/s} \\ -4.7042\,\text{rad/s} \end{pmatrix}, \begin{pmatrix} -1.8332\,\text{m/s} \\ 0.0000\,\text{m/s} \\ 6.1106\,\text{rad/s} \\ -0.1367\,\text{m/s} \\ 0.0000\,\text{m/s} \\ -11.7655\,\text{rad/s} \end{pmatrix}, \begin{pmatrix} -1.8801\,\text{m/s} \\ 0.0000\,\text{m/s} \\ 6.2669\,\text{rad/s} \\ 0.0049\,\text{m/s} \\ 0.0000\,\text{m/s} \\ -12.5500\,\text{rad/s} \end{pmatrix}.$$

$$\rho_k = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \end{pmatrix} = \begin{pmatrix} 0.1100 \\ -0.0000 \\ -0.2212 \\ -0.0000 \\ -0.7793 \end{pmatrix}\,\text{kg m/s}, \begin{pmatrix} 0.2090 \\ -0.0000 \\ -0.4202 \\ -0.0000 \\ -1.4807 \end{pmatrix}\,\text{kg m/s}, \begin{pmatrix} 0.2200 \\ -0.0000 \\ -0.4424 \\ -0.0000 \\ -1.5587 \end{pmatrix}\,\text{kg m/s}.$$

The change in the kinetic energy of the system $\Delta T = T^+ - T^-$ for the three values of the restitution coefficient $e = (0, 0.9, 1)$ are,

$$\Delta T = -1.4690\,\text{N m},\ -0.2791\,\text{N m},\ 0.0000\,\text{N m},$$

and the work W done by the contact impulse ρ_5 at the contact point is $\frac{1}{2}(1 - e)\rho_5\dot{C}_5^-$, for the three values of the restitution coefficient $e = (0, 0.9, 1)$ are,

$$W = 1.4690\,\text{N m},\ 0.2791\,\text{N m},\ 0\,\text{N m}.$$

Discussion: We already checked the velocities before impact. If we take the velocities after impact and calculate the time derivatives of the constraints after impact $\dot{C}_k^+ = C_{k,i}\dot{x}_i^+$, we see that they are all zero, except the relative

speed after impact at point C, thus $\dot{C}_5^+ = (0.0000, 3.3929, 3.7699)$m/s, which completely complies with Newton's impact law $\dot{x}_{rel}^+ = -e\dot{x}_{rel}^-$: check ok. Next, for the fully plastic case ($e = 0$) we expect point C to stay on the wall, and thus the angular speeds of the two bars should be identical in size but opposite in sign and the horizontal speed of the CoM's should be identical: check ok. Bars are vertical up and velocities are horizontal and impact direction is also horizontal, then all vertical constraint impulses should be zero: check ok. With increasing value of the restitution coefficient e the contact impulse should increase in magnitude: check ok. Due to the impact, kinetic energy will be lost when $e < 1$, and for $e = 0$ no kinetic energy is lost: check ok. The amount of work W done by the contact impulse should be equal to the magnitude of the kinetic energy change: check ok.

11.10 One could rewrite the equations of motion and take as new unknowns the velocity jumps $\Delta\dot{x}_i = \dot{x}_i^+ - \dot{x}_i^-$. Then, they read

$$\begin{pmatrix} M_{ij} & C_{k,i} \\ C_{k,j} & 0_{kk} \end{pmatrix} \begin{pmatrix} \Delta\dot{x}_j \\ \rho_k \end{pmatrix} = \begin{pmatrix} s_i \\ -(1+e)\, C_{k,j}\dot{x}_j^- \end{pmatrix}. \tag{G.297}$$

The change in kinetic energy is

$$\Delta T = T^+ - T^-, \tag{G.298}$$

with

$$T^+ = \tfrac{1}{2}\dot{x}_i^+ M_{ij}\dot{x}_j^+, \tag{G.299}$$
$$T^- = \tfrac{1}{2}\dot{x}_i^- M_{ij}\dot{x}_j^-. \tag{G.300}$$

The equations of motion can be multiplied by $\frac{1}{2}\dot{x}_i^+$ and $\frac{1}{2}\dot{x}_i^-$:

$$\underbrace{\tfrac{1}{2}\dot{x}_i^+ \left(M_{ij}\dot{x}_j^+\right)}_{T^+} + C_{k,i}\rho_k\big) = \tfrac{1}{2}\dot{x}_i^+ \left(M_{ij}\dot{x}_j^- + s_i\right), \tag{G.301}$$
$$\tfrac{1}{2}\dot{x}_i^- \left(M_{ij}\dot{x}_j^+ + C_{k,i}\rho_k\right) = \underbrace{\tfrac{1}{2}\dot{x}_i^- \left(M_{ij}\dot{x}_j^-\right)}_{T^-} + s_i\big). \tag{G.302}$$

This gives

$$T^+ = \tfrac{1}{2}\dot{x}_i^+ \left(-C_{k,i}\rho_k + s_i + M_{ij}\dot{x}_j^-\right), \tag{G.303}$$
$$T^- = \tfrac{1}{2}\dot{x}_i^- \left(C_{k,i}\rho_k - s_i + M_{ij}\dot{x}_j^+\right). \tag{G.304}$$

So, the difference in kinetic energy is

$$\Delta T = -\tfrac{1}{2}\rho_k C_{k,i}\left(\dot{x}_i^+ + \dot{x}_i^-\right) + \tfrac{1}{2}s_i\left(\dot{x}_i^+ + \dot{x}_i^-\right). \tag{G.305}$$

If we assume there are no applied impulses ($s_i = 0$),

$$\Delta T = -\tfrac{1}{2}\rho_k\,(1-e)\, C_{k,i}\dot{x}_i^-, \tag{G.306}$$
$$= -\tfrac{1}{2}\,(1-e)\,\rho_k\dot{C}_k^-, \quad k = 1\ldots c. \tag{G.307}$$

12.2 The generalized forces Q_s and Q_φ are drawn in the figure of the model in Figure G.30.

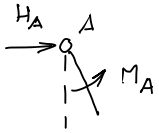


Figure G.30: Overhead track container crane generalized forces.

Q_s is the action-reaction force moving the support and Q_φ is the action-reaction torque driving the orientation of the pendulum. The physical units are $[Q_s] = \mathrm{N}$ and $[Q_\varphi] = \mathrm{N\,m}$, whereas the specific entries in the remaining terms of the equations of motion have units

$$\begin{pmatrix} \mathrm{kg} & \mathrm{kgm} \\ \mathrm{kg\,m} & \mathrm{kg\,m^2} \end{pmatrix} \begin{pmatrix} \mathrm{m/s^2} \\ \mathrm{rad/s^2} \end{pmatrix} = \begin{pmatrix} \mathrm{N} \\ \mathrm{N\,m} \end{pmatrix}.$$

Here we clearly see the effect of the different physical units of the generalized coordinates on the physical units in the terms in the equations.

12.3 Either replace the point mass in B by a rigid body with a finite mass moment of inertia, or add a mass to the moving support. In both cases the mass matrix will always be regular.

12.4 The double pendulum has two degrees of freedom, then for generalized independent coordinates we choose two angles, $\mathbf{q} = (\alpha, \beta)$, where α is the angle between bar 1 and the x-axis, positive when anti-clockwise, and β the angle between bar 2 and x-axis, also positive when anti-clockwise. These two angles can uniquely describe the configuration of the double pendulum. One can also make other choices, like the absolute angle of bar 1 and the relative angle between bar2 and bar 1. That is ok too, as long as your chosen coordinate uniquely can describe the configuration.

To derive the equations of motion by means of the Lagrange equations we first need to express the CoM coordinates $\mathbf{x}$ in terms of the generalized coordinates $\mathbf{q}$ and the system parameters.

$$x_i = \begin{pmatrix} x_1 \\ y_1 \\ \varphi_1 \\ x_2 \\ y_2 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} l/2\cos(\alpha) \\ l/2\sin(\alpha) \\ \alpha \\ l\cos(\alpha) + l/2\cos(\beta) \\ l\sin(\alpha) + l/2\sin(\beta) \\ \beta \end{pmatrix}$$

Then, with the mass matrix $M_{ij} = \mathrm{diag}(m, m, I, m, m, I)$, the kinetic energy of the system is simply, $T = 1/2\dot{x}_i M_{ij}\dot{x}_j$, and the potential energy is, $V = -mgx_1 - mgx_2$. The rest is algebra, see (12.31). Then, the resulting

equations of motion for the double pendulum expressed in independent generalized coordinates are,

$$\begin{pmatrix} I+5/4ml^2 & 1/2ml^2\cos(\alpha-\beta) \\ 1/2ml^2\cos(\alpha-\beta) & I+1/4ml^2 \end{pmatrix} \begin{pmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{pmatrix} = \begin{pmatrix} Q_\alpha - 3/2mgl\sin(\alpha) - 1/2ml^2\dot{\beta}^2\sin(\alpha-\beta) \\ Q_\beta - 1/2mgl\sin(\beta) + 1/2ml^2\dot{\alpha}^2\sin(\alpha-\beta) \end{pmatrix}$$

Finally, given the state and the accelerations of the generalized coordinates, the accelerations of the CoM coordinates can be calculated from $\ddot{x}_i = x_{i,j}\ddot{q}_j + x_{i,kl}\dot{q}_k\dot{q}_l$, which for this double pendulum problem is then,

$$\ddot{x}_j = \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\varphi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} -l/2\sin(\alpha)\ddot{\alpha} \\ l/2\cos(\alpha)\ddot{\alpha} \\ \ddot{\alpha} \\ -l\sin(\alpha)\ddot{\alpha} - l/2\sin(\beta)\ddot{\beta} \\ l\cos(\alpha)\ddot{\alpha} + l/2\cos(\beta)\ddot{\beta} \\ \ddot{\beta} \end{pmatrix} + \begin{pmatrix} -l/2\cos(\alpha)\dot{\alpha}^2 \\ -l/2\sin(\alpha)\dot{\alpha}^2 \\ 0 \\ -l\cos(\alpha)\dot{\alpha}^2 - l/2\cos(\beta)\dot{\beta}^2 \\ -l\sin(\alpha)\dot{\alpha}^2 - l/2\sin(\beta)\dot{\beta}^2 \\ 0 \end{pmatrix} \tag{G.308}$$

You can use these expressions to compare your results with those from Problem 11.3.

For the three different cases the accelerations of the generalized degrees of freedom are,

$$\text{b: } \begin{pmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{pmatrix} = \begin{pmatrix} -20.9903 \\ 6.9483 \end{pmatrix} \text{rad/s}^2, \text{ c: } \begin{pmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{rad/s}^2, \text{ d: } \begin{pmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{rad/s}^2.$$

Substitute these, together with the position and velocities in the expression (G.308) and then you can compare all results. A quick check is that the CoM angular accelerations are one-to-one: $(\ddot{\varphi}_1, \ddot{\varphi}_2) = (\ddot{\alpha}, \ddot{\beta})$.

12.5 Add the potential energy of the spring, $V_s = 1/2k\Delta l^2$ to the potential energy, with the elongation of the spring given by

$$\Delta l = \sqrt{(x_E - x_D)^2 + (y_E - y_D)^2} - l_0$$

The coordinates of point E of body 1 are now $(x_E, y_E) = (x_1 + 1/6l\cos(\alpha), y_1 + 1/6l\sin(\alpha))$. Then apply Lagrange equations of motion. The mass matrix does not change, and the forces on the right hand side of the equations have the additional terms due to the spring force,

$$\begin{pmatrix} Q_\alpha - \frac{3}{2}mgl\sin(\alpha) - \frac{1}{2}ml^2\dot{\beta}^2\sin(\alpha-\beta) + \frac{1}{3}kl^2\left(\sin(\alpha) - \frac{4}{\sqrt{25+24\cos(\alpha)}}\right) \\ Q_\beta - \frac{1}{2}mgl\sin(\beta) + \frac{1}{2}ml^2\dot{\alpha}^2\sin(\alpha-\beta) + 0 \end{pmatrix}$$

Substitution of the initial conditions results in the accelerations of the generalized coordinates, $(\ddot{\alpha}, \ddot{\beta}) = (34.8869, -76.7222)\text{rad/s}^2$, which comply with the results from Problem 11.5.

12.6 Add a motor with constant angular speed in joint B, such that the angular speed of body 2 with respect to body 1 is ω, where all angles and angular speeds are counter-clockwise positive. This can be done by adding a time dependent kinematic constraint, $C_1 = (\varphi_2 - \varphi_1) - \omega t$, to the Lagrange equations of motion as found under item b. The corresponding Lagrange multiplier λ_1 is then the motor torque. However, before using this constraint we first have to express the cm coordinates used in this constraint in terms of the generalized independent coordinates. This substitution is rather straightforward here and the result is: $C_1 = (\beta - \alpha) - \omega t$. Note that if you have defined your generalized coordinates differently that your constraint will have another form. With the initial values for the generalized degrees of freedom as $\alpha = \pi/2\,\text{rad}, \beta = \pi/2\,\text{rad}$ and $\dot{\alpha} = -\pi\,\text{rad/s}, \dot{\beta} = 0\,\text{rad/s}$ we solve for the accelerations $(\ddot{\alpha}, \ddot{\beta}) = (-12.2572, -12.2572)\text{rad/s}^2$ and Lagrange multiplier $\lambda_1 = 0.5205\,\text{N}$. These values comply with the results from Problem 11.6.

