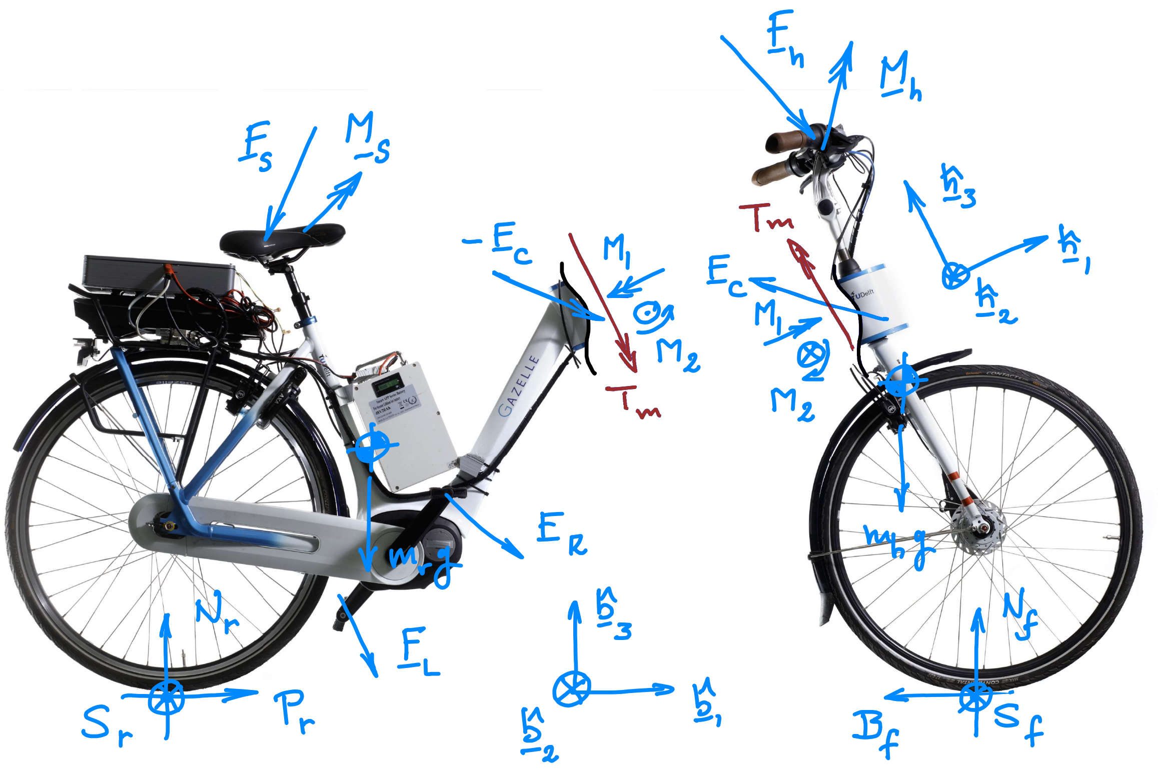
# Advanced Dynamics

Heike Vallery    Arend L. Schwab

Third fully revised edition, including
part I: Rigid-Body Dynamics
part II: Multibody Dynamics



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The cover image shows the steer-assist bicycle developed at the TU Delft in collaboration with the Dutch Royal Gazelle bicycle company. This prototype has a smart motor in the head tube, which can provide an actively controlled steer torque $T_{\mathrm{m}}$ (highlighted in red) to enhance the lateral stability of the bicycle at low-to-moderate forward speed. This active lateral-stability control is intended to reduce the number of falls in cycling. The prototype was built with funding support from the Sport Innovator Award 2016. Photo Sam Rentmeester.

Stichting Newton-Euler, Delft


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First edition, printed August 2017
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# Preface

Mechanics of rigid bodies moving in three-dimensional spaces represents a fundamental pillar for many engineering sciences.

Part I of this book is aimed particularly at undergraduate students who already have some background in statics and dynamics of particles and planar systems, but who wish to extend their knowledge towards three-dimensional problems involving rigid bodies, and towards variational methods.

Part II of this book is aimed particularly at graduate students interested in further deepening and extending their knowledge on these topics, applied to multibody systems.

For all introduced concepts, many practical examples and problems for self-study are included. Problems and examples are categorized according to their level. This level is depicted by a certain number of gimbals:

**Problem 99.1** Quick knowledge clarification or check; answers can (almost) directly be found in text.

**Problem 99.2** Full problem requiring several steps.

**Problem 99.3** Advanced problem, for readers who want extra knowledge or challenge.

For selected problems, answers and solutions are provided in the end of this book. Those problems are marked in the end with ▷ for an answer and with ▶ for a full solution. Also, some problems have hints, which are printed as footnotes and upside-down.

Particular emphasis is placed on gradually enabling readers to independently solve real engineering problems, with confidence in the correctness of their solution. Therefore, this book provides ample and detailed guidance on how to check plausibility and rigorously question own results. Explicit recommendations are provided at the end of the first five chapters of part I. Also many problems are provided that, instead of relying on worked-out exemplary solutions, are designed to guide readers in the process of verifying own results.

The initial basis for part I of this book were the lecture and lecture slides of the course "Rigid-Body Dynamics" (formerly named "Advanced Dynamics") at the Faculty of Mechanical, Maritime and Materials Engineering at TU Delft, by the authors. An earlier version of part I appeared as the first edition of this book in 2017.

The basis for part II of this book were the lecture and lecture slides of the course "Multibody Dynamics" at the same faculty at TU Delft, by the author Arend Schwab. This part was added in the second edition.

Both lectures continue to exist, and the two parts of this book function as readers. Therefore, the authors have taken care to keep both parts of this book readable independently, although readers may find it useful to also consult the respective other part of the book when looking for background or further information. This independence necessarily means that some of the content provided in part I is repeated in part II in slightly modified form.

This third, fully revised edition differs substantially from the previous ones. The notation in general has been improved, in particular the way in which 3-D coordinate systems are described with the usage of triads. Moreover, the notation used in part I, rigid-body dynamics, and part II, multibody dynamics, are now more in agreement. Finally, the number of problems throughout the whole book has been enlarged and now covers a larger range of difficulty level. In part I, many guided problems have been added. These guided problems find their origin in homework assignments and assist the reader, in a step-by-step manner, to solve a complex problem. In part II, a considerable number of advanced problems with answers and solutions have been added. These problems find their origin in homework assignments and exam questions. Extensive solutions are included to guide the reader in understanding the material at hand in self-study.

Editorial support and review for earlier editions of part I was provided by Pier de Jong, Bram Smit, Johan Schonebaum, and Frederik Lachmann. The initial transcription from the lecture was performed by Johan Schonebaum. Many problems and examples were created with contributions from Wouter Wolfslag, Daniel Lemus, Bram Smit, and Patricia Baines. Critical review was also performed by Wouter Wolfslag and Mukunda Bharatheesha. Several attentive students identified errata in the first edition and thereby contributed to the second edition, especially Roelf-Jilling Wolthuis, Louisa Preis, and Martijn van Veen.

Richard van der Linde contributed to the multibody dynamics lecture. Editorial support to revise and re-format the lecture notes was provided by Pier de Jong.

For this third edition, Daniel Lemus has contributed ideas for improved notations of triads. Pier de Jong continued to support the authors by answering formatting questions.

Jaap Meijaard has provided meticulous reading and excellent suggestions for improvement for all editions.

The authors would like to thank all contributors.

The authors have done their best to write carefully, and have placed no deliberate errors in the book, but are quite familiar with their own imperfections. They therefore ask the readers to be indulgent and assure them that e-mails from them calling attention to errors or containing suggestions for improvement of the book will be gratefully received and very much appreciated.

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Delft, The Netherlands
August, 2020

# Contents

# Part I

# Rigid-Body Dynamics

## Overview of Part I

The first chapter of this part contains a brief review of general mechanics background and also introduces some main notations used in the book.

Following up on these basics, Chapter 2 reviews basic dynamic concepts involving particles and systems of particles, using fixed coordinate systems.

When transitioning from 2-D to 3-D rigid-body dynamics, description of orientation requires particular attention. Therefore, two chapters are dedicated only to kinematics involving rotating bodies and coordinate systems, namely Chapter 3 and Chapter 4.

To derive the equations of motion of mechanical systems, four alternative methods will be introduced: Work and Energy in Chapter 5, Newton-Euler in Chapter 6, Virtual Work in Chapter 7, and Lagrange's method in Chapter 8. Practical guidance and advice is provided on when to choose which method for a particular problem at hand.

An introduction to numerical simulation of dynamic systems is provided in Chapter 9. This chapter can mostly be read independent of the previous, and some readers may prefer to read it first instead of last. That way, the different methods to derive the equations of motion can directly be translated to numerical simulations of the treated mechanical systems.

For further reading, we recommend classical mechanics and dynamics books, of which we like to mention here: Whittaker [75], Hamel [28], Sommerfeld [67], Pars [54], and Goldstein [25]. In particular we like to draw attention to the very readable book by Cornelius Lanczos on variational principles in mechanics [39].

# 1 Notations and Background

This chapter introduces some general notations used in this book, and it reviews a few selected concepts of mechanics that will be needed frequently in the following chapters. For readers unfamiliar with these topics, more detailed information is for example given in [4].

All following content relies on solid knowledge of the mathematical tools presented in Appendix A and Appendix B. Deeper coverage of the mathematical prerequisites can for example be found in [40, 57, 70].

## 1.1 Notations

### 1.1.1 Typesetting of Scalars, Vectors, and Matrices

Scalars, vectors, unit vectors and matrices in this book are typeset as depicted in Table 1.1.

Table 1.1: Notation of scalars, vectors, unit vectors and matrices

| | This book | Handwriting |
|---|---|---|
| Scalars | $F$ | $\mathsf{F}$ |
| Vectors | $\boldsymbol{F}$ | $\underline{\mathsf{F}}$ |
| Unit vectors | $\hat{\boldsymbol{e}}$ | $\underline{\hat{e}}$ |
| Matrices | $\mathbf{R}$ | $\mathcal{R}$ |

Note that we often use the same letter when we refer to a vector's magnitude, so for example $a$ could be used to refer to the magnitude $|\boldsymbol{a}|$ of a vector $\boldsymbol{a}$. However, such a relationship must still always be explicitly defined.

### 1.1.2 Sub- and Superscripts

A position vector that points to a point $A$ from a point $P$ will be denoted as $\boldsymbol{r}_{A/P}$ (which reads: "position of $A$ *with respect to* $P$"). Analogous notation will be used for other relative quantities, such as linear or angular velocity.

◯ **Problem 1.1** The *simplest walker* [19, 63], a 2-D model of a bipedal mechanism, consists of only three particles: two feet, each of mass $m$, and a hip mass $M$. Massless legs connect the particles and are hinged at the hip (see Figure 1.1). The model can walk down a slope of angle $\gamma$ under the influence of gravity $g$.

Use vector addition (see Appendix A.1) to calculate the location $\boldsymbol{r}_{D/A}$ of the swing foot with respect to the stance foot, as a function of the position vector $\boldsymbol{r}_{B/A}$ of the hip with respect to the stance foot $A$ and the relative position vector $\boldsymbol{r}_{D/B}$ of the swing foot with respect to the hip.[a]

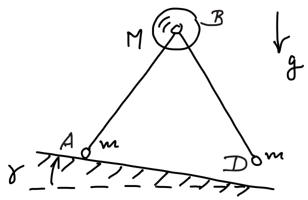



Figure 1.1: 2-D walker.

[a] *Hint*: Compare to Example A.1.

For components of $m$-dimensional vectors, we use the *index notation*:

$$\boldsymbol{r} = \begin{pmatrix} r_1 & r_2 & \dots & r_m \end{pmatrix}^{\mathrm{T}}. \tag{1.1}$$

Elements of $m \times n$-dimensional matrices receive two indices, for row and column:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}. \tag{1.2}$$

In case we operate in $\mathbb{R}^3$ and with Cartesian coordinate systems, we also often use the names of the axes as indices (for example $x$, $y$, $z$) instead of the numerical values 1, 2, 3, to allow quick association.

In this book, we will frequently make use of different coordinate systems and associated triads, which are a set of three unit vectors that define the axis directions. To indicate that the components of a vector are expressed in a particular triad, we use a left superscript. Thus, the vector ${}^{\mathcal{B}}\boldsymbol{r}$ is the same vector as ${}^{\mathcal{N}}\boldsymbol{r}$, just for the former the components are expressed in a triad $\mathcal{B}$, and for the latter the components are expressed in a triad $\mathcal{N}$. So, the difference is in the projection onto different axes (see also Appendix A.4). In case we use an inertial triad $\mathcal{N}$, we often drop the superscript and only write $\boldsymbol{r}$. This is sufficient for the first two chapters in this book, where we do not yet deal with rotating coordinate systems or triads. Mapping between different triads will be explained in Chapter 3. Also more sub- and superscript notations are introduced in the following chapters.

### 1.1.3 Drawing Vectors

If we label an arrow in a drawing with a vector symbol (so a boldface variable), for example $\boldsymbol{F}$, then this label refers to the definition of the vector including its magnitude *and* direction. The direction drawn for the arrow is merely an illustration. This notation is particularly used if the direction of the vector is unknown or changes.

In contrast, if we label an arrow with a scalar variable name $F$ (so typeset regular), this scalar value is the magnitude of the vector, and the direction of the vector is defined by the drawn direction of the arrow. We do allow this scalar value $F$ to take on a negative value (even though the magnitude of a vector is strictly seen always positive), in order to enable inversion of the vector's direction later on: When defining a force vector or vector component, we do not need to know in which direction it points, we only need to know its *line of action*. This implies that an arrow labeled with $-F$ is equivalent to an arrow pointing in the opposite direction and labeled $F$. Particularly when a vector is split into components along specific axis directions, we will use scalar labels for these components.