12.7 For the Lagrange impact equations we can use the mass matrix from the equations of motion as derived in Problem 12.4. The impact contact constraint, C_k where $k = 1$, expressed in the generalized coordinates is, $C_1 = l\cos(\alpha) + l\cos(\beta)$. Adding the jacobian $C_{1,i}$ to the system of equations results in the Lagrange impact equations,

$$\begin{pmatrix} I + 5/4ml^2 & 1/2ml^2\cos(\alpha-\beta) & -l\sin(\alpha) \\ 1/2ml^2\cos(\alpha-\beta) & I + 1/4ml^2 & -l\cos(\beta) \\ -l\sin(\alpha) & -l\cos(\beta) & 0 \end{pmatrix} \begin{pmatrix} \dot{\alpha}^+ \\ \dot{\beta}^+ \\ \rho_1 \end{pmatrix} = \begin{pmatrix} (I + 5/4ml^2)\dot{\alpha}^- + (1/2ml^2\cos(\alpha-\beta))\dot{\beta}^- \\ (1/2ml^2\cos(\alpha-\beta))\dot{\alpha}^- + (I + 1/4ml^2)\dot{\beta}^- \\ -e(-l\sin(\alpha)\dot{\alpha}^- - l\cos(\beta)\dot{\beta}^-) \end{pmatrix}$$

Substitution of the initial values for the three different cases of the restitution factor $e = (0, 0.9, 1.0)$ results in the generalized velocities after impact,

$$\begin{pmatrix} \dot{\alpha}^+ \\ \dot{\beta}^+ \end{pmatrix} = \begin{pmatrix} 4.7042 \\ -4.7042 \end{pmatrix}\text{rad/s}, \begin{pmatrix} 6.1106 \\ -11.7655 \end{pmatrix}\text{rad/s}, \begin{pmatrix} 6.2669 \\ -12.5500 \end{pmatrix}\text{rad/s},$$

together with the contact impulses,

$$\rho_1 = -0.7793\,\text{kg m/s}, -1.4807\,\text{kg m/s}, -1.5587\,\text{kg m/s}.$$

And with these results the change in kinetic energy easily follows from $\Delta T = 1/2(\dot{q}_i^+ \bar{M}_{ij} \dot{q}_j^+ - \dot{q}_i^- \bar{M}_{ij} \dot{q}_j^-)$, with the mass matrix $\bar{M}_{ij}$ from above. And of course the work done by the contact impulse is equal to minus the change in kinetic energy.

13.3 Figure G.31 shows a simplified 2-D model of a bipedal walking machine.

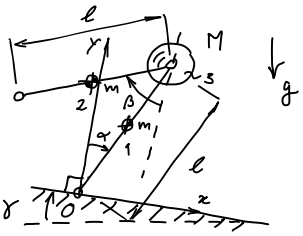


Figure G.31: 2-D walker model with coordinate system

a. Equations of motion using Lagrange's equations of motion. We use a rectangular x–y coordinate system, where the x-axis is along the slope downwards and the y-coordinate perpendicular to the slope upwards, see Figure G.31. The rotation φ is positive when counter-clockwise and zero at the x-axis. The origin is where the stance leg is fixed to the slope. Body 1 is the stance leg, body 2 is the swing leg, and the point mass is body number 3 (only position, no orientation). The generalized coordinates, $q_j = (\alpha, \beta)$, are described in the problem statement. To derive the equations of motion by either method, we first need to express the CoM coordinates x_i in terms of the generalized coordinates q_j and the system parameters.

$$x_i = \begin{pmatrix} x_1 \\ y_1 \\ \varphi_1 \\ x_2 \\ y_2 \\ \varphi_2 \\ x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} l/2\sin(\alpha) \\ l/2\cos(\alpha) \\ \pi/2 - \alpha \\ l\sin(\alpha) - l/2\sin(\beta) \\ l\cos(\alpha) - l/2\cos(\beta) \\ -\pi/2 - \beta \\ l\sin(\alpha) \\ l\cos(\alpha) \end{pmatrix} \tag{G.309}$$

Then, with the mass matrix $M_{ij} = \mathrm{diag}(m, m, I, m, m, I, M, M)$ (do not confuse the scalar mass M of body 3 with the mass matrix $\mathbf{M} = M_{ij}$), the kinetic energy of the system is simply, $T = 1/2\dot{x}_i M_{ij} \dot{x}_j$. Due to the choice of the coordinate system along and perpendicular to the slope, gravity is now working in both the x and y direction. The potential energy is now, $V = -g_x(mx_1 + mx_2 + Mx_3) - g_y(my_1 + my_2 + My_3)$, with the gravity components $g_x = g\sin(\gamma)$ and $g_y = -g\cos(\gamma)$. The derivation of the equations of motion by means of Lagrange' equations of motion is now just algebra, see (12.31). Then, the resulting equations of motion for the walker model, expressed in independent generalized coordinates are,

$$\begin{pmatrix} Ml^2 + 4/3ml^2 & -1/2ml^2\cos(\alpha-\beta) \\ -1/2ml^2\cos(\alpha-\beta) & 1/3ml^2 \end{pmatrix} \begin{pmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{pmatrix} = \begin{pmatrix} Q_\alpha + (M+3/2m)gl\sin(\alpha+\gamma) + 1/2ml^2\dot{\beta}^2\sin(\alpha-\beta) \\ Q_\beta - 1/2mgl\sin(\beta+\gamma) - 1/2ml^2\dot{\alpha}^2\sin(\alpha-\beta) \end{pmatrix} \quad \text{(G.310)}$$

Note: Here we have used $I = 1/12ml^2$, but you can also keep just I. We have added the generalized forces Q_α and Q_β on the right-hand side of the equations for generality, in the case of the walker they are zero.
Discussion: All generalized coordinates are angles, therefore all terms in the mass matrix have the dimension $\mathrm{kg\,m^2}$. If you only have a nonzero $\ddot{\beta}$ then the mass term is the mass moment of inertia of body 2 at the top hinge: $1/12ml^2 + m(l/2)^2 = 1/3ml^2$. If you only have a nonzero $\ddot{\alpha}$ then the mass term is the mass moment of inertia at the base hinge for body 1 plus the mass moment of inertia due to the added mass from body 2 and body 3 at the top, $1/12ml^2 + m(l/2)^2 + (M+m)l^2 = Ml^2 + 4/3ml^2$. The check on the off-diagonal term we leave for now because it is somewhat more complex to explain. The gravity forces on the righthand side of the equations can easily be checked by substitution of $\alpha = -\gamma$ and $\beta = -\gamma$ and noticing that the gravity torques are then zero, and by setting $\gamma = 0$ and seeing that the gravity torques have the correct sign. The magnitude follows from drawing the gravity forces in a sketch of the system and checking the equilibrium torques necessary to keep the system at that orientation. The same can be done for the convective acceleration terms, by drawing the centrifugal forces in a sketch of the system and checking the "equilibrium" torques necessary to keep the system at that orientation.

b. For the derivation of the equations of motion by means of the TMT-method we use the expressions the CoM coordinates x_i in terms of the generalized coordinates q_j (0.1), which we identify as $x_i = T_i(q_j)$. Next we apply (13.12), which is just algebra. For completeness, we present here the jacobian, $T_{i,j}$, of the coordinate mapping,

$$T_{i,j} = \begin{pmatrix} l/2\cos(\alpha) & 0 \\ -l/2\sin(\alpha) & 0 \\ -1 & 0 \\ l\cos(\alpha) & -l/2\cos(\beta) \\ -l\sin(\alpha) & l/2\sin(\beta) \\ 0 & -1 \\ l\cos(\alpha) & 0 \\ -l\sin(\alpha) & 0 \end{pmatrix},$$

and the applied CoM forces, f_i, together with the convective acceleration

terms g_k, being,

$$f_i = \begin{pmatrix} mg\sin(\gamma) \\ -mg\cos(\gamma) \\ 0 \\ mg\sin(\gamma) \\ -mg\cos(\gamma) \\ 0 \\ Mg\sin(\gamma) \\ -Mg\cos(\gamma) \end{pmatrix} \quad \text{, and} \quad g_k = \begin{pmatrix} -\dot{\alpha}^2 l/2\sin(\alpha) \\ -\dot{\alpha}^2 l/2\cos(\alpha) \\ 0 \\ -\dot{\alpha}^2 l\sin(\alpha) + \dot{\beta}^2 l/2\sin(\beta) \\ -\dot{\alpha}^2 l\cos(\alpha) + \dot{\beta}^2 l/2\cos(\beta) \\ 0 \\ -\dot{\alpha}^2 l\sin(\alpha) \\ -\dot{\alpha}^2 l\cos(\alpha) \end{pmatrix}.$$

Of course, the resulting equations of motion are identical to the ones from above, were we used Lagrange's equations of motion.

c. The equations of motion found at (a) should be identical to the ones found at (b).

d. For large systems the work done by using the Lagrange equations of motion is more, because from the level of force equations, we first have to go up to the level of energy, and form the scalar kinetic energy and potential energy expressions. Next, we differentiate these energy functions, to get back on the force equation level. Moreover, we also calculate extra terms $\partial T/\partial q_j$ in calculating $\mathrm{d}(\partial T/\partial \dot{q}_j)/\mathrm{d}t$, which we then have to subtract. The TMT-method stays on the level of forces, and no extra work is done.
It would be interesting to count the number of addition/subtraction and multiplication/division operations as applied in the two different methods and compare those results.

14.5 Numerical integration of the equations of motion for the double pendulum from Problem 10.2 with four different explicit methods, together with convergency plots to determine necessary step-size for a given global error bound.

a. Determine the max step-size for the angles at $T = 3$ sec to have a max global error of 1e-6 rad. We use the global error estimation method as described in Chapter 14.3. The tabulated results for the four different methods are given in Table G.3, Table G.4, Table G.5, and Table G.6
Your results can differ a little bit in the last digits because of a different implementation of the equations of motion, but the global results like the first two digits in max step size found should be the same.
From the tabulated results, we conclude that with Euler's method we cannot find a max stepsize that fulfills our criteria. The results are slowly converging but even for a small stepsize as 1.14e-5, we still have global errors in the order of 5e-3.

Table G.3: Numerical integration stepsize h and pendulum angles φ_1 and φ_2, together with global error estimates E at $T = 3\,\mathrm{s}$; Euler's method.

h/s	φ_1/rad	φ_2/rad	$E(\varphi_1)$/rad	$E(\varphi_2)$/rad
9.37500e-02	NaN	NaN	NaN	NaN
4.68750e-02	-0.51454773263997	8.69361309875534	NaN	NaN
2.34375e-02	0.29568848889725	29.61978272123748	8.10e-01	2.09e+01
1.17188e-02	-0.92209514878527	4.43521260084967	1.22e+00	2.52e+01
5.85938e-03	-1.25480890406251	-1.96390102281070	3.33e-01	6.40e+00
2.92969e-03	-0.93110187576018	-3.42747227650190	3.24e-01	1.46e+00
1.46484e-03	-1.15104734749807	-3.35307093670344	2.20e-01	7.44e-02
7.32422e-04	-1.19167152654867	-3.01760316835206	4.06e-02	3.35e-01
3.66211e-04	-1.20875859356599	-2.84636032856907	1.71e-02	1.71e-01
1.83105e-04	-1.21724809670651	-2.76462596532588	8.49e-03	8.17e-02
9.15527e-05	-1.22147758875182	-2.72508013406774	4.23e-03	3.95e-02
4.57764e-05	-1.22358456798668	-2.70567041966088	2.11e-03	1.94e-02
2.28882e-05	-1.22463545417821	-2.69605990060924	1.05e-03	9.61e-03
1.14441e-05	-1.22516015507608	-2.69127863498989	5.25e-04	4.78e-03

Taking smaller stepsizes takes too long, so we abandon Euler's method here. For Heun's method we are able to meet our criteria at $h =$ 2.28e-5. The max stepsize for the rk3 method is $h =$3.66e-4 and for the rk4 method we already hit our criterium at $h =$2.93e-3.

Of course, these stepsizes are conservative estimates, one could interpolate between the nearby results to get a more accurate estimate for the max stepsize h. We will show that in the next item, where we plot the results in a convergence plot and determine the crossing of the lines at $E =$1e-6.

Table G.4: Numerical integration stepsize h and pendulum angles φ_1 and φ_2, together with global error estimates E at $T = 3\,\mathrm{s}$; Heun's method.

h/s	φ_1/rad	φ_2/rad	$E(\varphi_1)$/rad	$E(\varphi_2)$/rad
9.37500e-02	NaN	NaN	NaN	NaN
4.68750e-02	-0.92544250483348	-3.37435934968173	NaN	NaN
2.34375e-02	-1.35325722544182	-2.20982765179883	1.43e-01	3.88e-01
1.17188e-02	-1.26012207188695	-2.54278080046031	3.10e-02	1.11e-01
5.85938e-03	-1.23454732125047	-2.64810405696499	8.52e-03	3.51e-02
2.92969e-03	-1.22794972786065	-2.67653865486204	2.20e-03	9.48e-03
1.46484e-03	-1.22625867543906	-2.68396666212280	5.64e-04	2.48e-03
7.32422e-04	-1.22582904684843	-2.68586967750184	1.43e-04	6.34e-04
3.66211e-04	-1.22572065628953	-2.68635163278068	3.61e-05	1.61e-04
1.83105e-04	-1.22569342715019	-2.68647292691122	9.08e-06	4.04e-05
9.15527e-05	-1.22568660286715	-2.68650335305250	2.27e-06	1.01e-05
4.57764e-05	-1.22568489463574	-2.68651097253758	5.69e-07	2.54e-06
2.28882e-05	-1.22568446730556	-2.68651287903504	1.42e-07	6.35e-07
1.14441e-05	-1.22568436043875	-2.68651335586372	3.56e-08	1.59e-07

Table G.5: Numerical integration stepsize h and pendulum angles φ_1 and φ_2, together with global error estimates E at $T = 3\,\mathrm{s}$; RK3 method.

h/s	φ_1/rad	φ_2/rad	$E(\varphi_1)$/rad	$E(\varphi_2)$/rad
9.37500e-02	-1.09891711190382	-1.71975872965837	NaN	NaN
4.68750e-02	-1.26490066729047	-2.34028260798716	2.37e-02	8.86e-02
2.34375e-02	-1.24060334340375	-2.60343571560555	3.47e-03	3.76e-02
1.17188e-02	-1.22704681733787	-2.67710418565241	1.94e-03	1.05e-02
5.85938e-03	-1.22584156983682	-2.68536957015933	1.72e-04	1.18e-03
2.92969e-03	-1.22570334979007	-2.68637256051201	1.97e-05	1.43e-04
1.46484e-03	-1.22568666872012	-2.68649602499906	2.38e-06	1.76e-05
7.32422e-04	-1.22568461581391	-2.68651133678182	2.93e-07	2.19e-06
3.66211e-04	-1.22568436106612	-2.68651324310011	3.64e-08	2.72e-07
1.83105e-04	-1.22568432933476	-2.68651348090890	4.53e-09	3.40e-08
9.15527e-05	-1.22568432537518	-2.68651351060494	5.66e-10	4.24e-09
4.57764e-05	-1.22568432488068	-2.68651351431499	7.06e-11	5.30e-10
2.28882e-05	-1.22568432481892	-2.68651351477875	8.82e-12	6.63e-11
1.14441e-05	-1.22568432481125	-2.68651351483651	1.10e-12	8.25e-12