**Example 1.1** A body is cut free from a frictionless surface, the ground. Figure 1.2 shows three different ways to draw the forces between body and ground.

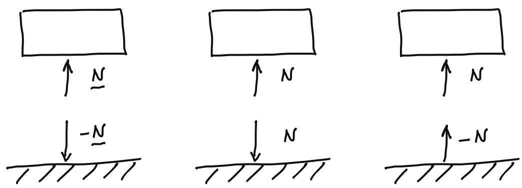



Figure 1.2: Equivalent drawings of force vectors, with vector (left) and scalar labels.

If the force vector acting on the body is labeled by $\boldsymbol{N}$, then the reaction force (following Newton's third law, see Section 1.2.5) to this vector, drawn as acting on the ground, *must* be labeled with $-\boldsymbol{N}$ (note the minus sign), to make clear that the direction is opposite (Figure 1.2, left). If the force vector acting on the body is labeled by a scalar $N$ and drawn upward, and its reaction force is drawn as pointing downward, then that reaction *must* be labeled by $N$ as well, not by $-N$ (Figure 1.2, center). Alternatively, and still formally correct, one could draw the same reaction force vector as pointing upward and label it by $-N$ (Figure 1.2, right). However, this is discouraged as it can be confusing.

A vector pointing into or out of of the paper plane (Figure 1.3, top row) is drawn following the analogy of a dart: We indicate vector components pointing into the plane by $\otimes$ and those pointing out by $\odot$.

Vectors that encode rotational information, such as moments (see Section 1.2.1), will be drawn as in the bottom row of Figure 1.3, with a double arrowhead in the plane. The *right-hand rule* for rotations states that if the thumb points in direction of the arrow, the curled fingers of one hand indicate the positive direction of a *moment* or angle (Figure 1.3, center).

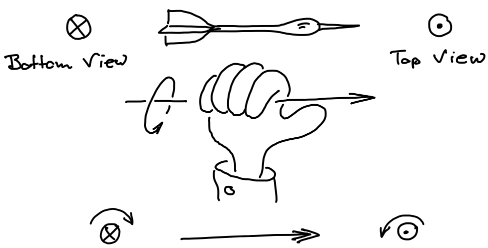



Figure 1.3: Notation for a vector that is perpendicular to the plane of drawing imagined as a dart (top); the right-hand rule to indicate positive direction of a rotation/moment about an axis (center and bottom).

### 1.1.4 Units

In mechanics, we often deal with quantities that have a measurement unit, like forces, velocities, or positions. Each such quantity is the *product* of a number and a measurement unit. This also holds for components of vectors or matrices. Generally, the letters of units are printed in roman (upright), those of scalar variables in italic.

◯ **Example 1.2** Force can be expressed as the product of a numerical value and the measurement unit Newton: $F = 3 \cdot \mathrm{N} = 3\,\mathrm{N}$

Expressions may contain mixtures of quantities given with their numerical value and quantities given as variables. Also in those cases, one must pay attention to consider both factors (numerical value and unit) that compose each quantity, and *not* to insert or omit units.

◯ **Example 1.3** For a particle of mass $m = 3\,\mathrm{kg}$ having acceleration vector $\boldsymbol{a}$, the resultant force vector $\boldsymbol{F}$ on the particle can be calculated as:
Correct is: $\boldsymbol{F} = m \cdot \boldsymbol{a} = 3\,\mathrm{kg} \cdot \boldsymbol{a}$.
*In*correct is: ~~$\boldsymbol{F} = 3 \cdot \boldsymbol{a}\,\mathrm{N}$~~.

◯ **Example 1.4** The derivative $\dot{\theta}$ of an angle $\theta$ is given as a constant value of $\dot{\theta} = 2\,\mathrm{rad/s}$. At time $t = 0$, the angle is zero as well. So, the angle $\theta$ is a function of time $t$:

$$\theta = \int \dot{\theta}\,\mathrm{d}t = 2\,\mathrm{rad/s} \cdot t. \tag{1.3E}$$

At $t = 3\,\mathrm{s}$, the angle is calculated as:

$$\theta = 2\,\mathrm{rad/s} \cdot 3\,\mathrm{s} = 2 \cdot 3 \cdot \mathrm{rad/s} \cdot \mathrm{s} = 6\,\mathrm{rad}. \tag{1.4E}$$

◯ **Problem 1.2** A particle performs a harmonic oscillation, such that its location coordinate $x$ is a function of amplitude $x_0$, angular frequency $\Omega$, and time $t$:

$$x = x_0 \sin(\Omega t). \tag{1.5P}$$

Fill in suitable units for $x$, $x_0$, $\Omega$, and $t$ and check the equation for consistency. Be aware that the argument of the sin-function does not have units.[a]

[a] *Hint*: Recall the definition of the unit radian, implying that 1 rad = 1.

Following the ISO norm [34] and the "Red Book" [13], for quantities that have a measurement unit, the numerical value of a quantity $Q$ is denoted by $\{Q\}$, and the measurement unit is denoted by $[Q]$, such that

$$Q = \{Q\} \cdot [Q]. \tag{1.6}$$

Therefore, the numerical value can also be denoted as $\{Q\} = Q/[Q]$.

◯ **Example 1.5** A mass $m = 3.5\,\mathrm{kg}$ has the numerical value $\{m\} = 3.5$ and the unit $[m] = \mathrm{kg}$. The same quantity value can be expressed as $3500\,\mathrm{g}$. Then, the numerical value is $\{m\} = 3500$, and $[m] = \mathrm{g}$.

Square brackets are a common source of mistakes. Such brackets only have meaning when placed around the physical quantity. Square brackets should *not* be placed around the unit.

◯ **Example 1.6** For time $t$,
good use is: $t/\mathrm{s} = 4$, or $[t] = \mathrm{s}$,
*not* good use is: ~~$t = 4$ [s]~~, or ~~$t$ [s]~~.

Incorrect square brackets are often found in axes labels of plots. The SI [5] advises division, as in a label like (for the example of time): $t/\mathrm{s}$. Also possible are round brackets [33, 56]: time $t$ (s).

A general advice is to work as long as possible with variables, and only to substitute numbers and units in the final solution. Units almost always help as part of plausibility checks for calculations.

◯ **Problem 1.3** Which of these expressions make correct use of units, for mass $m$, time $t$, force $F$, angular velocity $\omega$, and angle $\theta$?

A. $m = 20$
B. $m = 3\,\mathrm{N}$
C. $F = 3 \cdot m\,[\mathrm{N}]$
D. $\mathrm{F} = 20N$
E. $[m] = \mathrm{kg}$
F. $m/[m] = 20$
G. $t = 3\,\mathrm{s}$
H. $\omega = 2\,\mathrm{rad/s}$
I. $\theta = \omega \cdot t = 2 \cdot t\,[\mathrm{s}]$
J. $\theta = \omega \cdot t, [\omega] = \mathrm{rad/s}, [t] = \mathrm{s}, [\theta] = \mathrm{rad}$
K. $\{m\} = 20$
L. $\omega = t^2$
M. $\omega = 1\,\mathrm{rad/s} \cdot t^2$
N. $\omega = 1\,\mathrm{rad/s^3} \cdot t^2$
O. $f(t) = \cos(1\,\mathrm{rad/s} \cdot t)$
P. $g(t) = e^{-t}$
Q. $h(t) = \sin(\omega t) = \sin(2t)$

## 1.2 Basic Definitions and Tools Used in Mechanics

Readers are expected to have already knowledge on the branch of mechanics that is called *statics*. Statics deals with mechanical systems that are at rest or move with constant linear velocity. It defines conditions for forces and moments acting on or within these systems such that the systems remain in this state. Statics and other branches of mechanics share many important definitions and tools. In this section, some of these concepts will be reviewed.

### 1.2.1 Definition of a Moment

The *moment*, often also called *torque*[1], of a force $\boldsymbol{F}$ with respect to point $P$ is defined as:

$$\boldsymbol{M}_P := \boldsymbol{r}_{A/P} \times \boldsymbol{F}, \tag{1.7}$$

[1] The term torque is often used for a couple and for a torsional moment; a moment about a longitudinal axis. An example is the torque a motor applies to a shaft.

where the vector $\boldsymbol{r}_{A/P}$ is a vector from point $P$ to point $A$, the point of application of the force. Note that from the definition of the cross product, it follows that $A$ may in fact be any *arbitrary* point on the line of action of $\boldsymbol{F}$ (see Problem 1.5). Also note that the order in which this cross product is written matters, as $\boldsymbol{F} \times \boldsymbol{r}_{A/P} = -\boldsymbol{M}_P$.

**Problem 1.4** Calculate the moment about point $P$ using (1.7), for force $\boldsymbol{F} = (3,\, 4,\, 0)^{\mathrm{T}}\,\mathrm{N}$, if the position vector of the point of application of this force, $A$, with respect to point $P$ is $\boldsymbol{r}_{A/P} = (2,\, 3,\, 0)^{\mathrm{T}}\,\mathrm{m}$. Draw all three vectors in the $XY$-plane and apply the right-hand rule of Figure 1.2 to check the direction of the moment vector. Then, choose another point $A'$ on the line of action of the force to calculate the moment and compare results.

If two force vectors act on a rigid body and if these two forces have parallel lines of action, equal magnitude, but opposite direction, then the sum of these two forces (also called a *force couple*) is the zero vector. However, the forces still cause a moment vector. This *couple moment* vector does not depend on the point of reference $P$, so it is a *free vector*.

**Problem 1.5** Use the definition of the cross product to show that when calculating the moment vector $\boldsymbol{M}_P$ of a force vector $\boldsymbol{F}$ about point $P$ according to (1.7), it does not matter which vector $\boldsymbol{r}_{A/P}$ you choose from point $P$ to the line of action of the force.[a]

[a] *Hint*: Try these two approaches: a) Draw the parallelogram spanned by position and force vector and show that neither the magnitude nor the direction of the cross product changes when the force shifts along its line of action. Use (A.2) and the right-hand rule. b) Describe an arbitrary points on the line of action of a force as a linear combination of a position and a direction vector. Then, calculate the cross product of this point with the force vector.

## 1.2.2 Force System Resultants: Equipollence

When several forces and moments act on a rigid body, it is always possible to formulate an equivalent system that consists of 1) one single resultant force vector acting at an arbitrary point $O$ and 2) one couple moment vector. Equivalence, or more precisely *equipollence*, means that the effect of the original and of the reduced force/moment system on the rigid body is identical. This leads to the following steps:

1. To determine the magnitude and direction of the resultant force $\boldsymbol{F}_{\mathrm{r}}$, all $N$ original force vectors $\boldsymbol{F}_i$ need to be summed:

$$\boldsymbol{F}_{\mathrm{r}} = \sum_{i=1}^{N} \boldsymbol{F}_i. \tag{1.8}$$

2. The associated resultant couple moment vector $\boldsymbol{M}_\mathrm{r}$ equals the moment caused by the $N$ original forces and $K$ original couple moments about the same point $O$:

$$\boldsymbol{M}_\mathrm{r} = \sum_{j=1}^{K} \boldsymbol{M}_j + \sum_{i=1}^{N} \boldsymbol{r}_i \times \boldsymbol{F}_i, \tag{1.9}$$

where the $\boldsymbol{r}_i$ are the position vectors of the points of application of the $N$ original forces with respect to point $O$.

Simplifying even further, it is also always possible to find an equivalent system that contains only one single force vector and a single couple moment vector *that is parallel to the force vector's line of action.*

**Problem 1.6** For an original system of $N$ forces $\boldsymbol{F}_i$, $i = 1 \ldots N$, set up an algorithm (set of equations) that allows determining the resultant force vector, one point on its line of action, and a resultant couple moment vector that is parallel to this line of action. Test using an example.[a]

[a] *Hint*: The resultant force vector determines the direction of any possible couple moment. Therefore, express the couple moment vector as a product of its magnitude and direction, which contains only one scalar unknown. To determine the location of the line of action of this force as well as the magnitude of the couple moment, equate the moment of all original forces and moments about a point $O$ to the moment of the resultant force and the couple moment vector. Any point on the line of action of the force is a possible point of application. So, you could choose a position vector that is perpendicular to the line of action and only has unknown length.

### 1.2.3 Newton's First Law and Static Equilibrium

In 1687, Isaac Newton (1642 - 1727) stated his three laws of motion [49]. The first law reads:

1. *"Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon."*

So, a particle that moves at constant linear velocity $\boldsymbol{v}$ will keep the same velocity unless a force $\boldsymbol{F}$ acts upon it. Note that a particle at rest has a constant velocity of zero.

$$\sum \boldsymbol{F} = 0 \leftrightarrow \dot{\boldsymbol{v}} = 0. \tag{1.10}$$

Based on this law, static equilibrium of a rigid body is defined as a state where the sum of all forces acting on the system is zero, as is the sum of all moments acting on the body about any arbitrary point.

Note that the term equilibrium does not necessarily imply *stable* equilibrium. For example, a ball in a convex valley resides in a stable equilibrium, while the same ball balancing on top of a concave mountain peak is in an unstable equilibrium. Stability can be investigated by analyzing the system's response to infinitesimally small perturbation from its state of rest.

### 1.2.4 Center of Mass

The position vector $\boldsymbol{r}_{C/A}$ from a point $A$ to the overall *center of mass* (CoM) of a system of $N$ particles with masses $m_i$ is defined by:

$$\boldsymbol{r}_{C/A} := \frac{\sum_{i=1}^{N} m_i \boldsymbol{r}_{i/A}}{\sum_{i=1}^{N} m_i}, \tag{1.11}$$

where $\boldsymbol{r}_{i/A}$ is the position vector of the $i$-th particle with respect to the same point $A$.

When calculating the center of mass for a rigid body, the summation turns into an integral:

$$\boldsymbol{r}_{C/A} := \frac{\int_m \boldsymbol{r}_{P/A} \, \mathrm{d}m}{\int_m \mathrm{d}m}, \tag{1.12}$$

where $\boldsymbol{r}_{P/A}$ is the position vector of the center of mass $P$ of an infinitesimally small volume element of mass $\mathrm{d}m$ in the body. The integral resolves to a triple integral for the case of three-dimensional bodies.

In order to calculate the center of mass for a composite body where the mass and center of mass of each individual element are known, (1.11) can again be used. When dealing with hollow sections in systems with homogeneous mass distribution, the calculation can be simplified by considering first a solid structure and then considering each hollow section as a body with negative mass.

Note that the *centroid* (geometric center), the center of mass, and the *center of gravity* (point of application of the resultant gravitational force) may all be different points. This is the case if mass distribution is not homogeneous and/or if the gravitational field is not homogeneous.

**Example 1.7** Three particles of masses $m_1$, $m_2$ and $m_3$ are at locations with position vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$, respectively, with respect to a point $O$. The values are given as:

$$m_1 = 2\,\text{kg},\; m_2 = 3\,\text{kg},\; m_3 = 5\,\text{kg, and}$$
$$\boldsymbol{r}_1 = \begin{pmatrix}5 & 10 & 1\end{pmatrix}^{\mathrm{T}}\,\text{m},\; \boldsymbol{r}_2 = \begin{pmatrix}-10 & 0 & 6\end{pmatrix}^{\mathrm{T}}\,\text{m},\; \boldsymbol{r}_3 = \begin{pmatrix}2 & 0 & 4\end{pmatrix}^{\mathrm{T}}\,\text{m}.$$

Their combined center of mass is given by:

$$\begin{aligned}\boldsymbol{r}_C &= \frac{1}{m_1 + m_2 + m_3}\left(\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3\right)\\ &= \frac{1}{(2+3+5)\,\text{kg}}\left(2\,\text{kg}\begin{pmatrix}5\\10\\1\end{pmatrix}\text{m} + 3\,\text{kg}\begin{pmatrix}-10\\0\\6\end{pmatrix}\text{m} + 5\,\text{kg}\begin{pmatrix}2\\0\\4\end{pmatrix}\text{m}\right)\\ &= \begin{pmatrix}-1 & 2 & 4\end{pmatrix}^{\mathrm{T}}\cdot\text{m}.\end{aligned}$$

**Problem 1.7** Use integration to calculate the center of mass of several common bodies, such as a cone, a solid hemisphere, and a thin-walled hollow cone. Assume homogeneous mass distribution.

**Problem 1.8** A group of students constructs a top (Figure 1.4) from

- three orthogonally arranged wooden skewers, each of mass $m_{\mathrm{s}}$, length $l$, and negligible thickness, and
- five pieces of cork, each of mass $m_{\mathrm{k}}$ and negligible dimensions. The center cork is positioned at a distance $\Delta$ from the center $A$ of skewer 1.

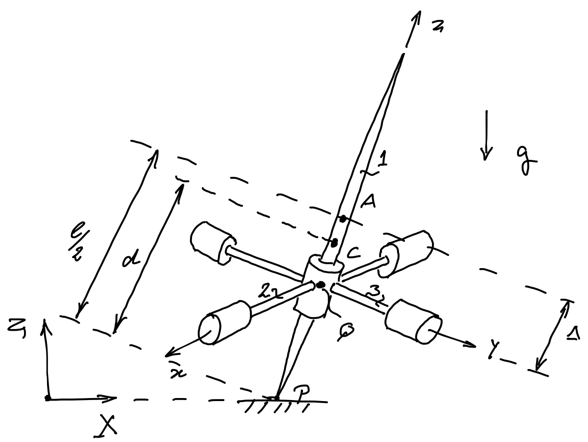



Figure 1.4: Top of skewers and corks.

(DIY-tip: use a screwdriver to drill a hole through the cork, after which it is easier to put the skewer through the cork)

The construction is idealized, such that the skewers intersect in point $Q$ and the top's center of mass $C$ is on the $z$-axis defined by skewer 1. The axes $x$ and $y$ point along skewers 2 and 3. Gravity with acceleration constant $g$ acts in negative vertical direction. An inertial $XYZ$ coordinate system has its origin on the ground and $Z$ points up. A triad $\mathcal{N}$ is associated with $XYZ$, and a triad $\mathcal{B}$ is associated with $xyz$. After building the top, students realize that the top's overall mass is $M^*$, and the center of mass is not on the $z$-axis, but at the location $C^*$ with relative position vector ${}^{\mathcal{B}}\boldsymbol{r}_{C^*/P} = (x_{C^*}, y_{C^*}, z_{C^*})^{\mathrm{T}}$ with respect to point $P$. The students want to add two more pieces of cork, each of mass $\frac{M^*}{10}$, to the skewers at yet unknown locations $D$ and $E$, in order to put the center of mass back on $z$.

Calculate the components of suitable relative position vectors ${}^{\mathcal{B}}\boldsymbol{r}_{D/P}$ and ${}^{\mathcal{B}}\boldsymbol{r}_{E/P}$ of these two pieces with respect to point $P$. Give these components as functions of $x_{C^*}$, $y_{C^*}$, $z_{C^*}$, $M^*$, $l$, and $\Delta$.

Remember that the cork pieces can only be put onto skewers, and that they have negligible dimensions. ▷▶

### 1.2.5 Newton's Third Law of Motion

Newton's third law of motion [49] states:

3. *"To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts."*

So, if an object $a$ exerts a force $\boldsymbol{F}_a$ on another object $b$, then $b$ exerts an equal and opposite force $\boldsymbol{F}_b$ on $a$:

$$\boldsymbol{F}_a = -\boldsymbol{F}_b. \tag{1.13}$$

The forces share the same line of action. Philosophically speaking: "A force is never alone"; the action-reaction forces form a unit, the force pair. When we discuss "generalized forces" in Chapter 7 and Chapter 8, we will see that a force pair often even forms a single generalized force.

### 1.2.6 Definition of Internal and External Forces

Two types of forces are distinguished which act on a particle that is part of a system of $N$ particles: *external* forces $\boldsymbol{F}_i$ and *internal* forces $\boldsymbol{f}_{ij}$, such that the sum of all forces acting on a particle is:

$$\boldsymbol{F}_i + \sum_{j=1}^{N} \boldsymbol{f}_{ij}. \tag{1.14}$$

The external force vector $\boldsymbol{F}_i$ is defined as the sum of all forces that act on particle $i$ by sources outside the system.

Internal forces $\boldsymbol{f}_{ij}$ are the forces that act between particles in a system. For example, if a spring interconnects two particles in one system, then the spring force is called an internal force. The subscript $ij$ means "acting on particle $i$, exerted by particle $j$". Note that $\boldsymbol{f}_{ij} = \mathbf{0}$ for $i = j$. Figure 1.5 illustrates the particles and forces.

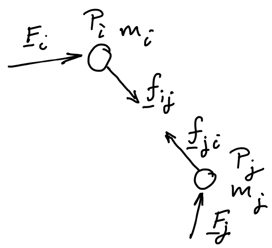


Figure 1.5: Two particles $i$ and $j$ with internal and external forces acting on them.

According to Newton's third law (1.13), there must be an equal and opposite reaction exerted on particle $j$, namely

$$\boldsymbol{f}_{ji} = -\boldsymbol{f}_{ij}. \tag{1.15}$$

Therefore, over the entire system of particles, the sum of all these internal forces is zero:

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \boldsymbol{f}_{ij} = 0. \tag{1.16}$$

Note that the interpretation of a force as being internal or external depends on the chosen definition of system boundaries. We tend to see gravity as an external force, because we consider Earth just in terms of providing us with an inertial coordinate system, but not as being part of our mechanical system. In case we do consider Earth as part of the system (like in a study on planetary motions), then gravity suddenly becomes an internal force. This demonstrates that the idea of internal and external is simply a matter of system definition.

### 1.2.7 Moments of Internal and External Forces

The resultant moment of internal forces acting on a system is zero, as will be shown in the following.

Figure 1.6 shows again two particles $i$ and $j$ of a system of particles with internal and external forces acting on them. We define a relative position vector $\boldsymbol{\rho}_i$ of the $i$-th particle with respect to an arbitrary reference point $Q$, such that

$$\boldsymbol{r}_i = \boldsymbol{r}_Q + \boldsymbol{\rho}_i. \tag{1.17}$$

This implies that $\boldsymbol{\rho}_i$ points from $Q$ to the particle. This is for brevity of notation, as $\boldsymbol{\rho}_i$ is shorter than the more consistent naming $\boldsymbol{r}_{i/Q}$ of the relative position of $i$ with respect to $Q$.

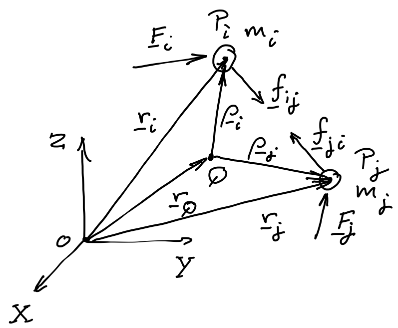


Figure 1.6: Definitions of the absolute and relative position of particles $i$ and $j$, as well as of internal and external forces acting on the particles.

To obtain the resultant moment of all forces acting on particle $i$ about the arbitrary point $Q$, we take the cross product of the particle's relative position vector $\boldsymbol{\rho}_i$ and the external and internal forces:

$$\boldsymbol{M}_{Q,i} = \boldsymbol{\rho}_i \times \left( \boldsymbol{F}_i + \sum_{j=1}^{N} \boldsymbol{f}_{ij} \right). \tag{1.18}$$

Summing over all particles gives the resultant moment of all forces acting on the system:

$$\begin{aligned} \boldsymbol{M}_Q &= \sum_{i=1}^{N} \boldsymbol{\rho}_i \times \left( \boldsymbol{F}_i + \sum_{j=1}^{N} \boldsymbol{f}_{ij} \right) \\ &= \sum_{i=1}^{N} (\boldsymbol{\rho}_i \times \boldsymbol{F}_i) + \sum_{i=1}^{N} \left( \boldsymbol{\rho}_i \times \sum_{j=1}^{N} \boldsymbol{f}_{ij} \right). \end{aligned} \tag{1.19}$$

Because of Newton's law of action and reaction (1.15), we know that for each internal force $\boldsymbol{f}_{ij}$, there is always a counter-force $\boldsymbol{f}_{ji}$ acting in opposite direction, with the same magnitude and on the same line of action. That means that whichever point is chosen as a reference point, the moment pairs about this point will cancel:

$$\boldsymbol{\rho}_i \times \boldsymbol{f}_{ij} = -\boldsymbol{\rho}_j \times \boldsymbol{f}_{ji}. \tag{1.20}$$

Therefore, the sum of moments of all internal forces is zero, and (1.18) simplifies to

$$\boldsymbol{M}_Q = \sum_{i=1}^{N} (\boldsymbol{\rho}_i \times \boldsymbol{F}_i). \tag{1.21}$$

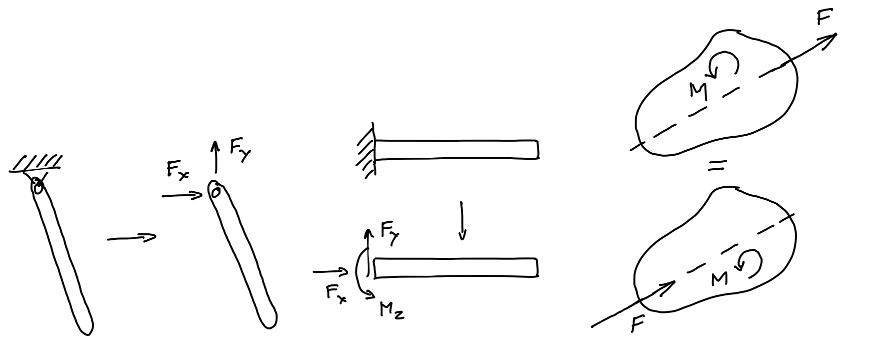
This proves that internal forces cannot cause net moments on a system, regardless of the choice of reference $Q$.

### 1.2.8 Free-Body Diagrams

A *free-body diagram* (FBD) helps to understand and visualize a problem and is used to define system boundaries and variables. It is a simplified drawing of (a part of) a mechanical system with all forces, moments and dimensions. The name is slightly misleading, as FBDs cannot only be drawn for single rigid bodies. They can also be established for parts of bodies or for systems of bodies or particles. These are the steps to draw a FBD:

1. Draw the system in a free state, i.e. "cut" the system at convenient locations and draw outlined shapes of the separate pieces. Each cut will introduce new external (formerly internal) action-reaction forces at the system boundaries. Some helpful guidelines:
   a) Always draw the system in a generic state. So for example if you draw the FBD of a pendulum, draw it with a generic angle, and not in upright or downright position, as those are special cases.
   b) Whenever possible, choose your system boundaries such that you expose only action-reaction forces that you are actually interested in calculating. Otherwise, extra equations and unknowns are introduced, complicating calculations.
2. Establish at least one coordinate system: Choose a clearly defined position for the origin. Check if rotating and/or translating the coordinate system can make things easier.