Table G.6: Numerical integration stepsize h and pendulum angles φ_1 and φ_2, together with global error estimates E at $T = 3\,\mathrm{s}$; RK4 method.

h/s	φ_1/rad	φ_2/rad	$E(\varphi_1)$/rad	$E(\varphi_2)$/rad
9.37500e-02	-1.30594520384171	-1.64233939773265	NaN	NaN
4.68750e-02	-1.26397841929786	-2.48243491021852	2.80e-03	5.60e-02
2.34375e-02	-1.22328083700147	-2.69028644719574	2.71e-03	1.39e-02
1.17188e-02	-1.22562159004160	-2.68657880541565	1.56e-04	2.47e-04
5.85938e-03	-1.22568177637754	-2.68651771611062	4.01e-06	4.07e-06
2.92969e-03	-1.22568421046631	-2.68651379013539	1.62e-07	2.62e-07
1.46484e-03	-1.22568431906995	-2.68651353260704	7.24e-09	1.72e-08
7.32422e-04	-1.22568432449497	-2.68651351597514	3.62e-10	1.11e-09
3.66211e-04	-1.22568432479174	-2.68651351491612	1.98e-11	7.06e-11
1.83105e-04	-1.22568432480895	-2.68651351484942	1.15e-12	4.45e-12
9.15527e-05	-1.22568432481003	-2.68651351484498	7.22e-14	2.96e-13
4.57764e-05	-1.22568432481004	-2.68651351484481	6.22e-16	1.09e-14
2.28882e-05	-1.22568432480999	-2.68651351484523	3.58e-15	2.77e-14
1.14441e-05	-1.22568432481011	-2.68651351484493	8.22e-15	1.97e-14

b. The convergence plots for the two angles are shown in Figure G.32 and Figure G.33.
We now can make a more accurate estimate of the max step size which meets our criterium of $E \leq$1e-6 by determining the intersection of the plots with the line $E =$1e-6. First note that the error in φ_2 is always larger than the one in φ_1, so we look only at the φ_2 plot. Again for Euler we can not find a max stepsize. For Heun's method the max stepsize is $h =$3.0e-5, for rk3 we get $h =$ 5.7e-4, and for rk4 we estimate $h =$4.0e-3. Note that around $h =$5e-5 the rk4 method hits the roundoff error and that the global error estimate increases with decreasing stepsize.

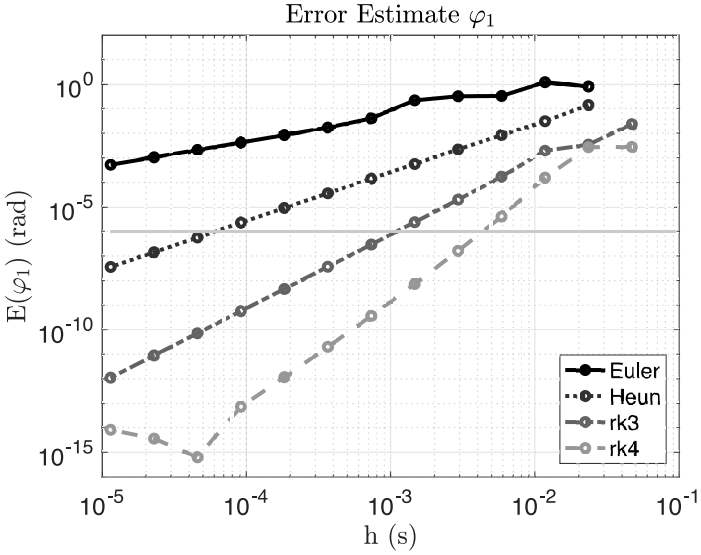


Figure G.32: Convergence plot with the global error estimates E for pendulum angle φ_1 at $T = 3$ sec for four different methods as a function of the stepsize h. The horizontal line is the error bound $E = 10^{-6}$, and the black lines parallel to the colored lines are power laws h^p with $p = 1, 2, 3, 4$.

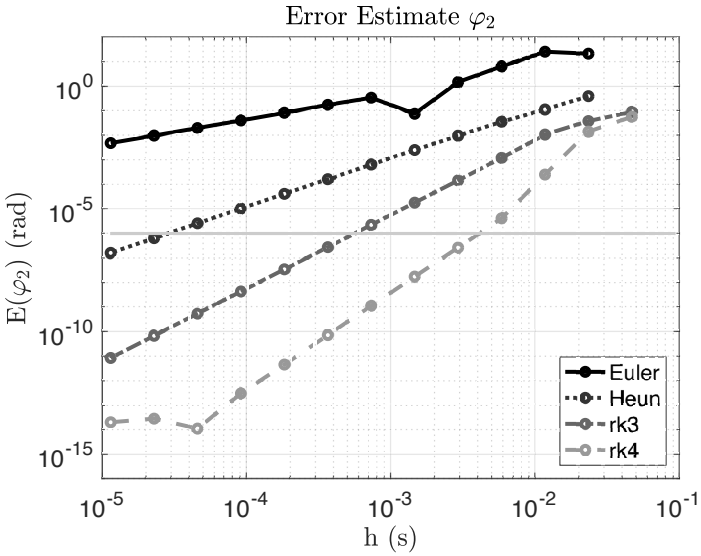


Figure G.33: Convergence plot with the global error estimates E for pendulum angle φ_2 at $T = 3$ sec for four different methods as a function of the stepsize h. The horizontal line is the error bound $E = 10^{-6}$.

c. In using the Matlab ODE solvers we have to decide what value for `AbsTol` and `RelTol` to set. Note that `AbsTol` and `RelTol` are local error bounds, the global error bound is the sum over the number of steps of these local errors. However, initially we do not know how many step will be taken, therefore we can only determine `AbsTol` and `RelTol` in an iterative way to meet our global error bound of E =1e-6 [rad]. The angles are of the order 1, so to make things easy we only use `AbsTol` and make `RelTol` very small. For instance, setting `AbsTol` to 1e-8 and keeping track of the number of steps can be done with,

```
ops = odeset('AbsTol',1e-8,'RelTol',1e-14,'Stats','on')
```

Then the call to the ode is something like,

```
odesol = ode23(@odehwa6,[t0 tend],y0,ops)
```

All results are in the struct `odesol`. For example, `odesol.x` is the time vector, and `odesol.y` are the state value vectors. The number of steps is part of the struct `odesol.stats`, namely `odesol.stats.nsteps`. The strategy is now to calculate the solutions for a number of decreasing values of `AbsTol`, check the number of steps taken, determine the average stepsize h, and plot the absolute difference between two subsequent solution, $D = |y_n - y_{n-1}|$, as a function of `AbsTol` and also as a function of the average stepsize to check the convergence rate (slope of the plot) and to have an estimate on the global error E. The difference between D and E is a constant factor of the order 1.

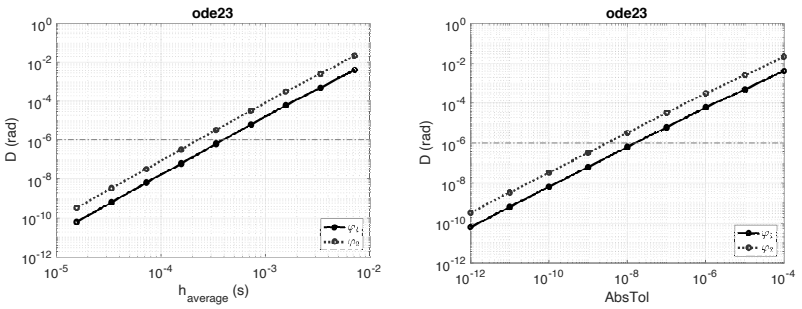


Figure G.34: Convergence plot for `ode23` with the global error estimates D for pendulum angles at $T = 3$ sec as a function of the average stepsize h (left) and as a function of `AbsTol` (right). The horizontal line is the error bound $D = 10^{-6}$.

For `ode23` we see in Figure G.34 that for h =2.0e-4 we meet the criterium, which means that we have to set `AbsTol` to 3.0e-9 (right figure). The convergence rate of D as a function of h is of the order $p = 3$ (slope of the line in the left figure), this complies with a local truncation error of order 4, which agrees with the documentation of `ode23`. And the max stepsize is somewhere between the one found for the constant step size rk3 and rk4 method from above.

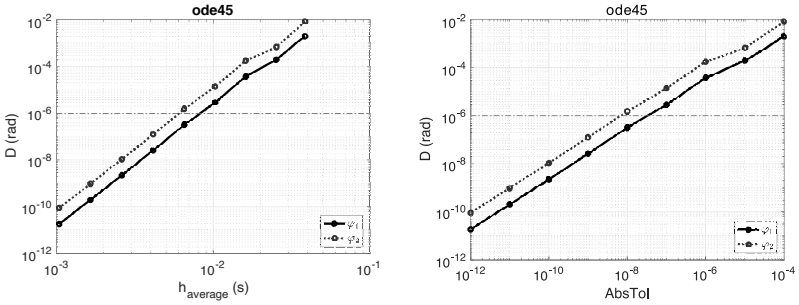


Figure G.35: Convergence plot for `ode45` with the global error estimates D for pendulum angles at $T = 3$ sec as a function of the average stepsize h (left) and as a function of `AbsTol` (right). The horizontal line is the error bound $D = 10^{-6}$.

For `ode45` we see in Figure G.35 (left) that for h =6.0e-3 we meet the criterium, which means that we have to set `AbsTol` to 7.0e-9 (right figure). The convergence rate of D as a function of h is of the order $p = 5$ (slope of the line in the left figure), this complies with a local truncation error of order 6, which agrees with the documentation of `ode45`. And the max stepsize is in the order of the one found for the constant step size rk4 method from above.

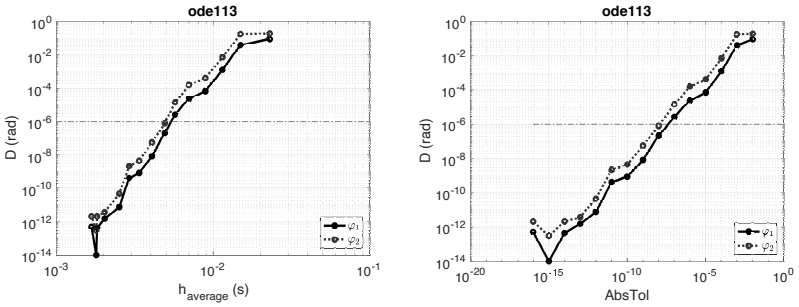


Figure G.36: Convergence plot for `ode113` with the global error estimates D for pendulum angles at $T = 3$ sec as a function of the average stepsize h (left) and as a function of `AbsTol` (right). The horizontal line is the error bound $D = 10^{-6}$.

For `ode113` we see in Figure G.36 (left) that for h =5.0e-3 we meet the criterium, which means that we have to set `AbsTol` to 1.0e-8 (right figure). This surprisingly large compared to the other two methods. If we read the documentation on `ode113` we read the following:

> `ode113` is a variable-step, variable-order (VSVO) Adams-Bashforth-Moulton PECE solver of orders 1 to 13. The highest order used appears to be 12, however, a formula of order

13 is used to form the error estimate and the function does local extrapolation to advance the integration at order 13. `ode113` may be more efficient than `ode45` at stringent tolerances or if the ODE function is particularly expensive to evaluate. `ode113` is a multistep solver – it normally needs the solutions at several preceding time points to compute the current solution [1], [2].

This explains a lot. Indeed, if we check the convergence rate of the global error we see that is is of the order $p = 11$, so it was probably using the highest order (=12). However, note that for a constant stepsize rk4 method we found a max stepsize of h =4.0e-3, which is about the same here. However, the rk4 method uses four function evaluations per step, whereas the `ode113` uses only two (one for advancing the solution and one for error estimation).
To summarize, to meet the tolerance criterium of a max absolute error of 10^{-6} after 3 seconds, we have found approximately the following values for `AbsTol` for the three ODE methods,

$$\begin{aligned} \texttt{ode23}: &\quad \texttt{AbsTol} = 3.0\text{e-}9 \\ \texttt{ode45}: &\quad \texttt{AbsTol} = 7.0\text{e-}9 \\ \texttt{ode113}: &\quad \texttt{AbsTol} = 1.0\text{e-}8, \end{aligned}$$

Next, we can compare the four fixed-step methods used before with the ones used here, as shown in Table G.7.

Table G.7: Numerical integration stepsize h and nfe = total number of function evaluations of the ODE $\dot{y} = f(t, y)$, for all methods used.

method	average stepsize h	nfe
Euler	NaN	NaN
Heun	3.0e-5	200000
rk3	5.7e-4	15789
rk4	4.0e-3	3000
ode23	2.0e-4	45000
ode45	6.0e-3	3000
ode113	5.0e-3	1200

In conclusion, rk4 and `ode45` perform about the same and take about the same effort (nfe). If evaluation the differential equations, $\dot{\boldsymbol{y}} = \boldsymbol{f}(t, \boldsymbol{y})$, is time-consuming, then it is advised to use higher-order methods like the `ode113`.

15.1 The first part $\frac{a}{b}(d\cos\alpha - a)$ has the dimension of length whereas it should be dimensionless, apparently the typo is that the lower case b should be an upper case B, and the correct expression is

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega\left[\frac{a}{B}(d\cos\alpha - a) - \frac{Ad}{S}\left(2 - \frac{D}{B}\right)\right].$$

15.2 Numerical integration of the equations of motion for a quick-return mechanism from literature, using generalized coordinates, cutting the closed loops, and using coordinate projection method to fulfill the constraints.

a. The algorithm described in words is:
- Make an open-loop structure by one cut in sliding joint A and one cut in sliding joint C, you only cut perpendicular to the sliding motion.
- Introduce three generalized coordinates $q_j = (\alpha, \beta, \gamma)$, where α is the angle of crank 2, β the angle of rocker 4, and γ the angle of connecting bar 5.
- Express the position and orientation of the rigid bodies and the positions of the point masses in terms of the generalized coordinates, $x_i = T_i(q_j)$.
- Define the two constraints $C_k(q_j) = 0$ which close the loops. The first constraint is the normal distance of point A to rocker 4, and the second constraint is the normal distance of point 6 to the top horizontal surface at $y_c = 0.9\,\mathrm{m}$.
- Form the constrained equations of motion according to (15.26)
- Make an m-file which implements these constrained equations of motion as an ODE, something like:
`function dy=daehwa7(t,y).`
- Write an m-file which does the coordinate projection method on the coordinates and the speeds (the state vector `y`), something like:
`function y = projhwa7(y).`
- Determine the initial conditions `y0` at `t0=0`, and use your coordinate projection method to make sure that these initial coordinates AND initial speeds fulfill the constraints.
- Now use a fixed step rk4 method to numerically integrate your ODE. We use a fixed step odesolver so you can do a coordinate projection after every numerical integration step. When you use a Matlab ode solver it will be a lot more difficult to implement that.
The algorithm is now something like this:

```
y=y0; t=t0; h=0.001;
while y(1)<4*pi do
  [t,y]=rk4step(@daehwa7,t,y,h);
  y = projhwa7(y);
  store your results for plotting later on
end while
```

b. Figure G.37 shows the angular speed of crank 2, rocker 4 and connecting bar 5.

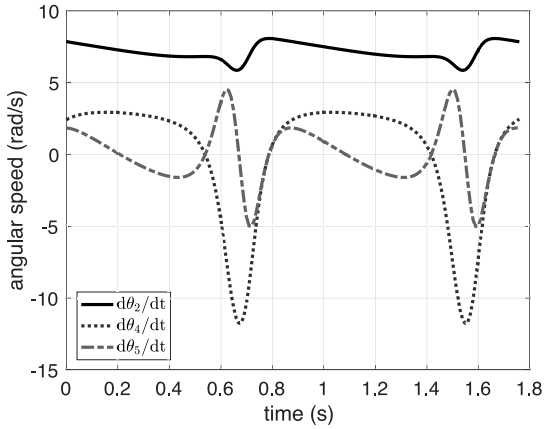


Figure G.37: The angular speed of crank 2, rocker 4 and connecting bar 5.

c. Figure G.38 shows the sliding speed of slider 3 with respect to rocker 4.

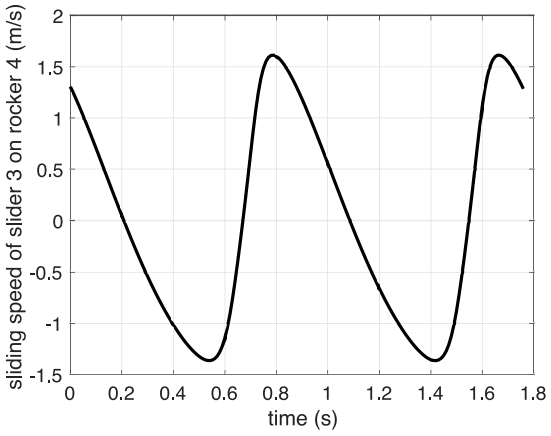


Figure G.38: The sliding speed of slider 3 with respect to rocker 4.

Figure G.39 shows the horizontal position, speed and acceleration of slider 6.

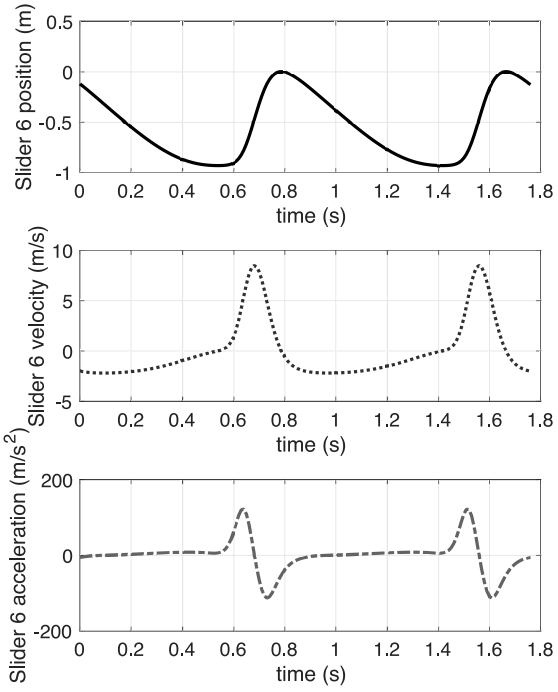


Figure G.39: The horizontal position, speed and acceleration of slider 6.

d. The normal force exerted by the slider 3 on the rocker 4 and the normal force exerted by slider 6 on the ground, see Figure G.40. Note that we have not defined + or - for the normal forces, so your plots can be mirrored, which is ok.

e. Clearly the motion is periodic. You can find the period T in seconds by interpolation of the angle θ_2 of rocker 2, when it crosses the 2π value from one numerical integration step to the next one. Let us use for simplicity a zero crossing variable $y = \theta_2 - 2\pi$ and the value just before crossing 0 is y_0 and after is y_1, such that $y_0 < 0$ and $y_1 > 0$. To find the time of zero crossing of y we use linear interpolation, and the time of zero crossing is then $T = (y_1 t_0 - y_0 t_1)/(y_1 - y_0)$. Using an rk4 scheme and a fixed step size of $h = 0.01\,\mathrm{s}$, we found $T = 0.878\,000\,\mathrm{s}$. To check the accuracy, reducing the step size by a factor 2, we found that $T = 0.877\,998\,\mathrm{s}$, so we conclude that with at least three significant digits, the period is $T = 0.878\,\mathrm{s}$.

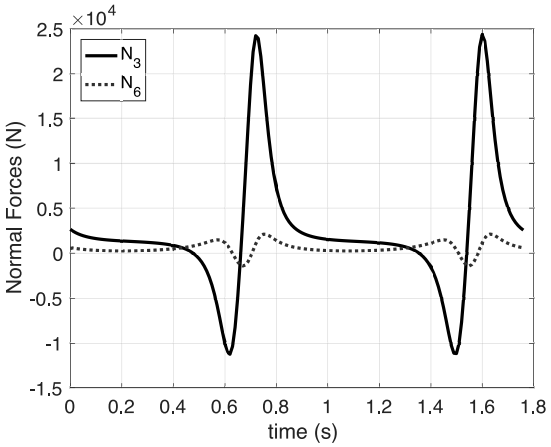


Figure G.40: The normal force exerted by the slider 3 on the rocker 4, and the normal force exerted by slider 6 on the ground. Note that + or - for the normal forces is not defined, so your plots can be mirrored, which is ok.

16.3 We formulate the equations of motion for a holonomic and non-holonomic constrained system in DAE form, the ezyroller. Numerical integration of the EOM of this ezyroller model to analyse the motions and to try and understand how the propulsion by oscillating steering torques works.

a. The equations of motion for this system in DAE form are presented in (16.29). With the CoM coordinates $x_i = (x_1, y_1, \varphi_1, x_2, y_2, \varphi_2)$, the mass matrix becomes $M_{ij} = \mathrm{diag}(m_1, m_1, I_1, m_2, m_2, I_2)$. We start by writing down the holonomic and the non-holonomic constraints and their derivatives.
Body 1 and body 2 are connected by a revolute joint in B, resulting in the holonomic constraints:

$$\begin{aligned} C_1 &= (x_2 - d\cos(\varphi_2)) - (x_1 + b\cos(\varphi_1)) = 0, \\ C_2 &= (y_2 - d\sin(\varphi_2)) - (y_1 + b\sin(\varphi_1)) = 0. \end{aligned}$$

Then the Jacobian matrix is,

$$C_{k,i} = \begin{pmatrix} -1 & 0 & b\sin(\varphi_1) & 1 & 0 & d\sin(\varphi_2) \\ 0 & -1 & -b\cos(\varphi_1) & 0 & 1 & -d\cos(\varphi_2) \end{pmatrix},$$

and the convective terms are,

$$h_k = C_{k,ij}\dot{x}_i\dot{x}_j = \begin{pmatrix} b\cos(\varphi_1)\dot{\varphi}_1^2 + d\cos(\varphi_2)\dot{\varphi}_2^2 \\ b\sin(\varphi_1)\dot{\varphi}_1^2 + d\sin(\varphi_2)\dot{\varphi}_2^2 \end{pmatrix}.$$

No side-slip conditions in body 1 at A and in body 2 at C, results in the non-holonomic constraints:

$$\begin{aligned} S_1 &= -\dot{x}_1 \sin(\varphi_1) + \dot{y}_1 \cos(\varphi_1) - a\dot{\varphi}_1 = 0, \\ S_2 &= -\dot{x}_2 \sin(\varphi_2) + \dot{y}_2 \cos(\varphi_2) + c\dot{\varphi}_2 = 0. \end{aligned}$$

Then the Jacobian matrix is,

$$S_{ki} = \begin{pmatrix} -\sin(\varphi_1) & \cos(\varphi_1) & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin(\varphi_2) & \cos(\varphi_2) & c \end{pmatrix},$$

and the convective terms are,

$$h_k = S_{ki,j}\dot{x}_i\dot{x}_j = \begin{pmatrix} -\dot{x}_1\dot{\varphi}_1 \cos(\varphi_1) - \dot{y}_1\dot{\varphi}_1 \sin(\varphi_1) \\ -\dot{x}_2\dot{\varphi}_2 \cos(\varphi_2) - \dot{y}_2\dot{\varphi}_2 \sin(\varphi_2) \end{pmatrix}.$$

Finally, the applied forces on the right-hand side are,

$$F_i = (f_{x1}, f_{y1}, M_1, f_{x2}, f_{y2}, M_2),$$

and their specific values depend on the problem at hand.

b. The algorithm for the coordinate projection method on the coordinates is something like this,

```
function y = CoordProjCart(y)
MAXITERAT = 10; TOL = 1e-12;
iterat = 0;
[Ck,Cki,hk] = HConCart(y);
while max(abs(Ck))>TOL
   y(1:6) = y(1:6) + pinv(Cki)*(-Ck);
   [Ck,Cki,hk] = HConCart(y);
   iterat = iterat + 1;
   iterat > MAXITERAT
      error('maximum number of iteration (10)');
   end
end
end
```

The coordinate project method on the velocities is just a one step thing, because the velocity constraints are linear in the velocities. Note that we also have to fulfill the constraints on the velocities which come from the holonomic constraints. The algorithm is something like this,

```
function y = SpeedProjCart(y)
[Ck,Cki,hck] = HConCart(y);
%time derivative of the holonomic constraints
dCkdt = Cki*y(7:12);
[sk,Ski,hsk] = NHConCart(y);
```

```
err = [dCkdt; sk];
Jki = [Cki; Ski];
y(7:12) = y(7:12) + pinv(Jki)*(-err);
end
```

c. The MATLAB^{TM}-implementation of the DAE, with the state vector defined as $\mathrm{y} = (x_i, \dot{x}_i)$, could be something like this:

```
function dy = DAECart(t,y)
% DAE for EzyRoller.
% y = [x1 y1 p1 x2 y2 p2 dx1 dy1 dp1 dx2 dy2 dp2].'

%     Arend L. Schwab 23-Apr-2016
%     Copyright (c) 2016 by TU-Delft, the Netherlands.
global A_ B_ C_ D_ M1_ I1_ M2_ I2_ FC_

M = diag([M1_ M1_ I1_ M2_ M2_ I2_]);
[Ck,Cki,hck] = HConCart(y);
[sk,Ski,hsk] = NHConCart(y);
C = [Cki; Ski];
h = [hck; hsk];
A = [ M   C.'
      C zeros(4)];
% determine the applied forces at time t
f = AppliedForces(t,y);
rhs = [f; -h];
lhs = A\rhs;
dy = zeros(12,1);
dy(1:6) = y(7:12);
dy(7:12) = lhs(1:6);
% store the constraint forces in a global vairable FC_
FC_ = lhs(7:10);
end
```

And the MATLAB^{TM}-implementation of the holonomic and the non-holonomic constraint functions could be something like this:

```
function [Ck,Cki,hk] = HConCart(y)
%function [Ck,Cki,hk] = HConCart(y)
% holonomic constraints Ck,
% jacobian Cki and convective terms hk
global A_ B_ C_ D_ M1_ I1_ M2_ I2_
% unpack the coordinates xi and the speeds dxi
x1 = y(1); y1 = y(2); p1 = y(3);
x2 = y(4); y2 = y(5); p2 = y(6);
dx1 = y(7); dy1 = y(8); dp1 = y(9);
```

```
dx2 = y(10); dy2 = y(11); dp2 = y(12);
% constraints Ck
Ck = [ x2 - x1 - B_*cos(p1) - D_*cos(p2)
       y2 - y1 - B_*sin(p1) - D_*sin(p2)];
% jacobian Cki
Cki = [ -1,  0,  B_*sin(p1), 1, 0,  D_*sin(p2)
          0, -1, -B_*cos(p1), 0, 1, -D_*cos(p2)];
% convective terms hk
hk = [B_*cos(p1)*dp1^2 + D_*cos(p2)*dp2^2
       B_*sin(p1)*dp1^2 + D_*sin(p2)*dp2^2];
end

function [sk,Ski,hk] = NHConCart(y)
%function [sk,Ski,hk] = NHConCart(y)
% nonholonomic constraints sk,
% jacobian Ski and convective terms hk
global A_ B_ C_ D_ M1_ I1_ M2_ I2_
% unpack the coordinates xi and the speeds dxi
x1 = y(1); y1 = y(2); p1 = y(3);
x2 = y(4); y2 = y(5); p2 = y(6);
dx1 = y(7); dy1 = y(8); dp1 = y(9);
dx2 = y(10); dy2 = y(11); dp2 = y(12);
% constraints sk
sk = [ dy1*cos(p1) - A_*dp1 - dx1*sin(p1)
       C_*dp2 + dy2*cos(p2) - dx2*sin(p2)];
% jacobian Ski
Ski = [ -sin(p1), cos(p1), -A_,        0,       0,  0
              0,       0,   0, -sin(p2), cos(p2), C_];
 % convective terms hk
hk = [ -dp1*(dx1*cos(p1) + dy1*sin(p1))
        -dp2*(dx2*cos(p2) + dy2*sin(p2))];
end
```

The MATLAB^{TM}-implementation of the constraint stabilization can be something like this:

```
function y = CoordProjCart(y)
% Coordinate Projection Method on EzyRoller.
% y = [x1 y1 p1 x2 y2 p2 dx1 dy1 dp1 dx2 dy2 dp2].'
% set the maximum number of iterations
% and the tolerance on the constraints
MAXITERAT = 10; TOL = 1e-12;

iterat = 0;
[Ck,Cki,hk] = HConCart(y);
% force allways one iteration?
```

```
while max(abs(Ck))>TOL
  y(1:6) = y(1:6) + pinv(Cki)*(-Ck);
  [Ck,Cki,hk] = HConCart(y);
  iterat = iterat+1;
  if iterat>MAXITERAT
    error('maximum number of iteration (10)')
  end
end
end

function y = SpeedProjCart(y)
% Coordinate Projection Method for velocities.
% y = [x1 y1 p1 x2 y2 p2 dx1 dy1 dp1 dx2 dy2 dp2].'
v = y(7:12);
[Ck,Cki,hk] = HConCart(y);
[sk,Ski,hk] = NHConCart(y);
Jki = [Cki; Ski];
slip = [Cki*v; sk];
v = v + pinv(Jki)*(-slip);
y(7:12) = v;
end
```

d. Demonstrate that the unpowered model of the EzyRoller is moving properly. The initial velocities on body 1 are $(\dot{x}_1, \dot{y}_1, \dot{\varphi}_1) = (1, 0, 0)$, which fulfill the non-holonomic constraint on body 1 in point A. And since we only have two degrees of freedom in the velocity space for this system, the initial velocities on body 2 are fully determined by the constraints. The velocity of point B, which is the mutual point of body 1 and 2, is $(\dot{x}_B, \dot{y}_B) = (1, 0)$m/s. And with the skate constraint in C, point C is momentarily at rest (zero speed), thus the center of rotation of body 2. The angular velocity of body 2 is now $\dot{\varphi}_2 = 1/(c + d) = 5\,\text{rad/s}$, and then the velocity of the CoM is $(\dot{x}_2, \dot{y}_2) = (c\dot{\varphi}_2, 0) = (0.5, 0)$m/s. The initial state vector is now `y0=`$(0.2, 0, 0, 0.8, 0.1, 1.5708, 1, 0, 0, 0.5, 0, 5)$.
In Figure G.41 the path of the CoM of body 1 and 2 are shown for a simulation period of 1 second. We clearly see that body 1 is moving forward and that body 2, the steering assembly, is aligning with body 1. In this aligning the system is steering a little bit away from the initial heading (x-axis). The distance traveled in the x-direction is a little bit less than 1 m; that is ok, because some kinetic energy from body 1 should be transferred to body 2 to make it move with the same forward speed. By checking the constraint forces in revolute joint B you can see that body 1 is pushing body 2 until they have the same speed.