3. Indicate all known and unknown external forces and moments that act on the system, on the correct locations (e.g. external loads, support reactions, weight). Some rules:
   a) Do not show internal forces or moments.
   b) If a connection prevents movement in a particular direction, then forces/moments are drawn in that direction, see Figure 1.7a and Figure 1.7b.
   c) If a segment is split in two, the forces and moments acting on the two segments in the separated FBDs are equal in magnitude and opposite in direction.
   d) Forces acting on a rigid body may be shifted along their lines of action (sliding vectors), see Figure 1.7c.
   e) Couple moments acting on a rigid body may be placed anywhere (they are free vectors), see Figure 1.7c.
4. Label all forces and moments with unique names.
5. Label all necessary dimensions (distances and angles) to calculate moments of forces with respect to relevant reference points like the center of mass.




(a) For a pendulum, the support is replaced by forces transmitted by the hinge joint.

(b) For a clamped beam, the supporting wall is replaced by the forces and moment it exerts.

(c) Forces may be shifted along their lines of action; moments are free vectors.

Figure 1.7: Examples for drawing components of FBDs (notice that these are *not* complete FBDs).

A FBD must include all relevant information of the original drawing, particularly all information to establish the sums of forces and moments.

**Example 1.8** Figure 1.8 shows a homogeneous block sliding down a slope (left), and its FBD (right).

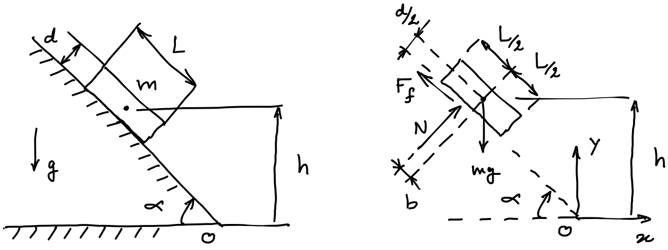



Figure 1.8: Sliding block and corresponding FBD.

A frequent mistake when drawing FBDs for dynamical systems is to already consider the specific movement the system performs. For example, if a particle rotates about one axis, one may be tempted to draw a force in radial direction based on previous knowledge on dynamics and the intuition that there must be a radial force sustaining this rotation. This is a critical mistake, as it defies the purpose of a FBD, reducing it to a confirmation of one's own intuition. Such a procedure exposes subsequent calculation to omissions. A FBD must contain all the *possible* reaction forces generated at the system boundaries, only based on the nature of the connection, and disregarding information that is given on the system's movement. Consequently, an FBD drawn to determine static equilibrium looks identical to one intended to derive a system's *equations of motion* (EoM), as will be done in later chapters.

**Example 1.9** A cylindrical bar (Figure 1.9) has a body-fixed coordinate system $uvw$ associated with triad $\mathcal{B}$. The bar can rotate with variable angular speed $\dot{\psi}$ about the $Z$-axis of an inertial coordinate system $XYZ$, associated with triad $\mathcal{N}$. The $Z$- and $w$-axis always coincide and the angle between the $Y$- and $v$-axis is denoted by $\psi$.

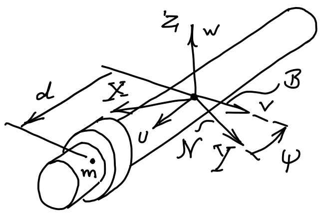



Figure 1.9: A rotating bar and sliding ring.

A ring with mass $m$ can slide without friction on the bar and has a position vector ${}^{\mathcal{B}}\boldsymbol{\rho} = (d,\, 0,\, 0)^{\mathrm{T}}$, with respect to the origin of the coordinate systems. Herein, $d = d(t)$ is a variable in time $t$. The radius, width, and thickness of the ring are *not* negligible. There is no gravity.

Draw a FBD of the ring, in a suitable projection into a 2-D plane.

*Exemplary solution*

It is convenient to choose an orthographic projection into the $uw$-plane. The FBD (Figure 1.10) reflects that except for sliding in $u$-direction and rotating about that axis, the ring cannot move relative to the bar. Representing these constraints for the other two translations and for the other two rotations, forces and moments need to be drawn in those directions, respectively.

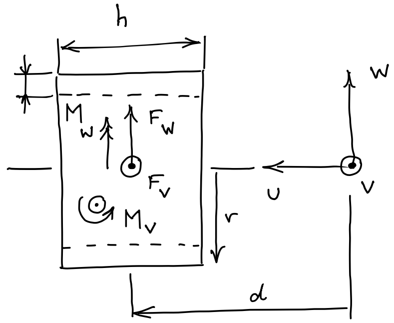



Figure 1.10: Free-body diagram of the ring.

**Problem 1.9** A solid wheel (Figure 1.11) rolls without slipping in a steady circular motion with constant angle $\theta$ and constant ground radius $R$ in positive direction about the $Z$-axis (Figure 1.11). Its center of mass $C$ has speed $v_C$. The wheel is modeled as a thin disk with radius $r$ and mass $m$.

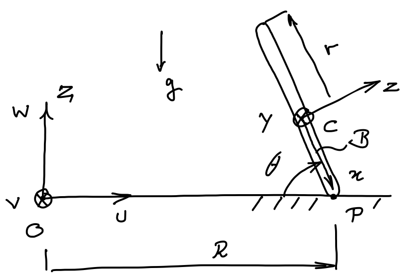



Figure 1.11: A disk rolling in a steady motion.

A coordinate system $uvw$ rotates around an inertial $Z$-axis, such that the disk's current ground contact point $P$ is always on the $u$-axis.

Draw a FBD of the wheel, projected into the $uw$-plane. Use only scalar labels by resolving all force and moment vectors into their components, in such a way that there is a minimal number of labels. ▶

Drawing FBDs often leaves a choice how to apply the principle of equipollence, as reviewed in Section 1.2.2. To indicate that a body can neither translate in one particular direction $x$ nor rotate about an axis $y$ perpendicular to $x$, there are two main choices to represent both constraints:

- Drawing a force in direction of $x$ at an unknown location, and specifying as unknowns the magnitude of the force and a parameter describing its location in $y$-direction. In that case, no additional moment vector needs to be drawn. However, one is making the implicit assumption that the force will never be zero in combination with a nonzero moment, which needs to be verified.
- Letting the force act at a pre-specified location (for example a rigid body's center of mass) and also adding a moment vector in $y$-direction. Then, the two unknowns are the magnitudes of the force and of the moment vector.

The second option (also chosen in Figure 1.10) is more generic and safer in terms of avoiding possible mistakes in assumptions. However, in the case of unilateral contact, where it is not possible to exert a pure moment *without* a force, it is advisable to choose the first option, particularly when aiming for easy possibilities to later check results. Note that this option was also chosen in Figure 1.8.

◎ **Example 1.10** A cone rolls on the ground (Figure 1.12). It has mass $m$, uniform density $\rho$, base radius $R$, height $H$ and center of mass $C$.

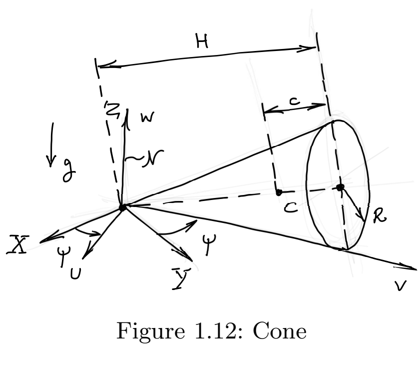



Figure 1.12: Cone

The figure indicates an inertial $XYZ$ coordinate system, and a rotating coordinate system $uvw$. The $v$-direction is always aligned with the cone's ground contact line, and $w$ is always vertical. The cone rolls (possibly with slipping) on the $XY$-plane, and its tip always remains in the origin $O$. It is subjected to gravity with field strength $g$ in negative $Z$-direction.

Draw a free-body diagram (FBD) of the cone projected onto the $vw$-plane. Show distributed loads only by their resultant, and leave out all components of these reactions which are zero.

*Exemplary solution*

A possible FBD for the cone is shown in Figure 1.13.

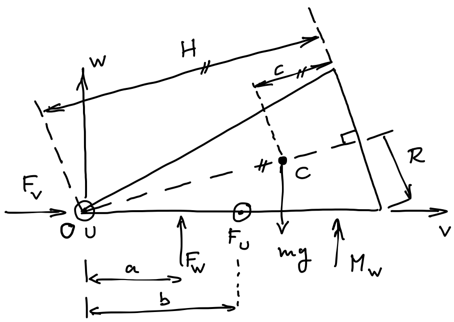



Figure 1.13: Exemplary FBD for the rolling cone.

One may ask why it contains a moment component about the $w$-axis, but not about the other axes. To understand this, one has to consider the type of constraint and use equipollence, as reviewed in Section 1.2.2: The resultant vertical ground reaction force $F_w$, representing the centroid of the distributed load acting on the cone, has been drawn at a (yet unknown) distance $a$ from the origin $O$. This way, there is no additional constraint moment about the $u$-axis to be considered. Alternatively, using equipollence one could draw this resultant force acting in the origin, in which case an additional moment component about the $u$-axis would appear.

The occurrence of a couple moment component about the $u$-axis without any resultant vertical force is impossible given the unilateral contact. Drawing the force at the unknown distance $a$ and solving for $a$ has a clear advantage for results checking: A value for $a$ being negative or larger than $\sqrt{H^2 + R^2}$ would be implausible, because the resultant ground reaction force can only be applied within the base of support of the cone.

A similar relationship holds for the force component in $u$-direction and the moment about the $w$-axis, although here it is possible that a pure (friction) couple moment can be transmitted without any resultant force $F_u$ being present. The resulting redundancy would need to be resolved for a given situation by an additional condition during the calculations. For example, by stating that the couple moment about the $w$-axis is only considered in case $F_u$ resolves to be zero, or by forcing $b$ to be zero.

About the $v$-axis, no moment can be transmitted as none of the potential distributed loads in the line contact has any lever arm.

**Problem 1.10** The cone of Example 1.10 is now at rest. Use the FBD and static equilibrium in that case to calculate the distance $a$ between the cone's tip and its *center of pressure*, so the resultant vertical ground reaction force.

## 1.3 Introduction to Dynamics

### 1.3.1 Kinematics and Kinetics

Dynamics is the branch of mechanics that deals with moving systems. It allows to analyze movements, as well as to determine the relationship between the movement and the forces and moments causing this movement. Accordingly, dynamics is split into two main branches, kinematics and kinetics:

- *Kinematics* is the domain in mechanics that describes motions, without considering the cause of these motions.
- *Kinetics* is the domain in mechanics that finds the relationships between motion and its causes (i.e. forces and moments). From these relationships, we will derive the equations of motion (EoM).

### 1.3.2 Kinematics in Fixed Coordinate Systems

In a fixed (or inertial) Cartesian coordinate system, the position vector $\boldsymbol{r}_P$, velocity $\boldsymbol{v}_P$ and acceleration $\boldsymbol{a}_P$ of a point $P$ are defined as follows:

$$\textit{Position} \qquad \boldsymbol{r}_P := X_P\hat{\boldsymbol{n}}_1 + Y_P\hat{\boldsymbol{n}}_2 + Z_P\hat{\boldsymbol{n}}_3, \tag{1.22}$$

$$\textit{Velocity} \qquad \boldsymbol{v}_P := \dot{\boldsymbol{r}}_P = \dot{X}_P\hat{\boldsymbol{n}}_1 + \dot{Y}_P\hat{\boldsymbol{n}}_2 + \dot{Z}_P\hat{\boldsymbol{n}}_3, \tag{1.23}$$

$$\textit{Acceleration} \qquad \boldsymbol{a}_P := \dot{\boldsymbol{v}}_P = \ddot{\boldsymbol{r}}_P = \ddot{X}_P\hat{\boldsymbol{n}}_1 + \ddot{Y}_P\hat{\boldsymbol{n}}_2 + \ddot{Z}_P\hat{\boldsymbol{n}}_3. \tag{1.24}$$

The vectors $\hat{\boldsymbol{n}}_1 = (1,0,0)^{\mathrm{T}}$, $\hat{\boldsymbol{n}}_2 = (0,1,0)^{\mathrm{T}}$, $\hat{\boldsymbol{n}}_3 = (0,0,1)^{\mathrm{T}}$ are unit vectors in the direction of respectively the inertial $X$-, $Y$- and $Z$-axes. These vectors are constant, their derivatives with respect to time are zero vectors. The scalars $X_P$, $Y_P$ and $Z_P$ are *coordinates* of $\boldsymbol{P}$ in $XYZ$.

Velocity and *speed* are not synonymous. Velocity is a vector with magnitude and direction. Speed is the (signed) magnitude of velocity.

○ **Problem 1.11** Calculate the velocity vector of the center of mass of a system of particles as a function of the particles' individual velocity vectors.[a]

[a] *Hint:* Do this by taking the derivative of (1.11) with respect to time.

◎ **Problem 1.12** A particle's movement is described by its Cartesian position vector as a function of time $t$:

$$\boldsymbol{r}(t) = \begin{pmatrix}\sin(3t/\mathrm{s}) & 3\cos(2t/\mathrm{s}) & \sin^2(2t/\mathrm{s})\end{pmatrix}^{\mathrm{T}} \mathrm{m}. \tag{1.25P}$$

Calculate the particle's acceleration $\ddot{\boldsymbol{r}}(t)$ as a function of time. Use the product and the chain rule, as described in Appendix B.1. Note that you can simplify your result using a trigonometric addition formula. ▷

### 1.3.3 Signs

Clear definition of signs is paramount in obtaining reliable results. Frequently, students believe that a sign mistake is a minor mistake, or that it can be fixed later on by intuition, when the final result appears to be in the wrong direction. Both assumptions are utterly untrue and dangerous for many real-world problems. Therefore, we will spend some time to revise rules for how "positive" and "negative" are determined.