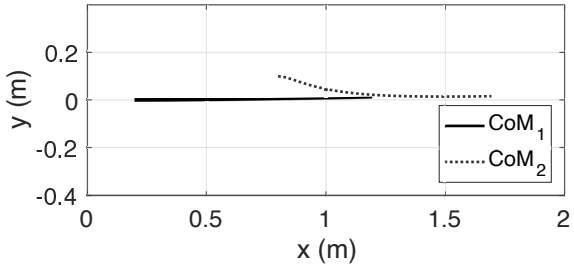


Figure G.41: Path of the CoM of body 1 and 2 for a simulation period of 1 second.

From here we analyse the motion of the ezyroller for 60 seconds starting from rest with a periodic forcing on the steering system.

f. Add an action-reaction torque $M(t) = M_0 \cos(\omega t)$ in revolute joint at B. The applied foreces are now $F_i = (0, 0, -M(t), 0, 0, M(t))$. The path of point A and C are shown in Figure G.42. We notice that the system is slowing gaining forward speed and that the heading chances during the first few seconds from 0 to about 60 degrees, which is of course due to the direction of the push-off at the wheels, which is lateral. When the system has gained some forward speed we see the path of the wheel contact points C to oscillate, just like the push-off in speed skating. And since body 2 oscillates, body 1 also does.

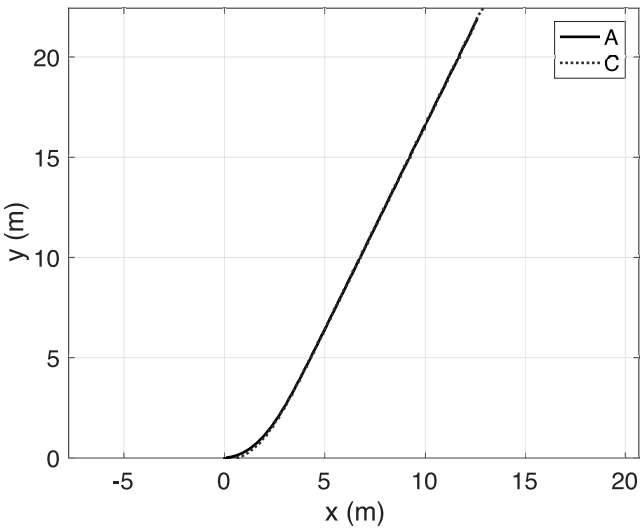


Figure G.42: Path of point A and C for a simulation period of 60 second with periodic forcing.

g. The linear and angular velocity of body 1 is shown in Figure G.43. We see that the system is gradually picking up forward speed, at first somewhat oscillating. The angular velocity of body 1 oscillates nicely with the same frequency as the forcing. The amplitude is the results of the amplitude of the applied oscillating torque, the combined rotational inertia at revolute joint B, and the frequency of the oscillation, a rough estimate would be $\frac{(a+b)-(c+d)}{(c+d)}\frac{M_0}{(I_1+I_2)}\frac{1}{\omega_0}$, which for this results in an amplitude of 0.19 rad/s, which seems fairly ok when looking at the figure. The oscillation of ω_1 is after some transient nicely around zero, which means that the heading is then constant.

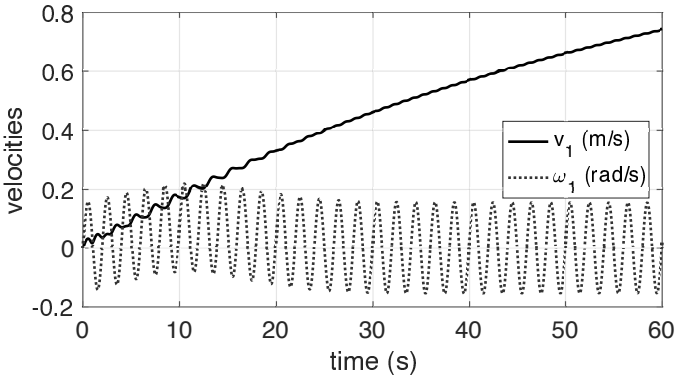


Figure G.43: Linear and angular velocity of body 1 as a function of time.

h. The Kinetic energy of the system is $T = \frac{1}{2}\dot{x}_i M_{ij} \dot{x}_j$, and is plotted in Figure G.44. We see that after some transient behavior the kinetic energy grows linearly with time. This is expected with a periodic forcing and a periodic relative angular velocity in the revolute joint. The kinetic energy is the integral of the product of the two with time, and therefore will grow linearly. This means that the forward velocity will grow proportional to the square root of time as we can see in Figure G.43. Notice the small oscillation in the kinetic energy, this is of course due to the kinetic energy in the oscillating orientation of the two bodies.

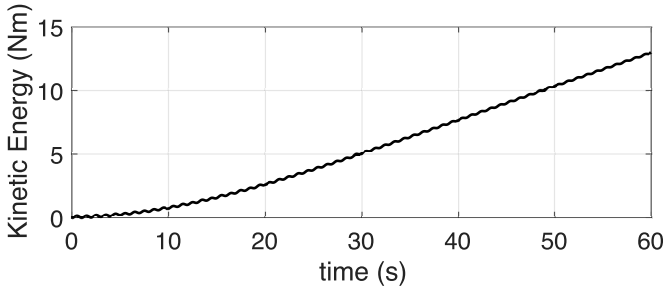


Figure G.44: Kinetic energy of the system as a function of time.

i. The applied torque M and the relative angular velocity $\omega = \dot{\varphi}_2 - \dot{\varphi}_1$, in revolute joint B between body 2 and body 1, are plotted in Figure G.45. The relative angular velocity ω has the same frequency as the applied torque. At the start of the motion they are around 90 deg out of phase but after some time they get into phase and the amplitude of the relative angular velocity settles to somewhat smaller value than at the start of the motion.

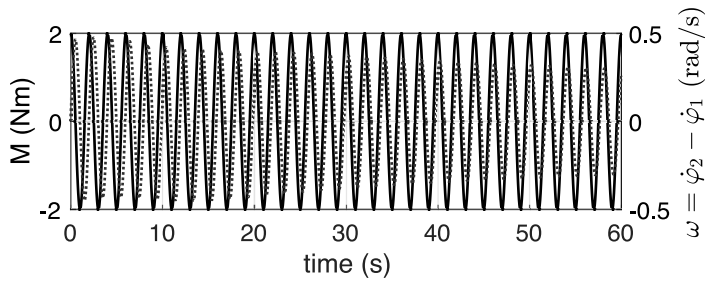


Figure G.45: Applied torque M and relative angular velocity ω in revolute joint B.

j. The work done by the applied action-reaction torque M in revolute joint B is by definition $W = \int M\omega dt$, with the relative angular velocity $\omega = \dot{\varphi}_2 - \dot{\varphi}_1$ in joint B. This work as a function time together with the kinetic energy of the system are plotted in Figure G.46. We clearly see that one is the opposite of the other, or in other words that amount of work applied to the system is equal to the kinetic energy of the system. Of course, the system is non-dissipative and therefore energy conservative.

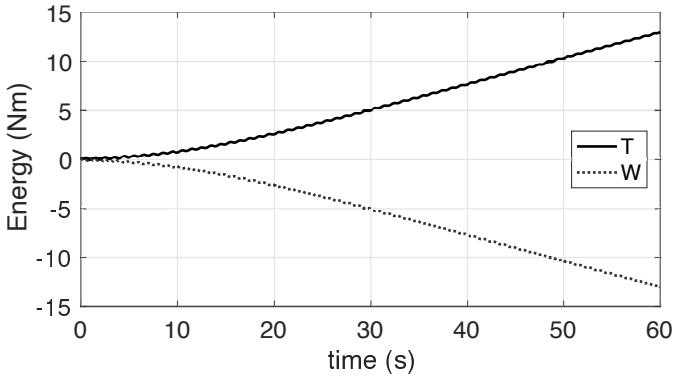


Figure G.46: Work W done by the applied torque M in B together with the kinetic energy T of the system as a function of time.

17.4 We formulate the equations of motion for a single rigid body, the T-shaped tool from the Salyut-7, in space. Then, we simulate its torque-free motion and interpret the results.

a. The CoM position, $\boldsymbol{r}_C$, can be determined from the first moment of mass equation, $\sum m_i \boldsymbol{r}_i = \sum m_i \boldsymbol{r}_C$ (Section 1.2.4). Clearly the CoM is in the xy-plane, and given the symmetry on the x-axis, the distance is $d = 1/150\,\mathrm{m} \approx 0.006\,666\,7\,\mathrm{m}$.

b. The mass moments of inertia for the system result from the sum of parts, where we use the parallel-axis theorem (Section 5.2.1) to shift the mass moments of inertia of the individual parts from their CoM to the system's CoM. For body 1, we obtain

$$\mathbf{I1}_{cm} = \begin{pmatrix} \frac{1}{12}m_1 l_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12}m_1 l_1^2 \end{pmatrix} + m_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & d^2 & 0 \\ 0 & 0 & d^2 \end{pmatrix},$$

and for body 2,

$$\mathbf{I2}_{cm} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{m_2 l_2^2}{12} & 0 \\ 0 & 0 & \frac{m_2 l_2^2}{12} \end{pmatrix} + m_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & (\frac{1}{2}l_2 - d)^2 & 0 \\ 0 & 0 & (\frac{1}{2}l_2 - d)^2 \end{pmatrix}$$

Substituting the values of the parameters and adding the contributions gives,

$$\mathbf{I}_{cm} = \mathbf{I1}_{cm} + \mathbf{I2}_{cm} = \begin{pmatrix} 6.4 & 0 & 0 \\ 0 & 2.4 & 0 \\ 0 & 0 & 8.8 \end{pmatrix} 10^{-6}\,\mathrm{kg\,m^2}.$$

c. The equations of motion for the angular velocity ${}^{\mathcal{B}}\boldsymbol{\omega}$ are the Euler equations of motion for a rigid body with zero applied torque, which results in,

$${}^{\mathcal{B}}\dot{\boldsymbol{\omega}} = -\mathbf{I}_{cm}^{-1}\left({}^{\mathcal{B}}\boldsymbol{\omega} \times (\mathbf{I}_{cm}\,{}^{\mathcal{B}}\boldsymbol{\omega})\right)$$

One can program this directly in Matlab with vector-matrix expressions, irrespective of the specific values and zeros in the mass moments of inertia matrix. One could make compuations more efficient by noting that the inertia matrix has a special structure, because the rigid body has no extension in the z-direction, therefore $I_{zz} = I_{xx} + I_{yy}$, which follows directly from (5.13). Using these specific properties and expanding the equations of motion results in,

$$\dot{\omega}_x = -\omega_y\omega_z, \tag{G.311}$$
$$\dot{\omega}_y = \omega_z\omega_x, \tag{G.312}$$
$$\dot{\omega}_z = (I_{xx} - I_{yy})/(I_{xx} + I_{yy})\omega_x\omega_y, \tag{G.313}$$

where all ω-components are in the $\mathcal{B}$-triad.

d. For numerical integration in Matlab, we used `ode45` with the standard error tolerance settings. Figure G.47 shows the plots of the three components of the angular velocity vector, $\omega_x, \omega_y, \omega_z$, together with the magnitude of the angular velocity vector, $|{}^{\mathcal{B}}\boldsymbol{\omega}|$, as expressed in the body-fixed triad $\mathcal{B}$.

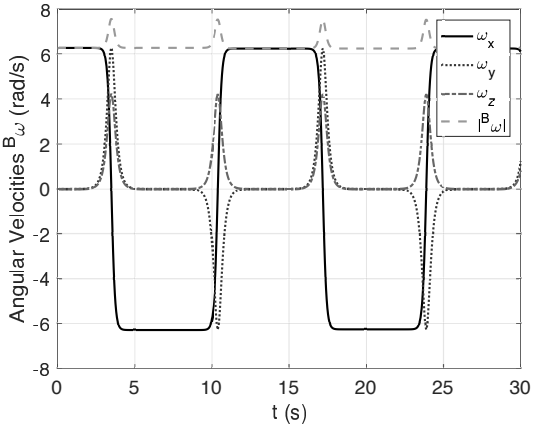


Figure G.47: Angular velocities ${}^{\mathcal{B}}\boldsymbol{\omega}$ as a function of time.