For drawing vectors and labeling them, we already covered some principles in 1.1.3. These are particularly relevant for FBDs.

○ **Example 1.11** The cover of this book shows (almost) complete FBDs of the rear frame and of the handlebar assembly of a bicycle with a special steer-assist system. The action-reaction forces and moments that are drawn *between* the two parts illustrate the difference between vector labels and scalar labels on arrows in terms of signs, as explained in Section 1.1.3. For example, while the reaction to force $\boldsymbol{F}_C$ must be $-\boldsymbol{F}_C$, with a minus sign, the reaction to the motor moment $T_\mathrm{m}$ is drawn in the opposite direction and also labeled with a scalar $T_\mathrm{m}$.

For components of position, velocity and acceleration vectors, special care must be taken to keep definitions consistent, such that indeed $\boldsymbol{v}_P = \dot{\boldsymbol{r}}_P$ and $\boldsymbol{a}_P = \dot{\boldsymbol{v}}_P$, following (1.23) and (1.24).

Sign mistakes can be avoided by following these rules:

1. *Variable* distances or angles are defined by a line or arc with a single arrowhead. The arrowhead ends where the point or line segment is in a generic state. The arrow starts at a clearly defined reference, for example a resting position for a distance, or a vertical or horizontal line for an angle.

2. *Constant* distances or angles are generally measured by lines or arcs, respectively, with arrowheads on *both* sides. This is in fact not a strict rule but a guideline. There are some exceptions, for example when indicatig a radius, following rules of technical drawing. In such cases, only one arrowhead is drawn.
3. Axis names should not be used to also denote dimensions. A new name should be chosen for each specific dimension. For example, the $X$-coordinate of point $P$ should not be called $X$ as well (like the axis direction), a better name is $X_P$.
4. Coordinates, so elements of a position vector, are defined positive in positive axis direction, so if an arrow is to be labeled with a positive coordinate, the arrowhead must be pointing in positive axis direction. In case a point $P$ is drawn at a location having a negative $X$-coordinate, we still follow rule 1, which means that the label of the arrow includes a negative sign, as in $-X_P$.
5. Variables that are *not* coordinates or derivatives of already defined variables may be defined positive in any direction[2]. This includes angles.

Redundant measures are allowed, for example to measure the position of a point with respect to two different references. However, one should be aware of such redundancy to not miscount the number of unknowns and equations for a given problem.

◯ **Example 1.12** In the Kenyan savannah, a lion (simplified as point $L$) and a rhinoceros (point $R$) are approaching a waterhole at a speed $v_L$ and $v_R$, respectively, as depicted in Figure 1.14. An antelope (point $A$) is simultaneously running away from the waterhole at speed $v_A$.

The three animals have the respective distances $d_L, d_R$, and $d_A$ from the waterhole. They are not changing directions but running in straight lines. All the above-mentioned distances and speeds are positive in the given situation.

For now, we focus only on the lion and want to express its position and velocity vectors as functions of the above variables.

*Exemplary solution*

We start by defining an inertial coordinate system. A convenient origin is the center $O$ of the waterhole. As directions, we choose the $X$-axis pointing East, and the $Y$-axis pointing North. So, the right-hand rule dictates that the $Z$-axis points vertically up.

[2] We will see later that variables that describe kinematic configuration without being Cartesian coordinates can still be interpreted as *generalized coordinates*.

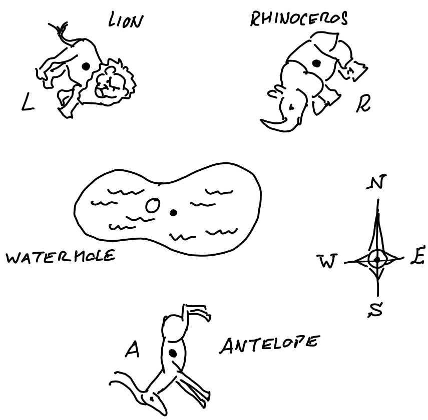



Figure 1.14: A lion, a rhinoceros, and an antelope at a waterhole.

In the first drawing of Figure 1.15, we label the distance $d_L$, using an arrow with its tail connected to the fixed reference $O$, and its arrowhead ending at the variable location of point $L$, the lion, following rule 1 above.

The arrowhead for the angle $\beta$ follows the same rule. We choose to measure the angle from the horizontal in such a way that clockwise movement of the lion leads to an increase in $\beta$. This choice of direction is allowed following rule 5. Even though the animal does not change direction, there is no constraint that keeps it from doing so. So, we choose not to follow rule 2, but to label the angle also with only a singe arrowhead. This also enables later extension of the problem to more generic movement with non-constant angle.

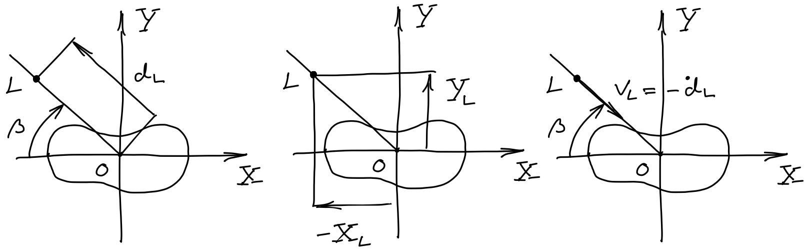



Figure 1.15: Describing kinematics of the lion. Left: Location in terms of distance and angle. Center: Cartesian coordinates. Right: Velocity

In the second drawing, we label the coordinates of the lion, $X_L$ and $Y_L$, following rule 3. Since the lion is on the left of the origin, while $X$ points right, the arrow has to be labeled with a negative sign, following rule 4.

The third drawing shows the lion's velocity. By definition, the derivative of a variable is positive if that variable increases (See Appendix B.1). Therefore, a vector labeled with $\dot{d}_L$ needs to point outward. However, the lion's speed $v_L$ was defined positive for inward movement, so an arrow labeled with this variable needs to be drawn pointing inward.

The position and velocity vectors can be populated with the correct components. For position, we obtain the alternative expressions:

$$\boldsymbol{r}_L = \begin{pmatrix} X_L \\ Y_L \\ Z_L \end{pmatrix} = \begin{pmatrix} -d_L \cos\beta \\ d_L \sin\beta \\ 0 \end{pmatrix} \tag{1.26E}$$

For velocity, there are various alternative expressions, which all need to be consistent regarding their signs:

$$\boldsymbol{v}_L = \begin{pmatrix} v_L \cos\beta \\ -v_L \sin\beta \\ 0 \end{pmatrix} = \dot{\boldsymbol{r}}_L = \begin{pmatrix} \dot{X}_L \\ \dot{Y}_L \\ \dot{Z}_L \end{pmatrix} = \begin{pmatrix} -\dot{d}_L \cos\beta + d_L \cancelto{0}{\dot{\beta}} \sin\beta \\ \dot{d}_L \sin\beta + d_L \cancelto{0}{\dot{\beta}} \cos\beta \\ 0 \end{pmatrix} \tag{1.27E}$$

The drawing in Figure 1.15 also shows that $v_L = -\dot{d}_L$, so this is consistent.

**Problem 1.13** Describing the movement of the rhinoceros and the antelope in Example 1.12 requires careful handling of signs.

First, complete the three drawings in Figure 1.15 with the equivalent variables describing the kinematics of these two animals. Use the angle $\alpha$ for the rhino. There is no need to consider an angle for the antelope.

Then, replace the $\overset{?}{\pm}$ in the below equations by a "+" or a "−"-sign:

The rhino's position vector is:

$$\boldsymbol{r}_R = \begin{pmatrix} \overset{?}{\pm} X_R \\ \overset{?}{\pm} Y_R \\ \overset{?}{\pm} Z_R \end{pmatrix} = \begin{pmatrix} \overset{?}{\pm} d_R \cos\alpha \\ \overset{?}{\pm} d_R \sin\alpha \\ 0 \end{pmatrix} \tag{1.28P}$$

The rhino's velocity vector is:

$$\boldsymbol{v}_R = \begin{pmatrix} \overset{?}{\pm} v_R \cos\alpha \\ \overset{?}{\pm} v_R \sin\alpha \\ 0 \end{pmatrix} = \overset{?}{\pm} \dot{\boldsymbol{r}}_R = \begin{pmatrix} \overset{?}{\pm} \dot{X}_R \\ \overset{?}{\pm} \dot{Y}_R \\ \overset{?}{\pm} \dot{Z}_R \end{pmatrix} = \begin{pmatrix} \overset{?}{\pm} \dot{d}_R \cos\alpha \overset{?}{\pm} d_R \cancelto{0}{\dot{\alpha}} \sin\alpha \\ \overset{?}{\pm} \dot{d}_R \sin\alpha \overset{?}{\pm} d_R \cancelto{0}{\dot{\alpha}} \cos\alpha \\ 0 \end{pmatrix} \tag{1.29P}$$

The antelope's position vector is:

$$\boldsymbol{r}_A = \begin{pmatrix} \overset{?}{\pm} X_A \\ \overset{?}{\pm} Y_A \\ \overset{?}{\pm} Z_A \end{pmatrix} = \begin{pmatrix} 0 \\ \overset{?}{\pm} d_A \\ 0 \end{pmatrix} \tag{1.30P}$$

The antelope's velocity vector is:

$$\boldsymbol{v}_A = \begin{pmatrix} 0 \\ \overset{?}{\pm} v_A \\ 0 \end{pmatrix} \overset{?}{=}\pm \dot{\boldsymbol{r}}_A = \begin{pmatrix} \overset{?}{\pm} \dot{X}_A \\ \overset{?}{\pm} \dot{Y}_A \\ \overset{?}{\pm} \dot{Z}_A \end{pmatrix} = \begin{pmatrix} 0 \\ \overset{?}{\pm} \dot{d}_A \\ 0 \end{pmatrix} \tag{1.31P}$$

▶

◯ **Problem 1.14** For all coordinates $X_L, X_R, X_A, Y_L, Y_R, Y_A$ and their derivatives in Example 1.12 and Problem 1.13, indicate whether their values are positive, zero, or negative in the described situation. ▷

◯ **Problem 1.15** The animals of Example 1.12 and Problem 1.13 are not running at constant speed. The lion and the rhino are decelerating, so reducing their speed, at a rate of $a_L$ and $a_R$, respectively. The antelope is accelerating at a rate of $a_A$. The three rates are all defined positive, so $\{a_L, a_R, a_A\} > 0$.

First, draw the acceleration with the above labels into Figure 1.15.

Second, write down the acceleration vector components as functions of $a_L, a_R, a_A$. Also specify for all variables $\dot{v}_L$, $\dot{v}_R$, $\dot{v}_A$, $\ddot{X}_L$, $\ddot{X}_R$, $\ddot{X}_A$, $\ddot{Y}_L$, $\ddot{Y}_R$, $\ddot{Y}_A$ whether they currently have positive, zero, or negative values.

### 1.3.4 Newton's Second Law of Motion

*Newton's second law of motion* [49] is specifically relevant to the field of dynamics, forming its fundament:

2. *"The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed."*

This means that when a resultant force acts upon an object, it will evoke a change of the object's velocity vector that is proportional in magnitude to the force and that also occurs in the direction in which the force acts. The constant of proportionality is defined as the mass $m$ of the object. So, Newton's second law resolves to the known equation:

$$\sum \boldsymbol{F} = m\dot{\boldsymbol{v}} = m\boldsymbol{a}. \tag{1.32}$$

Departing from the object being a single particle, we will show in the next chapter that the formula also holds for a system of particles or rigid bodies, where $\boldsymbol{a}$ is the acceleration of the CoM of the system.

## 1.4 Solving Problems in Mechanics

### 1.4.1 Question Answering Strategy

For most problems in mechanics, the recommended strategy is to:

- start the answer to each question by defining the strategy, using only symbolic equations. Only fill in details (such as vector components or even numerical values) when really needed to compute/simplify the answer.
- start at the end: With equations that contain the variable *asked for*. Looking at those equations, and identifying which terms are known and unknown, typically works better than trying to find equations that contain parameters *given* in the problem and then hoping to reach the correct answer.
- draw a FBD, even if it is not directly asked for. It will help to precisely understand the question.
- define *all* variables not mentioned in the text explicitly by a figure or algebraically, in order to be fully aware of their meaning.
- include units in all calculations with numerical values.

### 1.4.2 Checking Plausibility

In the end or during the calculations one needs to check if the obtained (intermediate) results are correct. Some specific ideas for checking plausibility and identifying inconsistencies are to:

- use units as a quick check: if units on both sides of an equal sign do not match, the equation is incorrect.
- use clearly separate notations for vectors as a quick sanity check: if a calculation involves division by a vector, or taking the cross product of two scalars, it is incorrect.
- do the same question with a *different coordinate system*.
- check *special cases* where the answer is easier to find, for example when some values or derivatives are 0. An intuitive special case for dynamic systems is often static equilibrium.

- solve the problem *numerically*, e.g. via MATLAB™, and compare results.
- check vector *directions* independent of magnitudes
- check *signs*.
- differentiate a previously integrated term again.
- check dimensions of a variable, e.g. are vectors and scalars added?
- check different approaches (e.g. to determine the equations of motion, this book introduces four different approaches).
- compare to peers and discuss.

◎ **Example 1.13** This Example is about the ring on a rotating bar of Example 1.9 on page 18. A student is uncertain about the correctness of the equation of motion she derived in the body-fixed $u$-direction:

$$F_u = -m(\ddot{d}\sin\psi + 2\dot{d}\dot{\psi}\sin\psi + d\ddot{\psi}\sin\psi + d\dot{\psi}^2\cos\psi), \tag{1.33E}$$

where $F_u$ is the sum of all forces acting on the ring in $u$-direction.

Check the plausibility of this equation with three different, *independent* plausibility checks. Base one of the checks on FBD of the ring. For each check, first state a precise condition you expect to hold for the equation, and a reasoning why you expect this to hold. Only then, perform the check.

*Exemplary solution*

- A first check can be done on units.
  *Prediction and Reasoning*: Units must be identical for any two terms that are added or equated, so all terms in the equation must have the same unit (Note that sine or cosine of an angle have no units.):

  $$[F_u] = [m]\cdot[\ddot{d}] = [m]\cdot[\dot{d}]\cdot[\dot{\psi}] = [m]\cdot[d]\cdot[\ddot{\psi}] = [m]\cdot[d]\cdot[\dot{\psi}^2]. \tag{1.34E}$$

  *Execution and Conclusion*: It is easiest to check with SI units,

  $$[F_u] = \mathrm{N}, [m] = \mathrm{kg}, [d] = \mathrm{m}, [\dot{d}] = \mathrm{m/s}, [\ddot{d}] = \mathrm{m/s^2},$$
  $$[\dot{\psi}] = \mathrm{rad/s} = \mathrm{s^{-1}}, [\ddot{\psi}] = \mathrm{s^{-2}}, [\dot{\psi}^2] = \mathrm{s^{-2}}.$$

  Substituting these units, the left side of (1.34E) is a force with unit N, and all terms on the right side of the equation have the same unit, which also resolves to N (as $\mathrm{N} = \mathrm{kg\,m/s^2}$).
  So, this is plausible.
- A second check can be done for the special case of static equilibrium:
  *Prediction and Reasoning*: Special case: In static equilibrium, so if all derivatives are zero ($\ddot{d} = 0, \dot{d} = 0, \dot{\psi} = 0, \ddot{\psi} = 0$), the sum of forces (so also the component $F_u$) acting on the ring must be zero.

*Execution and Conclusion*: Substituting zeros for all derivatives gives

$$F_u = -0 \cdot \sin\psi - 2 \cdot 0 \cdot 0 \cdot \sin\psi - d \cdot 0 \cdot \sin\psi - d \cdot 0^2 \cdot \sin\psi = 0. \quad (1.35E)$$

This fulfills the prediction and is therefore plausible.

- A third check can be done based on the FBD:
  *Prediction and Reasoning*: The FBD shows that the sum of forces in $u$-direction must always be zero, because there is no friction: $F_u = 0$. This must hold for any physically plausible kinematic parameters.
  *Execution and Conclusion*: A possible combination could be $\psi = 0$, $d > 0$, and $\dot{\psi} > 0$. We do not even need to think about what feasible accelerations and velocities in this case could be, because they do not appear anymore once we substitute $\psi = 0$ in (1.33E):

$$F_u = -md\dot{\psi}^2 < 0. \quad (1.36E)$$

This contradicts the prediction, so (1.33E) must be incorrect.

### 1.4.3 Frequent Mistakes

Some mistakes are particularly frequent, so it is likely that a calculation failed a plausibility check because:

- Coordinate system were not clearly defined, and vectors expressed in different coordinate systems mixed incorrectly. To remedy, it is important to define all used coordinate systems (origin and axis directions) and to *draw them* in a general state.
- Formulae were applied without checking their underlying assumptions. The first step is awareness that many formulae are true for special cases only, and the second is to thoroughly check whether the given problem really falls into that special case.
- Variables were implicitly (and inconsistently) defined. Variable definitions should be made explict.
- Vectors and scalars were not cleanly separated. To remedy, clear notations and typesetting often help.
- Units were omitted throughout calculations, and only later re-generated based on prior assumptions on units of the resulting variable. This hinders the use of units to spot mistakes.
- scalar equations from 2-D dynamics were used, thereby omitting terms that only exist in 3-D.
- out-of-plane reaction forces and moments were omitted in FBDs.

## 1.5 Summary

The importance of making and adhering to proper definitions, particularly of signs, can hardly be overestimated. Lack of explicit definitions is one of the most common sources of uncertainty and mistakes.

Free-body diagrams are the first step to solving most problems in mechanics. An FBD must 1) be in a generic state, and include 2) a coordinate system, 3) all external forces and moments, 4) unique labels for these, and 5) dimensions needed to calculate moments.

When drawing FBDs, actual movement of the system must *not* be considered. The focus should instead be on representing any cut kinematic constraints of the system via forces and moments.

If the line of action passes of a force $\boldsymbol{F}$ passes through point $A$, the moment with respect to point $P$ is:

$$\boldsymbol{M}_P := \boldsymbol{r}_{A/P} \times \boldsymbol{F}, \tag{1.37}$$

With this and the principle of equipollence, equivalent force/moment systems acting on a rigid body can be calculated.

The center of mass with respect to point $A$ of a system of mass $m$, which could be a system of $N$ particles or a rigid body, is defined as

$$\boldsymbol{r}_{C/A} := \frac{\sum_{i=1}^{N} m_i \boldsymbol{r}_{i/A}}{\sum_{i=1}^{N} m_i}\ , \text{ or } \boldsymbol{r}_{C/A} := \frac{\int_m \boldsymbol{r}_{P/A}\, \mathrm{d}m}{\int_m \mathrm{d}m}\ , \text{ respectively.} \tag{1.38}$$

Newton's First Law states that a particle that moves at constant linear velocity $\boldsymbol{v}$ will keep the same velocity unless a force $\boldsymbol{F}$ acts on it.

Newton's Third Law is the law of action and reaction: If an object $a$ exerts a force on another object $b$, then $b$ exerts an equal and opposite force on $a$.

Kinematics is the domain of dynamics that describes motion. Velocity and acceleration of a point $P$ are defined as derivatives of the position vector with respect to time:

$$\boldsymbol{v}_P := \dot{\boldsymbol{r}}_P\,, \quad \boldsymbol{a}_P := \dot{\boldsymbol{v}}_P = \ddot{\boldsymbol{r}}_P. \tag{1.39}$$

In the special case where the position vector components are coordinates in an inertial Cartesian coordinate system, the derivatives can be calculated by taking derivatives of the components.

## 1.6 Problems

### 1.6.1 Guided Problems

**Problem 1.16** A Tippe Top [68] (Figure 1.16) is a special kind of top [12]. When it is spun on its spherical end under the influence of gravity with field strength $g$, it will invert itself and will continue to spin on the small peg.

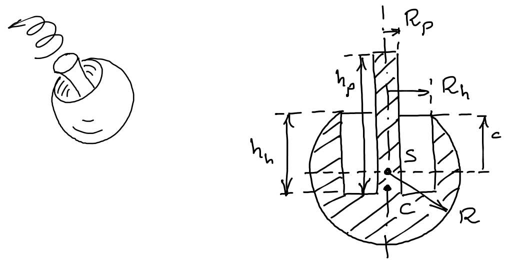



Figure 1.16: Tippe Top.

The top is modeled as shown in the section view. The spherical part has radius $R$ and has a cap cut off at $a$ from the sphere center. The cylindrical hole has radius $R_\mathrm{h}$ and height $h_\mathrm{h}$, and the peg has radius $R_\mathrm{p}$ and height $h_\mathrm{p}$. The complete top has mass $M$.

a. Calculate the position vector $\boldsymbol{r}_{K/C}$ from the center of mass $C$ to the contact point $K$ between ground and top as a function of the position vectors $\boldsymbol{r}_{C/S}$ and $\boldsymbol{r}_{S/K}$. Make a drawing in a suitable 2-D-projection that shows the three vectors and the three points in a generic configuration of the top.
b. Draw a free-body diagram (FBD) of the top. Do this in a generic state, and choose a projection that allows you to draw all vector components in the plane or orthogonal to it. Use only scalar labels. Choose a rotating coordinate system $xyz$ to define your projection. Let the coordinate axis directions define a triad $\mathcal{F}$. Clearly define the origin of your coordinate system, and state whether it translates or not, for general motion of the top. Label all relevant points and dimensions.
c. Use equipollence and the cross product to derive an equivalent system where all forces and moments act in the center of mass of the top. Calculate the components of the resultant force and moment vectors as functions of the given constants $M, g, R$, the distance $d$ between the center $S$ of the sphere and the center of mass $C$ of the top, and the variables you defined in your own FBD. Use your chosen coordinate system, so express all vector components in $\mathcal{F}$.

d. Defining that a solid sphere with a cap of height $(R-a)$ cut off has a remaining volume of $V_s$, and that its center of mass is at a distance $b$ below the sphere's center, calculate the distance $d$ between the center $S$ of the sphere and the center of mass $C$ of the tippe top as a function of $V_s$, $b$, $h_h$, $h_p$, $R_h$, $R_p$, $a$ and $R$.

e. For the volume $V_{\mathrm{s}}$ of a solid sphere that has a cap of height $(R-a)$ cut off, one can use the formula

$$V_{\mathrm{s}} = \tfrac{1}{3}\pi\,(R+a)^2\,(2R-a)\,. \tag{1.40P}$$

Check (1.40P) for plausiblity using at least three different checks, without re-calculating the equation yourself yet.

f. In an unknown triad $\mathcal{G}$, someone calculated three vectors: The gravitational acceleration ${}^{\mathcal{G}}\boldsymbol{g} = (0,\,-g,\,0)^{\mathrm{T}}$, the contact total friction force ${}^{\mathcal{G}}\boldsymbol{F}_{\mathrm{f}} = (F_{\mathrm{f}x},\,0,\,F_{\mathrm{f}z})^{\mathrm{T}}$ acting on the top, and the contact normal force ${}^{\mathcal{G}}\boldsymbol{F}_{\mathrm{n}} = (0,\,F_{\mathrm{n}y},\,0)^{\mathrm{T}}$. Show mathematically that the gravitational acceleration vector is parallel to the normal force vector. Then show that the friction force is perpendicular to the normal force.

g. In dynamics, we often encounter Greek symbols. Someone labels the angle between the peg axis and the vertical $\theta$ (so-called *nutation*), the *spin* angle around the peg axis with $\psi$, and the rotation angle about the vertical axis (so-called *precession*) with $\varphi$. Furthermore, an angular speed is labeled with $\omega$. These letters will be used especially in Chapter 3 and Chapter 4.
How are these Greek lowercase letters pronounced and what are the uppercase symbols? What is the alternative symbol for $\varphi$?

h. Figure 1.17 shows the three angles in three different *orthographic projections* of the tippe top into two-dimensional planes. Four (incompletely drawn) moving coordinate systems are defined: $XYZ$, $x'y'z'$, $x'', y'', z''$, and $x''', y''', z'''$. For each of the projections, apply the right-hand rule to draw all missing axes of the coordinate system $x''y''z''$ that are contained within the drawn plane or orthogonal to it.

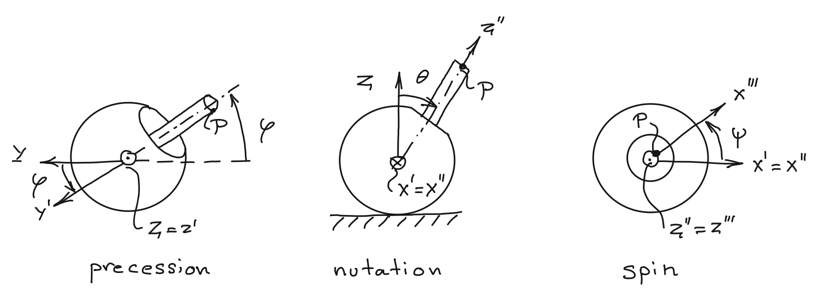



Figure 1.17: Tippe Top projections.

i. A student applies the Newton-Euler method (a technique to be treated in Chapter 6) and derives the equations of motion of the top. One of these equations results to be

$$Md(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta - \ddot{\varphi}\sin\theta - 2\dot{\varphi}\dot{\theta}\cos\theta) = N - Mg, \qquad (1.41P)$$

with variables as defined before, gravitational acceleration $g$, and $N$ the normal force exerted on the top by the ground in upward direction. The student wants to verify the result. Conduct at least three different, independent plausibility checks on this equation.

j. Calculate the volume $V_s$ via integration, and compare your result to (1.40P). Make one or multiple drawings where you include and label *all* variables used in your calculations (including those that are infinitesimally small).

k. Use integration to calculate the distance $b$ as a function of $R$ and $a$. Make again drawings to define all variables.

**Example 1.14** A force $\boldsymbol{F}$ applied at point $P$ causes a moment $\boldsymbol{M}_O = \boldsymbol{r}_{P/O} \times \boldsymbol{F}$ about point $O$ with position vector $\boldsymbol{r}_{P/O}$ (see Figure 1.18). The vectors have components

$$\boldsymbol{M}_O = \begin{pmatrix} M_X \\ M_Y \\ M_Z \end{pmatrix}, \qquad \boldsymbol{r}_{P/O} = \begin{pmatrix} X_P \\ Y_P \\ Z_P \end{pmatrix}, \qquad \boldsymbol{F} = \begin{pmatrix} F_X \\ F_Y \\ F_Z \end{pmatrix}. \qquad (1.42E)$$

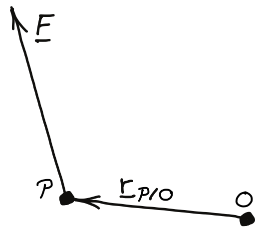



Figure 1.18: Illustration of a force and position vector of its point of application.