We see that initially, the body is rotating mainly about the x-axis, but after about 2.5 seconds there is a sudden change, and within a second, the rotational speed about the x-axis flips from $+2\pi$ to -2π. So within a second, the rotational speed is reversed! This pattern repeats roughly every seven seconds. This behavior fully complies with the results from Example 17.1, where we have shown that rotation about the axis which has the intermediate mass moment of inertia is unstable. Note that the magnitude of the angular velocity is not constant but rises somewhat during the transition.

This instability can also be understood by just looking at the equations of motion (G.311)-(G.313). From these equations, we see that with only *one* non-zero angular speed about one of the three axes, either x, y or z, all angular accelerations are zero, and there is steady motion. However, adding a small perturbation about any of the other axes leads to non-zero angular accelerations. To be specific, starting with $\omega_x = \omega_0$ and $\omega_y = 0$ and $\omega_z = \epsilon$ (a small number), the system has a non-zero $\dot{\omega}_y = \epsilon\omega_0$. So, ω_y will grow and lead to a non-zero $\dot{\omega}_z$ of the order ϵ and to an ω_z of the order ϵ, and then to a decrease in ω_x according to a now non-zero $\dot{\omega}_x$, etc. etc. This is how the unstable motion initiates.

e. The plot of the three components H_x, H_y, H_z of the angular momentum vector, together with the magnitude of the angular momentum vector, $|{}^{\mathcal{B}}\boldsymbol{H}|$, as expressed in the body-fixed triad $\mathcal{B}$, as a function of time is shown in Figure G.48.

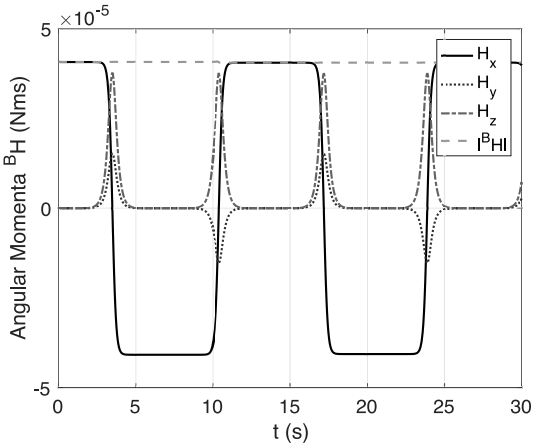


Figure G.48: Angular momenta ${}^{\mathcal{B}}\boldsymbol{H}$ as a function of time.

These closely resemble the angular velocities, which is of course correct, because they are these angular velocities scaled by the mass moments of inertia, as in $\boldsymbol{H} = \mathbf{I}\boldsymbol{\omega}$. One might be surprised to see that the magnitude of the angular momentum vector looks constant over time. However, this complies fully with Euler's law written in terms of angular momenta with respect to an inertial triad: the sum of the applied torques is equal to the change in angular momenta with respect to time, $\sum \boldsymbol{M} = \dot{\boldsymbol{H}}$. And thus, in the inertial triad, the angular momentum vector is a constant vector. We calculate everything in a moving triad, so the vector components are not constant. Still, the magnitude of the angular momentum vector does not change when we express this vector in another triad, and therefore the magnitude of the angular momentum vector here is also constant.

f. The kinetic energy of the body is given by $E_{\text{kin}} = \frac{1}{2}\,{}^{\mathcal{B}}\boldsymbol{\omega}^{\mathrm{T}}\mathbf{I}_{cm}\,{}^{\mathcal{B}}\boldsymbol{\omega}$ and plotted in Figure G.49.
We see that the kinetic energy is almost constant over time. This complies with the theory, because in the absence of applied torques and dissipation, this is an energy-conservative system. We do see a small decrease around $t = 10$ seconds, this has to do with the sudden change of rotation and is an effect of the finite accuracy of the numerical integration. Decreasing the Matlab `ode` error tolerances `RelTol` and `AbsTol` to `1e-6` reduces the maximum deviation from the constant value of `1.2802e-04` to the order of `1e-10`, and the resulting plot is an almost straight line.

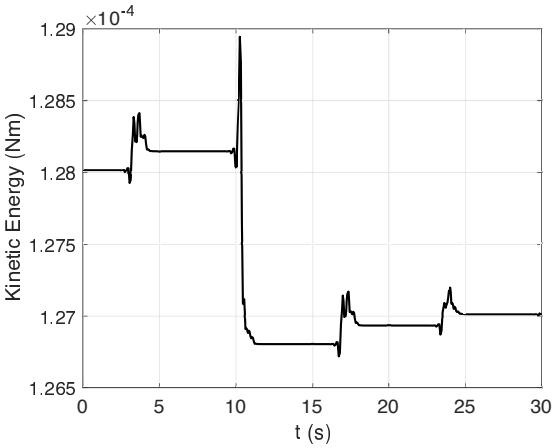


Figure G.49: Kinetic energy E_{kin} as a function of time.

g. Figure G.50 shows the absolute values of the angular velocity component about the body-fixed z-axis, $|\omega_z|$, as a function of time in a semi-logarithmic plot.

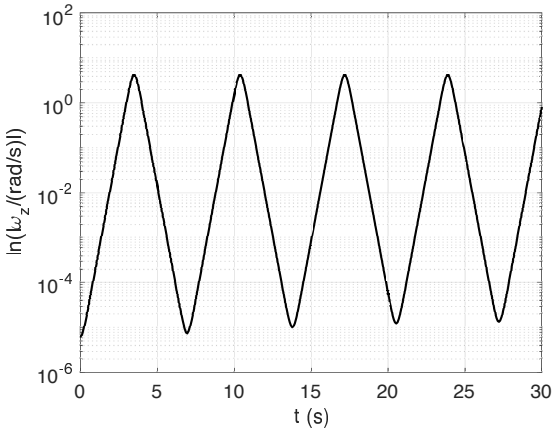


Figure G.50: Absolute value of the angular velocities $|\omega_z|$ as a function of time, with a logarithmic scale for the y-axis.

The logarithmic rise and decay of ω_z is a line with approximately constant slope, as in $\log|\omega_z/(\text{rad/s})| \approx c_1 + c_2 t$. From the graph, we estimate the slope constant c_2 to be $\pm 6/3.257\,1/\text{s} = \pm 1.8422\,1/\text{s}$ and that the exponential behavior is something like $\omega_z/(\text{rad/s}) = c_3 10^{\pm 1.8422 t/\text{s}}$. However, exponential behavior is usually written in terms of $\mathrm{e}^{\lambda t}$ and not in powers of 10. Using such a natural logarithm instead, the exponent lambda becomes $\lambda = \pm 1.8422\,1/\text{s}\ln 10 = \pm 4.2418\,1/\text{s}$. (Note: It is

confusing that Matlab uses the name `log` for the natural or e-based logarithm function which we usually write as `ln`. The name `log` is usually reserved for 10-based logarithms). We see that the small perturbation in the angular velocity grows exponentially with time. After it reaches its maximum value, in the midst of the flip, it starts decaying again in the same manner. This pattern repeats periodically.

h. The stability analysis in Example 17.1 showed that the eigenvalues of the small perturbed motions are $\lambda = \pm\omega_0\sqrt{(I_{xx} - I_{yy})(I_{zz} - I_{xx})/(I_{yy}I_{zz})}$, where ω_0 is the initial angular velocity about the intermediate axis, which is here the x-axis. Substitution of our parameters results in $\lambda = \pm 4.2361\,1/\text{s}$. The estimated values from the time series analysis, as found in the item above, compare very well with the model-based calculated values here.

18.4 We investigate the rotational motion of a torque-free satellite in deep space.

a. The rotation matrix ${}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}$ is formed by the triad direction vectors expressed in the space-fixed triad,

$$ {}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}} = \begin{pmatrix} {}^{\mathcal{N}}\hat{\boldsymbol{b}}_1 & {}^{\mathcal{N}}\hat{\boldsymbol{b}}_2 & {}^{\mathcal{N}}\hat{\boldsymbol{b}}_3 \end{pmatrix} = \begin{pmatrix} 0.7680 & -0.4240 & -0.4800 \\ 0.0240 & 0.7680 & -0.6400 \\ 0.6400 & 0.4800 & 0.6000 \end{pmatrix}. $$

b. To determine the associated Euler angles, we use the rotation matrix expanded into sine and cosine of the three Euler angles, see (18.33). We will start with entry R(3,3), from we which we determine θ. Next we determine φ from the combination of entries R(1,3) and R(2,3), and likewise we determine ψ from entries R(3,1) and R(3,2). Thus, the algorithm is:

```
theta = acos(R(3,3))
phi = atan2(R(1,3),-R(2,3))
psi = atan2(R(3,1),R(3,2))
```

The determination of θ from $\cos\theta$ would actually be two-valued on a complete interval length of 2π; or in other words, $\theta = \pm\arccos(R(3,3))$. Here we choose to use the positive value for θ. Still, the algorithm works perfectly well because we use the four-quadrant arctangent function `atan2(y,x)`, which takes into account the different sign of the multiplier $\sin\theta$. For the rotation matrix from item (a) we then get the following Euler angles,

$$ (\varphi, \theta, \psi) = (-0.6435, 0.9273, 0.9273)\,\text{rad}. $$

c. The angular velocity components in the body-fixed triad $\mathcal{B}$ follow directly from the coordinate transformation ${}^{\mathcal{B}}\boldsymbol{\omega} = {}^{\mathcal{B}}\mathbf{R}_{\mathcal{N}}\,{}^{\mathcal{N}}\boldsymbol{\omega}$, where the

inverse rotation matrix is the transpose from item (a), ${}^{\mathcal{B}}\mathbf{R}_{\mathcal{N}} = ({}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}})^{\mathrm{T}}$. This results in,

$$ {}^{\mathcal{B}}\boldsymbol{\omega} = (10, 0, 0.001)\,\mathrm{rad/s}. $$

d. The rate of change of the Euler angles can best be determined from $\dot{\boldsymbol{a}} = \boldsymbol{B}^{-1}\,{}^{\mathcal{B}}\boldsymbol{\omega}$, see (18.64). One can either take the matrix $\boldsymbol{B}$ from (18.61) and do the inverse numerically, or use (18.64).

e. The inertia tensor at the center of mass for this rectangular box, expressed in the body-fixed triad $\mathcal{B}$, is simply a diagonal matrix, because of the symmetries,

$$ {}^{\mathcal{B}}\boldsymbol{I}_C = \frac{1}{12}m \begin{pmatrix} l_y^2 + l_z^2 & 0 & 0 \\ 0 & l_z^2 + l_x^2 & 0 \\ 0 & 0 & l_x^2 + l_y^2 \end{pmatrix} = \begin{pmatrix} 7.65 & 0 & 0 \\ 0 & 1.25 & 0 \\ 0 & 0 & 8.00 \end{pmatrix}. $$

f. The Euler equations of motion expressed in the body-fixed triad $\mathcal{B}$ are given in (17.27) on page 423. The state of the system is defined by the Euler angles $\boldsymbol{a} = (\varphi, \theta, \psi)$ and the angular velocity expressed in the body-fixed triad, resulting in $\boldsymbol{y} = (\boldsymbol{a}, {}^{\mathcal{B}}\boldsymbol{\omega})$. The state equations describe the rate of change of the state variables, in general terms, $\dot{\boldsymbol{y}} = \boldsymbol{f}(t, \boldsymbol{y})$, which here breaks down into the decoupled systems of equations,

$$ \dot{\boldsymbol{a}} = \boldsymbol{B}^{-1}\,{}^{\mathcal{B}}\boldsymbol{\omega}, \quad \text{and} \quad {}^{\mathcal{B}}\dot{\boldsymbol{\omega}} = -({}^{\mathcal{B}}\boldsymbol{I}_C)^{-1}({}^{\mathcal{B}}\boldsymbol{\omega} \times ({}^{\mathcal{B}}\boldsymbol{I}_C\,{}^{\mathcal{B}}\boldsymbol{\omega})). $$

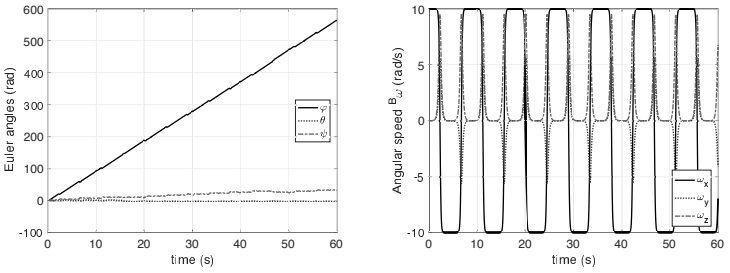
Exemplary MATLAB$^{\mathrm{TM}}$ code for this is the following,

```
function dy=odehw9ea(t,y)
global I
phi = y(1); theta = y(2);psi = y(3);
w = y(4:6);
B=[sin(psi)*sin(theta)  cos(psi) 0
   cos(psi)*sin(theta) -sin(psi) 0
         cos(theta)        0     1];
udot = B\w;
wdot = I\(-cross(w,I*w));
dy = y;
dy(1:3) = udot(1:3);
dy(4:6) = wdot(1:3);
end
```

g. To determine the motion of the satellite, we numerically integrate the state equations; using the MATLAB$^{\mathrm{TM}}$`ode45` function, the program could look like this,

```
...
tend = 60;
y0=[phi; theta; psi; wlocal];
[T,Y] = ode45(@odehw9ea,[0 tend],y0);
...
```

The motion of the satellite as a function of time is show in Figure G.51.



(a) (b)

Figure G.51: Euler angles (left) and angular velocities (right) as a function of time.

From the Euler angles, we see that φ and ψ increase linearly over time, whereas the middle angle θ oscillates around zero. Looking at these Euler angles as functions of time, it is not really clear what the motion is. However, if we look at the angular velocity components in the body-fixed triad $\mathcal{B}$, we see that the initially large angular rate about the x-axis is constant for a few seconds and then suddenly flips from $+10$ to $-10\,\text{rad/s}$. During this flip, the angular rates about the other two axes momentarily become large. The reason for this unstable rotary motion is that we started spinning with a large angular velocity about the axis with the *intermediate* mass moment of inertia, see Example 17.1.
Note: These unstable motions behave in time like exponential growth, $x = x_0 e^{\lambda t}$, where the eigenvalue of this unstable motion can be calculated from (17.44E)). For our problem, $\lambda \approx 4.71/\text{s}$, and the time it will take for such an initially small motion to double in time is $t = \ln(2)/\lambda$, which is $\approx 1.5\,\text{s}$. Zooming in on the plot on the right where the velocity starts to flip, we see that this seems correct (check this!).

h. The time history of the Euler angles θ and ψ are shown in Figure G.52 (left), whereas the 3-D path of point P is shown in the plot on the right.

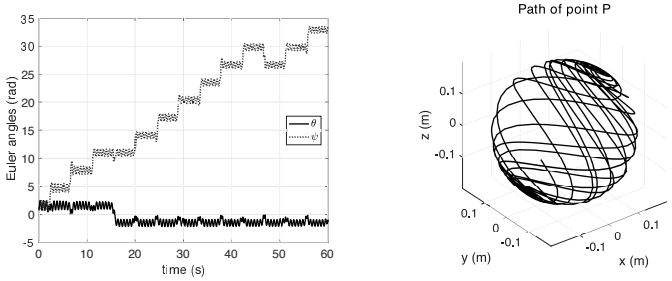


Figure G.52: Two Euler angles as functions of time (left) and 3-D-path of point P (right).

The coordinates of point P in the inertial space are found by transform-

ing the body-fixed coordinates ${}^{\mathcal{B}}\boldsymbol{p} = (l_x/2, 0, 0)$, which are constant in time, to the space-fixed triad, as in ${}^{\mathcal{N}}\boldsymbol{p} = {}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}\,{}^{\mathcal{B}}\boldsymbol{p}$, for every instant in time.

From the 3-D-path (Figure G.52 right) is it very clear that the satellite initially spins about the x-axis (dense painted path at the top), but that this spinning is about the unstable axis and that after some seconds it flips 180 degrees. This process repeats itself about every 4.5 seconds (see Figure G.52 on the left). Although this flipping repeats itself at these regular intervals, the values of the two Euler angles θ and ψ show some irregular behaviour. Around $t = 15\,\mathrm{s}$, the angle θ suddenly becomes negative and ψ stays on the same level. Then around $t = 47\,\mathrm{s}$, the angle ψ suddenly jumps down instead of up, whereas θ remains negative. These erratic flips all happen around $\theta = 0$ and clearly are caused by the singularity of the kinematic mapping between the angular velocities and the rate of change of the Euler angles, $\dot{\boldsymbol{a}} = \boldsymbol{B}^{-1}\,{}^{\mathcal{B}}\boldsymbol{\omega}$, in combination with the finite accuracy of the numerical integration step. Changing the initial conditions a little bit, or the error tolerances on the numerical integration, will make these irregularities happen at other instances in time.

i. The angular velocity vector ${}^{\mathcal{N}}\boldsymbol{\omega}$ and the angular momentum vector ${}^{\mathcal{N}}\boldsymbol{H}_C$, both expressed in $\mathcal{N}$, are shown in Figure G.53. These angular momenta are calculated by transforming the body-fixed angular momenta, ${}^{\mathcal{B}}\boldsymbol{H}_C = {}^{\mathcal{B}}\boldsymbol{I}_C\,{}^{\mathcal{B}}\boldsymbol{\omega}$, to the space-fixed triad, as in ${}^{\mathcal{N}}\boldsymbol{H}_C = {}^{\mathcal{N}}\mathbf{R}_{\mathcal{B}}\,{}^{\mathcal{B}}\boldsymbol{H}_C$.

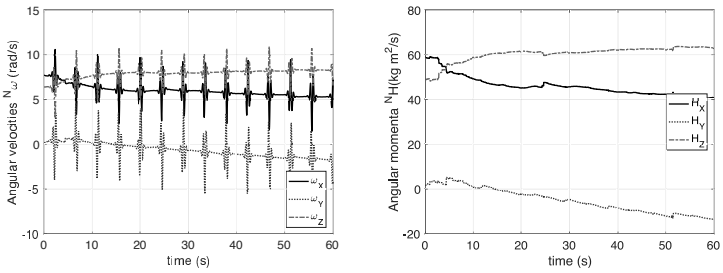


Figure G.53: Angular velocities (left) and angular momenta (right) expressed in the space-fixed triad $\mathcal{N}$ as a function of time, numerically integrated with `ode45` and the default error tolerances.

In the angular velocities, we clearly see again the flipping of the axis at regular intervals, and it looks as if the angular velocities expressed in the space-fixed triad always return to about the same values between this flipping. This complies with the angular velocity components in the body-fixed triad, as shown in Figure G.51(left), where we see that the large rotation alternates between the $+x$–axis and the $-x$–axis.

Besides the region around the flipping, it looks as if the angular velocities are about constant over time, possibly besides the y-component of ${}^{\mathcal{N}}\boldsymbol{\omega}$,

which shows a slow trend downwards.
For the angular momenta ${}^{\mathcal{N}}\boldsymbol{H}_C$, Figure G.53 (right), we also get the impression that they are approximately constant over time. Note also that the transient behavior around the flipping is not visible in the angular momenta. This may at first seem strange, because the angular momenta are the product of the inertia tensor times the angular velocities. However, remember that the body also changes orientation and that therefore the mass moments of inertia matrix expressed in the space-fixed triad also changes. In the product, these two effects cancel each other out, and make the angular momenta constant in time.
Indeed, according to Euler's second law of motion, $\sum \boldsymbol{M}_C = \dot{\boldsymbol{H}}_C$, and in the absence of applied torques, the inertial-triad angular momentum vector ${}^{\mathcal{N}}\boldsymbol{H}_C$ should be constant. However, we see in the plot that the components of ${}^{\mathcal{N}}\boldsymbol{H}_C$ are not really constant. This is likely a numerical error, and if we decrease the error tolerances on the numerical integration like in:

```
...
opts = odeset('abstol',1e-6,'reltol',1e-6);
[T,Y] = ode45(@odehw9ea,[0 tend],y0,opts);
...
```

we should see a better result, as shown Figure G.54.

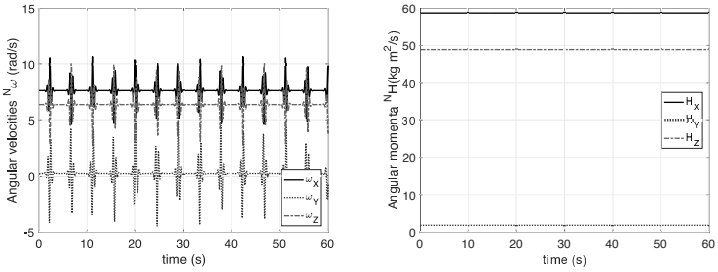


Figure G.54: Angular velocities (left) and angular momenta (right) in the space-fixed triad $\mathcal{N}$ as functions of time, numerically integrated with `ode45` but now with an error tolerance of 1e-6.

j. The system is energy-conservative; therefore, we expect the kinetic energy to be constant over time. The kinetic energy results from $E_\mathrm{k} = 1/2\, {}^{\mathcal{B}}\boldsymbol{\omega}^\mathrm{T}\, {}^{\mathcal{B}}\boldsymbol{I}_C\, {}^{\mathcal{B}}\boldsymbol{\omega}$ and is shown in Figure G.55.
Indeed, the kinetic energy is approximately constant over time. Notice the small jumps in the energy whenever the satellite flips, which can probably be traced back to the low accuracy of the kinematic speed mapping in the vicinity of $\theta = 0$, in combination with a modest error tolerance on the numerical integration step. Between the flips, the kinetic energy is fairly constant. The relative error in the kinetic energy, for this period of 60 s, is about 9.8e-4. If we decrease the error toler-

ances on the numerical integration from 1e-3 (default) to 1e-6, we obtain a relative kinetic energy error of 1.5e-6, which confirms our suspicion. Another set of three invariants are of course the one we already saw in item (i), the three inertial-based components of the angular momentum vector ${}^{\mathcal{N}}\boldsymbol{H}_C$.

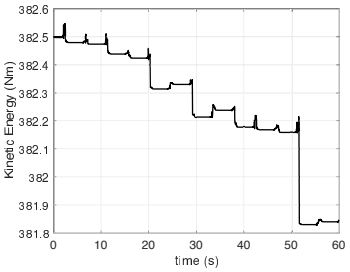


Figure G.55: Kinetic energy of the satellite as a function of time.

Follow-up question: With the four invariants as mentioned above, and the state of the system described by six variables, one could think that there are actually only six minus four is two degrees of freedom for this system. Or, in other words, the dynamics of this system can be described by two generalized coordinates. What are the generalized coordinates you could or would use?

18.5 Simple mechanical model of the human arm.

a. A sketch of the model using cans-in-series to depict the hinges, and marking out the origin O including the inertial triad, the elbow location E, the location of the hand H, and the locations of the CoMs, is shown in Figure G.56.

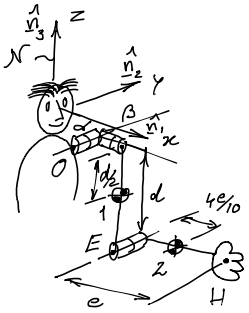


Figure G.56: Sketch of the model of the human arm.

b. The CoM coordinates of the two bodies in terms of the generalized

coordinates $x_i = T_i(q_j)$ are,

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} -d/2\sin(\alpha)\cos(\beta) \\ d/2\sin(\beta) \\ -d/2\cos(\alpha)\cos(\beta) \end{pmatrix} = \boldsymbol{T}_1(\boldsymbol{q}),$$

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} -d\sin(\alpha)\cos(\beta) + 4e/10(\cos(\alpha)\cos(\gamma) - \sin(\alpha)\cos(\beta)\sin(\gamma)) \\ d\sin(\beta) + 4e/10\sin(\beta)\sin(\gamma) \\ -d\cos(\alpha)\cos(\beta) - 4e/10(\sin(\alpha)\cos(\gamma) + \cos(\alpha)\cos(\beta)\sin(\gamma)) \end{pmatrix}.$$
$$= \boldsymbol{T}_2(\boldsymbol{q}).$$

c. To determine the static equilibrium torques we can apply virtual power. The virtual power of the applied CoM forces plus the virtual power from the generalized forces should be zero for arbitrary generalized velocities,

$$\delta P = F_i \delta \dot{x}_i + Q_j \delta \dot{q}_j = 0, \forall\, \delta \dot{q}_j \neq 0.$$

Substitution of the virtual velocities of the CoMs in terms of the virtual generalized velocities, $\delta \dot{x}_i = T_{i,j}(q_k)\delta \dot{q}_j$, and taking arbitrary variations results in,

$$Q_j = -T_{i,j} F_i.$$

Substitution of the initial configuration results in,

$$Q_j = \begin{pmatrix} M_\alpha \\ M_\beta \\ M_\gamma \end{pmatrix} = -\begin{pmatrix} -d/2 & 0 & 0 & -d & 0 & -4e/10 \\ 0 & d/2 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & -4e/10 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -m_1 g \\ 0 \\ 0 \\ -m_2 g \end{pmatrix}$$
$$= \begin{pmatrix} -4e/10 m_2 g & 0 & -4e/10 m_2 g \end{pmatrix}^{\mathrm{T}}$$

Substitution of the values of the system parameters then results in,

$$\begin{pmatrix} M_\alpha \\ M_\beta \\ M_\gamma \end{pmatrix} = \begin{pmatrix} -2.2073 \\ 0 \\ -2.2073 \end{pmatrix} \mathrm{N\,m}.$$

Clearly, if you draw this configuration with the applied forces and the joint reaction forces then you can easily check that the system is in static equilibrium, see the pencil and paper calculation in Figure G.57

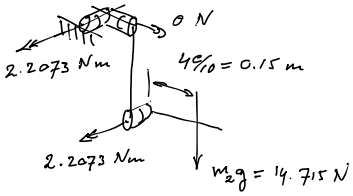


Figure G.57: Static equilibrium, pencil and paper calculation.

d. The angular velocities of the the two bodies expressed in their respectively body-fixed triads as a function of the generalized coordinates and the generalized velocities can be found by adding the individual joint angular speeds and transforming them to the body-fixed triads, see Figure G.58.

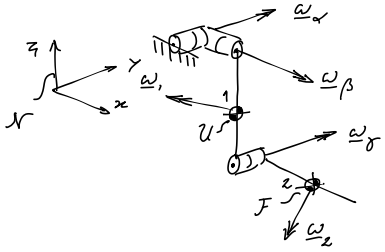


Figure G.58: Angular velocities.

We start from the general vector expressions,

$$\begin{aligned} \boldsymbol{\omega}_1 &= \boldsymbol{\omega}_\alpha + \boldsymbol{\omega}_\beta \\ \boldsymbol{\omega}_2 &= \boldsymbol{\omega}_\alpha + \boldsymbol{\omega}_\beta + \boldsymbol{\omega}_\gamma. \end{aligned}$$

Next we express the components in their individual triads, (u, v, w) for $\mathcal{U}$ and (r, s, t) for $\mathcal{F}$, and transform them to the respectively body frames as in,

$$\begin{aligned} {}^{\mathcal{U}}\boldsymbol{\omega}_1 &= \boldsymbol{R}(\beta)^{\mathrm{T}} \begin{pmatrix} 0 \\ \dot{\alpha} \\ 0 \end{pmatrix} + \begin{pmatrix} \dot{\beta} \\ 0 \\ 0 \end{pmatrix} \\ {}^{\mathcal{F}}\boldsymbol{\omega}_2 &= \boldsymbol{R}(\gamma)^{\mathrm{T}} \left(\boldsymbol{R}(\beta)^{\mathrm{T}} \begin{pmatrix} 0 \\ \dot{\alpha} \\ 0 \end{pmatrix} + \begin{pmatrix} \dot{\beta} \\ 0 \\ 0 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ \dot{\gamma} \\ 0 \end{pmatrix}. \end{aligned}$$

Expanding and rearranging terms results in,

$$\begin{aligned} \begin{pmatrix} \omega_{1u} \\ \omega_{1v} \\ \omega_{1w} \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ \cos(\beta) & 0 & 0 \\ -\sin(\beta) & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \boldsymbol{B}_1 \dot{\boldsymbol{q}}, \\ \begin{pmatrix} \omega_{2r} \\ \omega_{2s} \\ \omega_{2t} \end{pmatrix} &= \begin{pmatrix} \sin(\beta)\sin(\gamma) & \cos(\gamma) & 0 \\ \cos(\beta) & 0 & 1 \\ -\sin(\beta)\cos(\gamma) & \sin(\gamma) & 0 \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \boldsymbol{B}_2 \dot{\boldsymbol{q}}. \end{aligned}$$

e. The derivation of the equations of motion, expressed in terms of the independent generalized coordinates and their derivatives, starts with the Newton-Euler equations for every separate body,

$$\boldsymbol{F}_b = \boldsymbol{M}_b \ddot{\boldsymbol{x}}_b, \quad \text{and} \quad \boldsymbol{M}' = \boldsymbol{I}'_b \dot{\boldsymbol{\omega}}'_b + \boldsymbol{\omega}'_b \times (\boldsymbol{I}'_b \boldsymbol{\omega}'_b), \quad b = 1, 2.$$