Consider two cases:

- Case I: $\boldsymbol{r}_{P/O}$ known, $\boldsymbol{F}$ unknown,
- Case II: $\boldsymbol{r}_{P/O}$ unknown, $\boldsymbol{F}$ known.

a. Show that for both cases we cannot determine the unknown vector unambiguously. Do this by rewriting the cross product to a representation with the tilde matrix and try to solve for the respective unknown vector components. Alternatively, directly use the rank of the tilde matrix to come to a conclusion.

b. For a unique solution, we choose the constraint that the force is perpendicular to the position vector, $\boldsymbol{r}_{P/O} \perp \boldsymbol{F}$. Describe this constraint by a vector equation involving $\boldsymbol{r}_{P/O}$ and $\boldsymbol{F}$.

c. Combine the moment equation with the constraint from part b and rewrite the equations to the form $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$. Do this for both cases, such that $\boldsymbol{x}$ is $\boldsymbol{r}_{P/O}$ or $\boldsymbol{F}$, respectively [a].

d. Now the numerical values of the moment are known:

$$\boldsymbol{M}_O = \begin{pmatrix} 4 & 2 & 1 \end{pmatrix}^{\mathrm{T}} \,\mathrm{N\,m}. \tag{1.43E}$$

Also one of the other vectors' values are known:

- Case I: $\boldsymbol{r}_{P/O} = (1,\, 2,\, 4)^{\mathrm{T}}\,\mathrm{m}$,
- Case II: $\boldsymbol{F} = (-2,\, 6,\, -3)^{\mathrm{T}}\,\mathrm{N}$.

Substitute the values in the expression from part c and solve for the unknown vector to find the numerical values[b].

[a] *Hint*: Matrix **A** should contain (among others) a tilde matrix.
[b] *Hint*: Use MATLAB™'s \-operator.

**Example 1.15** A camera of mass $m$ is suspended on three cables, as shown in Figure 1.19.

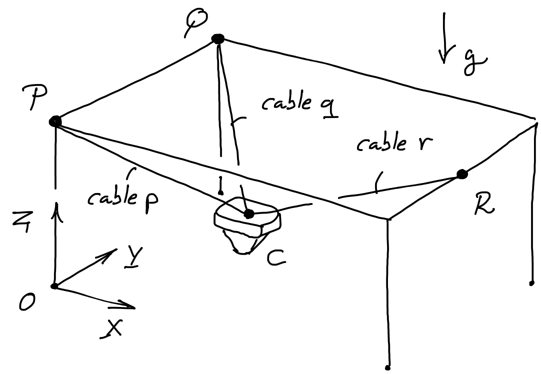



Figure 1.19: Camera suspension system

The position of the camera with respect to the origin $O$ of the $XYZ$ coordinate system is given as $\boldsymbol{r}_{C/O} = \boldsymbol{r}_C = (\frac{2l}{3},\, \frac{w}{3},\, \frac{h}{2})^{\mathrm{T}}$, in the indicated $XYZ$ coordinate system, with constant parameters $l$, $w$, and $h$ describing the dimensions of the suspending frame. The positions of the endpoints of cables p, q and r with respect tot $O$ are given as $\boldsymbol{r}_P = (0,\, 0,\, h)^{\mathrm{T}}$, $\boldsymbol{r}_Q = (0,\, w,\, h)^{\mathrm{T}}$ and $\boldsymbol{r}_R = (l,\, \frac{w}{2},\, h)^{\mathrm{T}}$, respectively. Gravity, with constant $g$, acts in the negative $Z$ direction. Simplify the camera as a particle, and at rest.

We want to determine the force in cable r. To increase reliability of the result, we use two different methods. The first is straightforward but requires more computational effort, the second is more elegant and costs less effort, but requires more insight.

a. Draw the free-body diagram of the camera.
**Method 1**
b. Use vector addition to calculate the vectors $\boldsymbol{r}_{P/C}$, $\boldsymbol{r}_{R/C}$, and $\boldsymbol{r}_{Q/C}$, with components expressed as functions of $l$, $w$, and $h$.
c. Calculate the length of cable p as a function of $l$, $w$, and $h$.
d. Calculate the components of the direction vector $\hat{\boldsymbol{u}}_\mathrm{r}$ in direction of cable r, as functions of the given constants, and such that the vector has a positive $Z$ component. Do the same for the two other cables.
e. Express the force vector $\boldsymbol{F}_\mathrm{r}$ that cable r exerts on point $C$ as a function of $\hat{\boldsymbol{u}}_\mathrm{r}$ and the (yet unknown) magnitude $F_\mathrm{r} = |\boldsymbol{F}_\mathrm{r}|$ of the force in cable r. Do the same for the two other cables.
f. Express the resultant force vector $\Sigma\boldsymbol{F}_\mathrm{c}$ acting on the camera as a function of the cable force magnitudes $F_\mathrm{p} = |\boldsymbol{F}_\mathrm{p}|$, $F_\mathrm{q} = |\boldsymbol{F}_\mathrm{q}|$, $F_\mathrm{r} = |\boldsymbol{F}_\mathrm{r}|$, and the constants $w$, $h$, $l$, $m$ and $g$.
g. Rewrite the vector equation $\Sigma\boldsymbol{F}_\mathrm{c} = \mathbf{0}$ for static equilibrium such that it takes the form $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, with $\boldsymbol{x} = (F_\mathrm{p},\ F_\mathrm{q},\ F_\mathrm{r})^\mathrm{T}$.
h. Solve $\Sigma\boldsymbol{F}_\mathrm{c}$ to determine $F_\mathrm{p}$, $F_\mathrm{q}$, and $F_\mathrm{r}$ as functions of $l$, $w$, $h$, $m$, $g$. The MATLAB™ symbolic toolbox can simplify calculations.
**Method 2**
i. Calculate a unit vector $\hat{\boldsymbol{e}}$ that is orthogonal to cables p and q, as a function of the given constants $l$, $w$, $h$.
j. Determine the projection $\Sigma F_{ce}$ of $\Sigma\boldsymbol{F}_C$ in direction of $\hat{\boldsymbol{e}}$ as a function of $F_\mathrm{p}$, $F_\mathrm{q}$, $F_\mathrm{r}$ and of the given constants. Explain why two of the unknowns do not appear in the function.
k. Use $\Sigma F_{ce} = 0$ to determine $F_\mathrm{r}$ as a function of $l$, $w$, $h$, $m$, $g$.
l. Given the two methods above, how would you generally choose coordinate directions to minimize calculation effort if some force vectors are irrelevant?

## 1.6.2 Practice Problems

○ **Problem 1.17** This Problem is about the top of corks and skewers of Problem 1.8 on page 12. Calculate the vector $\boldsymbol{r}_{A/P}$ as a function of the vector $\boldsymbol{r}_{C/P}$ and the two scalars $l$ and $d$. ▷

◎ **Problem 1.18** Three particles form a T-shaped rigid body floating in space. The particles have mass $m_1, m_2$, and $m_3$, respectively, and are connected by two massless rods (Figure 1.20). There is no gravity. A body-fixed $xyz$ coordinate system is drawn where the $x$-direction is aligned with the long rod of length $L$, and the $y$-direction is aligned with the short rod of length $L/2$. The origin of $xyz$ is in the center of mass of the body. The axis directions define a body-fixed triad $\mathcal{B}$.

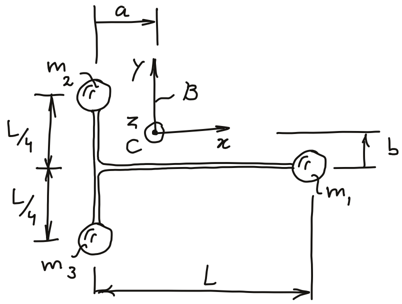



Figure 1.20: T-shaped rigid body consisting of three particles.

Calculate the distances $a$ and $b$ that define the location of the body's center of mass $C$, as functions of the given geometry and the particles' masses. ▷

◯ **Problem 1.19** Which statement(s) make(s) correct use of units for mass $m$, speed $v$, time $t$, and length $h$? Check all that apply.

A. $m =$kg
B. $m =$[kg]
C. $v = 20h\,$m/s
D. $\cos t = 1$
E. None of the above

▷

◎ **Problem 1.20** The thin-walled cylindrical rotor in Figure 1.21, of mass $m$ and radius $R$, is in *static unbalance*. This means that the rotor's center of mass $D$ is not on the $x$-axis. Static unbalance can cause undesired vibrations when the rotor is in motion.

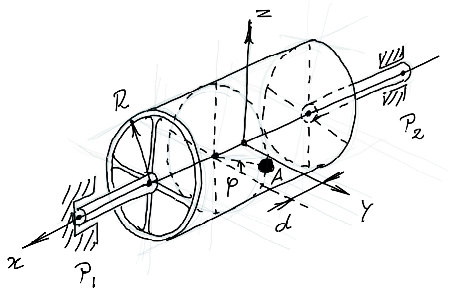



Figure 1.21: Unbalanced rotor.

A weight of mass $m_a$ is to be added to the rotor at a strategic place on the shell of the rotor $A$, described by the angle $\varphi$ and the distance $d$. The weight should compensate for static unbalance, such that the center of mass of the combined body (rotor plus mass $m_a$), with position vector $\boldsymbol{r}_C$, is on the $x$-axis. This is done so that the bearings $P_1$ and $P_2$ do not have to support any net force due to rotation of the rotor.

Point $D$ has the position vector $\boldsymbol{r}_\mathrm{D} = R \cdot (\frac{1}{40}, \frac{1}{50}, \frac{1}{50})^\mathrm{T}$ in the rotor-fixed $xyz$. The weight is considered a particle that is rigidly attached to the rotor.

Determine $m_a$ and *one possible* location $A$ to compensate for static unbalance of the rotor. Provide $m_a, \varphi$, and $d$ as functions of $R$ and $m$. ▶

**Problem 1.21** For all bodies shown below, free-body diagrams are to be drawn. Choose one or multiple projections that allow you to draw components of vectors either contained in or orthogonal to the paper plane, such that there is no ambiguity in direction. Also, use only scalar labels, so split vectors into multiple components when needed.

a. Draw the FBD of a spacecraft (Figure 1.22) of mass $m$ floating in outer space.

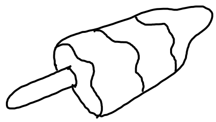


Figure 1.22: Spacecraft.

b. Three gimbal rings are mounted with friction-free rotational joints (Figure 1.23). An external moment about the vertical axis is applied on the outer gimbal ring. Draw the FBD of the *middle* gimbal ring.

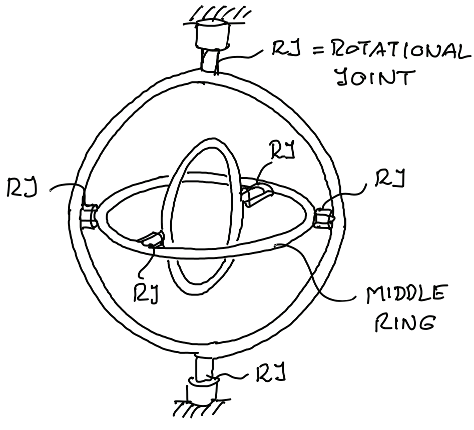



Figure 1.23: Gimbal rings.

c. A grasper is opened and empty at the moment as drawn in Figure 1.24. Each of the grasper's beaks is controlled by an electromotor on its rotational hinge. Draw the FBD of the upper beak.

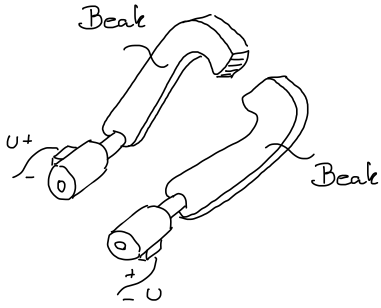



Figure 1.24: Grasper.

d. Draw the FBD of a piston as shown in Figure 1.25.

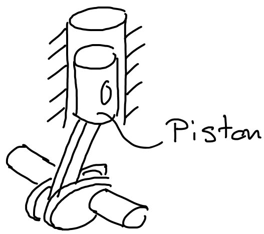



Figure 1.25: Piston.

e. A trombone is a brass musical instrument. The player can regulate the pitch by sliding a telescoping mechanism to a position (see Figure 1.26). Draw the FBD for the sliding element when the player is playing the instrument.

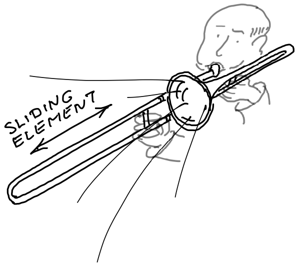
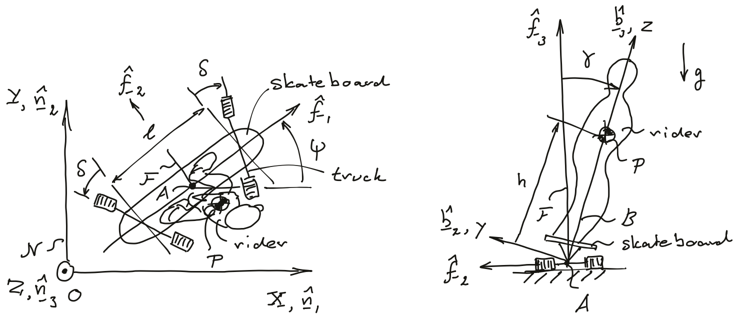



Figure 1.26: Trombone.

◎ **Problem 1.22** A person is riding a skateboard (Figure 1.27). The rider can steer the skateboard by leaning sideways over a tilt angle $\gamma$.




(top view) (rear view)

Figure 1.27: A person riding a skateboard.

The rider is modeled as rigidly attached to the board and positioned at the center of the board $A$.

The arrangement of the *trucks* (hinged axles), which connect the wheels to the board, are such that there is a fixed kinematic relation between the tilt angle $\gamma$ of the board and the steer angle of the front and rear truck $\delta$ (Figure 1.27). For small angles, this kinematic relation can be approximated by $\delta = k\gamma$, were the constant $k$ only depends on the fixed geometric arrangement of pivot axes of the trucks with respect to the skateboard.