Next, we can transform these equations of motion with the help of the Jacobians $\boldsymbol{T}_{b,\boldsymbol{q}}$ and $\boldsymbol{B}_b$, and sum over all the bodies, as explained in Chapter 13. This then results in the equations of motion expressed in terms of the generalized coordinates and their derivatives,

$$\sum_b \left[\boldsymbol{T}_{b,\boldsymbol{q}}^{\mathrm{T}} \boldsymbol{M}_b \boldsymbol{T}_{b,\boldsymbol{q}} + \boldsymbol{B}_b^{\mathrm{T}} \boldsymbol{I}_b \boldsymbol{B}_b\right] \ddot{\boldsymbol{q}} = \boldsymbol{Q} + \sum_b [\boldsymbol{T}_{b,\boldsymbol{q}}^{\mathrm{T}} (\boldsymbol{F}_b - \boldsymbol{M}_b \boldsymbol{g}_b) + \boldsymbol{B}_b^{\mathrm{T}} \left(\boldsymbol{M}_b' - \boldsymbol{I}_b' \boldsymbol{h}_b - \boldsymbol{\omega}_b' \times (\boldsymbol{I}_b' \boldsymbol{\omega}_b')\right)],$$

with the angular velocities expressed in terms of the generalized coordinates and their derivatives as,

$$\boldsymbol{\omega}_b' = \boldsymbol{B}_b \dot{\boldsymbol{q}},$$

and the convective acceleration terms,

$$\boldsymbol{g}_b = \boldsymbol{T}_{b,\boldsymbol{qq}} \dot{\boldsymbol{q}} \dot{\boldsymbol{q}}, \quad \text{and} \quad \boldsymbol{h}_b = \boldsymbol{B}_{b,\boldsymbol{q}} \dot{\boldsymbol{q}} \dot{\boldsymbol{q}}.$$

f. The MATLAB$^{\mathrm{TM}}$-code to implement the equations of motion could look something like this,

```
function dy = odehw10(t,y,par)
% unpack
a = y(1); b = y(2); c = y(3);
da = y(4); db = y(5); dc = y(6);
e = par.e; d = par.d;
Mij = par.Mij;
Iij = par.Iij;
Fi = par.Fi;
Mi = par.Mi;
Qj = par.Qj;
% transformations and convective terms
Tij = Tijfunc(a,b,c,d,e);
gk = gkfunc(a,b,c,d,da,db,dc,e);
Bij = Bijfunc(b,c);
hk = hkfunc(b,c,da,db,dc);
% eom
Mbar = Tij'*Mij*Tij+Bij'*Iij*Bij;
qd = [da;db;dc];
w = Bij*qd;
w1 = w(1:3); w2 = w(4:6);
I1 = Iij(1:3,1:3); I2 = Iij(4:6,4:6);
crossterm = [cross(w1,I1*w1); cross(w2,I2*w2)];
Qbar = Qj+Tij'*(Fi-Mij*gk)+Bij'*(Mi-Iij*hk-crossterm);
qdd = Mbar\Qbar;
dy = [y(4:6); qdd];
end
```

The functions that calculate the respective transformations and the convective terms are derived from symbolic code. They are transformed to MATLAB^{TM} with the help of the function `'matlabFunction'`, for example for $T_{i,j}$:

```
matlabFunction(Tij,'File','Tijfunc');
```

Next, this is how the implemented equations of motion are used in some MATLAB^{TM} script to check the results for some simple configurations,

```
t0 = 0;
y0 = [0; 0; pi/2; 0; 0; 0];
dy0 = odehw10(t0,y0,par)
```

We can check all arms vertical down $(\alpha, \beta, \gamma) = (0, 0, \pi/2)$ and the resulting accelerations are $(\ddot{\alpha}, \ddot{\beta}, \ddot{\gamma}) = (-0.1641, 0, 0.6304)10^{-14}\,\mathrm{rad/s^2}$, indeed all zero as expected. Next we try both arms vertical up, and again the joint accelerations are all zero ($< 10^{-14}$). Finally, for the case where the arm is horizontal, $(\alpha, \beta, \gamma) = (-\pi/2, 0, \pi/2)$, the resulting angular accelerations at the joints are

$$(\ddot{\alpha}, \ddot{\beta}, \ddot{\gamma}) = (41.6217, 0, -53.0380)\mathrm{rad/s^2}.$$

It is not easy to predict what the exact numbers should be, but clearly the arm is falling down, $\ddot{\alpha} > 0$, the elbow angle is becoming smaller, $\ddot{\gamma} < 0$ (like with the double pendulum from Problem 10.2), and the arm is not moving sideways, $\ddot{\gamma} = 0$. If we calculate the accelerations of the CoM's, $\ddot{x}_i = T_{i,j}\ddot{q}_j$, we find

$$(\ddot{x}_1, \ddot{y}_1, \ddot{z}_1, \ddot{x}_2, \ddot{y}_2, \ddot{z}_2) = (0, 0, -6.2432, 0, 0, -10.7740)\mathrm{m/s^2}.$$

Again, like in the 2-Ddouble pendulum, we see that the forearm is falling faster than the upper arm, that the downward acceleration of the CoM of the forearm is larger than g, and that the values are about the same as in the 2-Ddouble pendulum problem. Checking if all the velocity terms are correct is somewhat more difficult. You can check configurations where you expect an initial velocity to have no effect on the accelerations, f.i. the arm vertical down and only an angular speed in joint 2, and indeed all accelerations are still zero.

g. In the initial configuration the reduced mass matrix is given by,

$$\bar{\boldsymbol{M}} = \begin{pmatrix} 0.2445 & 0 & 0.0528 \\ 0 & 0.1937 & 0 \\ 0.0528 & 0 & 0.0528 \end{pmatrix} \mathrm{kg\,m^2}$$

h. In the ball catch posture, $(\alpha, \beta, \gamma) = (-70°, 70°, -30°)$, the equilibrium torques are,

$$\boldsymbol{Q} = \begin{pmatrix} M_\alpha \\ M_\beta \\ M_\gamma \end{pmatrix} = \begin{pmatrix} -2.616446911341033 \\ 1.962663582881516 \\ 0.813461200911191 \end{pmatrix} \mathrm{N\,m}$$

i. The MATLAB^{TM} code for the dynamic analysis of the equilibrium posture is something like this,

```
t0 = 0;
y0 = [-70*pi/180; 70*pi/180; -30*pi/180; 0; 0; 0];
par.Qj = [ -2.616446911341033
            1.962663582881516
            0.813461200911191];
tend = 5;
opts = [];
[T,Y] =  ode45(@odehw10,[t0 tend],y0,opts,par);
```

Then, in Figure G.59 the time history of the three joint angles are shown for the first 5 seconds (right), starting from the equilibrium ball catch posture at rest (left).

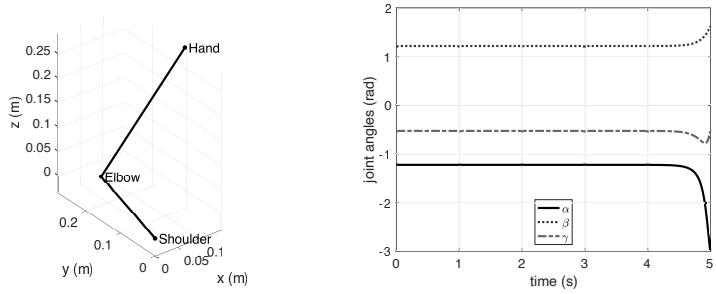


Figure G.59: Human arm model, ball catch posture (left) and joint angles as a function of time starting from an equilibrium position (right).

Clearly, the system starts moving after 4.5 seconds. This motion can be explained as follows. The key issue is that in this upright configuration the system is in an unstable equilibrium position. You can compare this to an inverted pendulum. Any small disturbance from the vertical up position will result in a falling down. Likewise with the human arm model in an upright like equilibrium position. In real life the perturbation, which starts the falling, can be wind, vibrating ground etc. In our simulation model it's round-off errors which start off the motion, even when we copy and paste the equilibrium torques with high precision.

j. In the equilibrium position where $(\alpha, \beta, \gamma) = (30°, 20°, 60°)$, the arm is hanging down, the elbow and hand a little behind the shoulder and the elbow a little bit bent from the straight, see Figure G.60(left). The equilibrium torques in this configuration are,

$$\boldsymbol{Q} = \begin{pmatrix} M_\alpha \\ M_\beta \\ M_\gamma \end{pmatrix} = \begin{pmatrix} 3.330116523969437 \\ 2.701885177317773 \\ 0.057639820172398 \end{pmatrix} \text{N m}$$

The time history of the three joint angles from the dynamics analysis for

this configuration, starting from equilibrium, is shown in Figure G.60 on the right. Clearly the system is in a stable equilibrium condition, just like a pendulum hanging down.

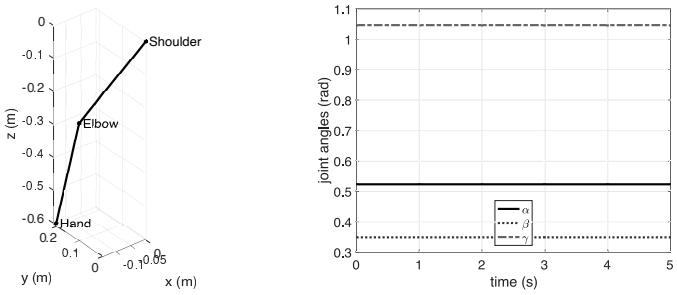


Figure G.60: Human arm model, hanging down (left) and joint angles as a function of time starting from that equilibrium position (right).

A.3 To solve the problem, we use vector addition in combination with representation of the vectors via their magnitude and direction:

$$\boldsymbol{a} + b\hat{\boldsymbol{e}}_b = c\hat{\boldsymbol{e}}_c. \tag{G.314}$$

This gives two scalar equations (we are in 2-D) for the two unknowns b and c.

A.9 The distance is given by the projection of the given vector onto the vertical axis, so it is its Z-component:

$$h = \hat{\boldsymbol{n}}_3 \cdot \boldsymbol{r}_{Q/P} = (0,\, 0,\, 1)\, \boldsymbol{r}_{Q/P}. \tag{G.315}$$

Note there is no need to take the absolute value to obtain a positive value for the distance, because point Q is always above the XY-plane.

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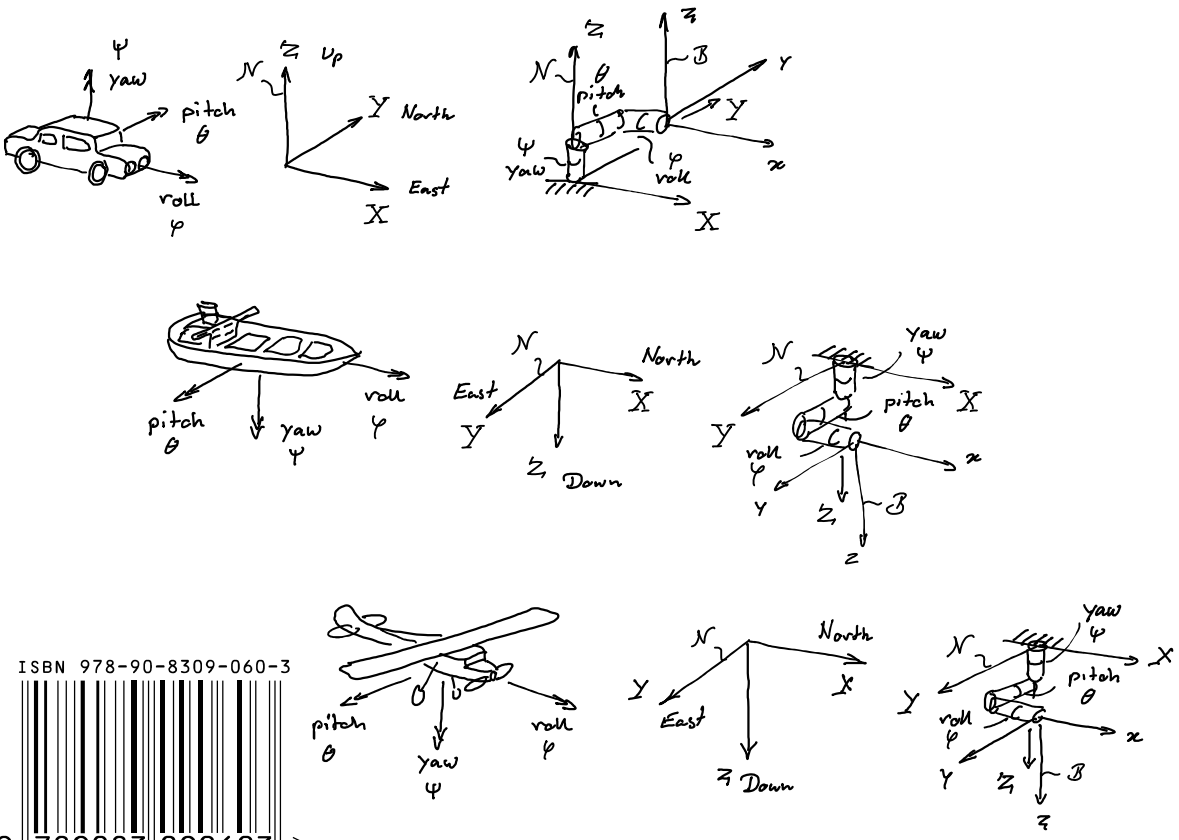
Dynamics of rigid bodies and multibody systems in three-dimensional spaces represents a fundamental pillar for many engineering sciences. This book is aimed at undergraduate and graduate students who already have some background in statics and dynamics of particles and planar systems, but who wish to extend their knowledge towards three-dimensional problems involving one or multiple rigid bodies, and towards variational methods. To facilitate application of this knowledge to real-world problems, the book also treats several methods and examples for the numerical simulation of dynamic systems. For all introduced concepts, practical examples and problems for self-study are included.



ISBN 978-90-8309-060-3