The wheels roll and do not slip in either lateral or longitudinal direction[a]. The distance between the centers of the wheel axles is $l$. The distance between $A$ and $P$ is $h$. We neglect friction in the bearings.

A person-fixed triad $\mathcal{B}$ is defined such that $\hat{\boldsymbol{b}}_3$ points upwards from the feet to the head, $\hat{\boldsymbol{b}}_1$ points in direction of the nose (in anatomical terms, the *anterior* direction), and $\hat{\boldsymbol{b}}_2$ points sideways towards person's left.

The inertial $XYZ$-coordinate system has directions of $\mathcal{N}$ and origin $O$.

An intermediate triad $\mathcal{F}$ describes the motion of the skateboard on the plane. The unit vector $\hat{\boldsymbol{f}}_1$ points in the direction of the forward speed of the board, and $\hat{\boldsymbol{f}}_3$ is aligned with the inertial triad vertical $\hat{\boldsymbol{n}}_3$.

Gravity with acceleration constant $g$ acts in negative $Z$-direction. The ground is flat.

Draw a FBD of the skateboard including the rigid rider in a top view, like the left picture in Figure 1.27.

Use only scalar labels for your vectors. Also include all dimensions that would be needed to calculate relevant moments for Euler's second law. Do not actually calculate the moments. ▶

[a]This is not fully realistic; to enable turning with wheels of finite width, there has to be local slip

◯ **Problem 1.23** This Problem is about the skateboarder of Problem 1.22 on page 39. Which statement(s) is/are always true about the (relative) position vectors $\boldsymbol{r}_{P/A}, \boldsymbol{r}_{P/O}, \boldsymbol{r}_{A/O}$? Check all that apply.

A. $\boldsymbol{r}_{P/A} = \boldsymbol{r}_{P/O} - \boldsymbol{r}_{A/O}$
B. $\boldsymbol{r}_{P/A} \times \boldsymbol{r}_{A/O} = \mathbf{0}$
C. $\boldsymbol{r}_{P/A} \cdot \boldsymbol{r}_{A/O} = 0$
D. None of the above

▷

**Problem 1.24** A simple 3-axis accelerometer (Figure 1.28) consists of a mass that is elastically suspended within a cage. This cage is attached to a rotating and translating object. Acceleration of the mass is related to deformations of the tension springs, which can be used to measure the object's movements. However, the springs also deform due to gravity acting on the mass.

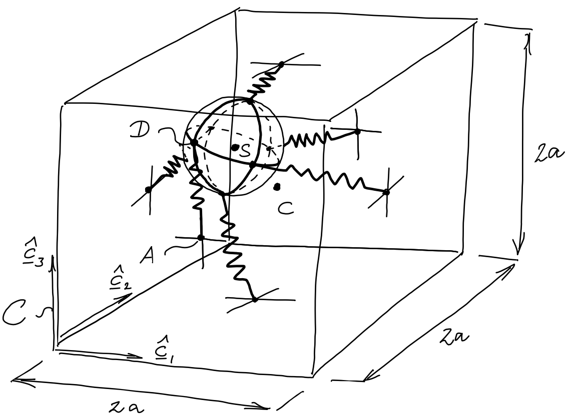



Figure 1.28: An accelerometer, consisting of a spherical mass suspended in a (transparent) cube by six springs.

We model the cage as a massless hollow cube with center $C$ and edge length $2a$. The cage-fixed triad $\mathcal{C}$ has the direction vectors $\hat{\boldsymbol{c}}_1, \hat{\boldsymbol{c}}_2, \hat{\boldsymbol{c}}_3$.

We model the mass as a rigid homogeneous sphere of mass $m$, radius $R$, and center $S$. The sphere-fixed triad $\mathcal{K}$ has the direction vectors $\hat{\boldsymbol{k}}_1, \hat{\boldsymbol{k}}_2, \hat{\boldsymbol{k}}_3$. When all springs have equal length, triad $\mathcal{C}$ and triad $\mathcal{K}$ are the same.

Point $O$ is the origin of the inertial axes $X, Y, Z$, which define triad $\mathcal{N}$.

Gravity with acceleration constant $g$ acts in negative $Z$-direction.

Draw FBDs of the cage and of the sphere, assuming that the cage is rigidly attached to another body at its face that contains point $A$. Here, we consider the case where there is *no* gravity. For the FBD, you may make the simplification that the springs are very stiff, such that their directions do not change with respect to the cage. Remark: Do *not* make this simplification in later uses of this problem. Use only scalar labels for your vectors. Show only *orthographic projections*, where all forces and moments are either contained in the paper plane or orthogonal to it. ▶

**Problem 1.25** This Problem is about the accelerometer of Problem 1.24 on page 41. The spring between points $A$ and $D$ has linear characteristics, with spring constant $k$. When no force acts on it, the spring has length $l_n$, and in static equilibrium (neglecting gravity), the spring has a different length $l_p$, because all springs are in pre-tension.

Calculate the force vector that this spring exerts on the sphere, as a function of the vector $\boldsymbol{r}_{D/A}$ and the constants $k, l_p, l_n$. ▷

**Problem 1.26** This Problem is about the accelerometer of Problem 1.24 on page 41. The accelerometer is mounted on a robot arm, which is subjected to the gravitational field of Earth, with acceleration constant $g$ acting in negative $Z$-direction. The sensors measure that the six springs apply a resultant spring force of $\boldsymbol{F}$ on the cage.

The sphere's center has an acceleration $\boldsymbol{a}_S$ of (check all that apply):

A. $\boldsymbol{a}_S = -\boldsymbol{F}/m - g\hat{\boldsymbol{n}}_3$
B. $\boldsymbol{a}_S = -\boldsymbol{F}/m + g\hat{\boldsymbol{n}}_3$
C. $\boldsymbol{a}_S = -\boldsymbol{F}/m$
D. $\boldsymbol{a}_S = \boldsymbol{F}/m - g\hat{\boldsymbol{n}}_3$
E. $\boldsymbol{a}_S = \boldsymbol{F}/m + g\hat{\boldsymbol{n}}_3$
F. $\boldsymbol{a}_S = \boldsymbol{F}/m$
G. None of the above

▷

**Problem 1.27** This Problem is about the accelerometer of Problem 1.24 on page 41. The accelerometer is simplified and used in a one-dimensional movement. It is oriented as shown in the kinematic diagram in Figure 1.29, and it does not change this special-case orientation. The coordinates $Z_C$, $x_S$, and $d$ are defined in the figure.

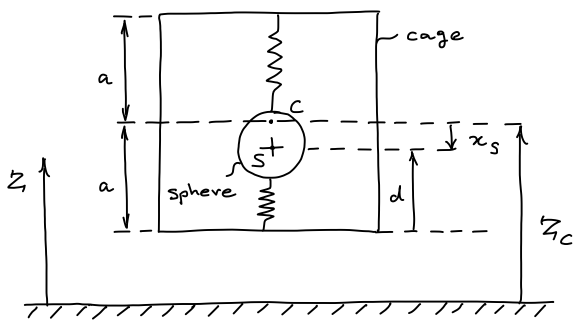



Figure 1.29: Kinematic diagram of simplified accelerometer in 1-D.

Someone already drew a FBD and determined that the sum of external forces acting on the mass in upward direction $Z$ is $\Sigma F_u$. The following equation(s) is/are correct (Check all that apply):

A. $\Sigma F_u = m\ddot{x}_S$
B. $\Sigma F_u = m\ddot{Z}_C$
C. $\Sigma F_u = 0$
D. $\Sigma F_u = m\ddot{d}$
E. $\Sigma F_u = -m\ddot{d}$
F. None of the above

▷

**Problem 1.28** Re-Do Example 1.12 with a North-East-Down (NED) coordinate system, where the $X$-axis points North and the $Y$-axis points East. Focus again on the signs and conduct plausibility checks.

**Problem 1.29** A person is performing tricks on a scooter with springboard. Figure 1.30 shows the system in 3-D(left), and a detailed sideview of the scooter (right).

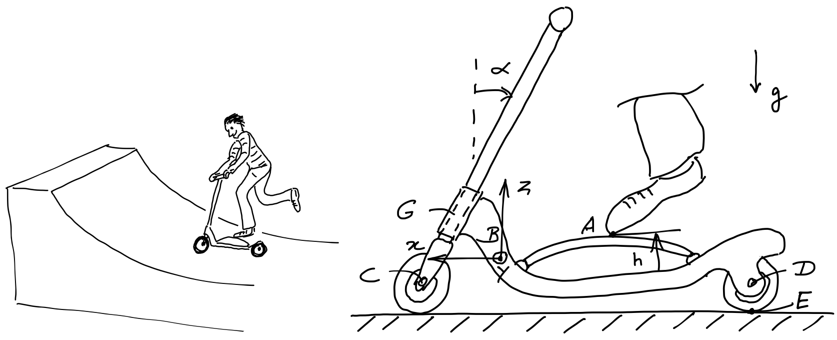



Figure 1.30: Springboard scooter.

The scooter consists of one massless leafspring and four rigid bodies that all have mass and inertia: Two wheels of equal radius and mass $m_{\mathrm{w}}$, a footboard of mass $m_{\mathrm{f}}$, and a steering handle. A local $xyz$ coordinate system is connected to the footboard and has its origin in the footboard's center of mass $B$.

The four bodies are all connected to each other by friction-free hinges: The wheels are connected to the footboard such that they can only rotate about their respective axles $C$ and $D$, which are parallel to the $y$-axis and pass through the centers of mass of the wheels. The steering handle is connected to the footboard by a hinge $G$ that allows only rotation about an axis that is inclined by an angle $\alpha$ with respect to the $z$-axis.

The scooter's wheels roll on the ground without slip and have only point contact, the rear wheel in point $E$. The person is touching the leafspring only with one toe, idealized as one point $A$. The person also holds the handle of the scooter.

Gravity acts in the downward direction with field strength $g$.

Draw the FBDs of two systems, namely of the rear wheel and of a system consisting of the footboard and the leafspring connected to each other. Do this by completing Figure 1.31, which already contains the outlined shapes of these two systems in a projection into the $xz$-plane.

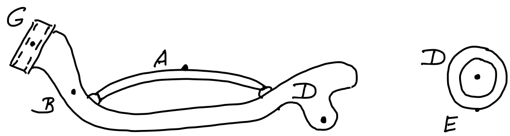



Figure 1.31: Scooter's footboard and rear wheel as free bodies.

Represent distributed loads by their resultant and split vectors into their components such that you use only scalar labels for all drawn arrows. Remember to use clear, unique labels. ▶

# 2 Particle Dynamics in Fixed Coordinate Systems

This chapter will deal with dynamics of particles and systems of particles using fixed coordinate systems[1], which forms the basis for the later analysis of more complex systems with rotating reference systems.

The Newton-Euler approach is explained for particles and systems of particles in Section 2.1 (The principle will be extended to rigid bodies in Chapter 6). The principle of work and energy is explained for particles and systems of particles in Section 2.2 (It will be extended to rigid bodies in Chapter 5).

## 2.1 The "Newton-Euler" Approach

This section reviews a fundamental approach to derive the equations of motion of particles and systems of particles.

### 2.1.1 Linear Momentum

The right side of Newton's second law of motion (1.32) can also be rewritten using *linear momentum*. Linear momentum is defined as the product of mass $m$ and velocity $\boldsymbol{v}$. Note that *momentum* is not the same as *moment*. For a particle $i$, located at $\boldsymbol{r}_i$, having mass $m$ and velocity $\boldsymbol{v}_i$, the linear momentum $\boldsymbol{p}_i$ is defined as

$$\boldsymbol{p}_i := m_i \boldsymbol{v}_i = m_i \dot{\boldsymbol{r}}_i. \tag{2.1}$$

For a system of $N$ particles, the individual linear momenta $\boldsymbol{p}_i$ according to (2.1) can be summed to an overall vector $\boldsymbol{p}$ of the system:

$$\boldsymbol{p} := \sum_{i=1}^{N} \boldsymbol{p}_i. \tag{2.2}$$

[1] In this book, only right-handed coordinate systems are used, and we recommend the reader to do the same to ensure that all formulae are valid.

### 2.1.2 Principle of Linear Impulse and Momentum

Newton's second law (1.32) states that the force $\boldsymbol{F}_i$ on a particle equals the rate of change of the particle's linear momentum:

$$\boldsymbol{F}_i = \dot{\boldsymbol{p}}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \boldsymbol{v}_i\right). \tag{2.3}$$

If the particle is part of a system of particles, we can apply summation to both sides of the equation. Because the sum of internal forces is zero (see (1.16)), Newton's second law of motion shows that the system's overall rate of change of the linear momentum $\boldsymbol{p}$ with respect to time equals the sum of the *external* forces acting on it:

$$\sum_{i=1}^{N} \boldsymbol{F}_i = \sum_{i=1}^{N} \dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}. \tag{2.4}$$

As a special case, this implies that if there are no external forces acting on a mechanical system in a particular direction, the system's linear momentum in this direction remains constant. Using the definition (1.11) of the center of mass of a system of particles and the definition $m := \sum_{i=1}^{N} m_i$, we see that for constant masses $m_i$,

$$\sum_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_i = m\ddot{\boldsymbol{r}}_C = m\boldsymbol{a}_C. \tag{2.5}$$

Combined with (2.4) and (2.3) for mass $m$, this yields

$$\sum_{i=1}^{N} \boldsymbol{F}_i = \sum_{i=1}^{N} m_i \boldsymbol{a}_i = m\boldsymbol{a}_C. \tag{2.6}$$

So, Newton's second law tells us that the center of mass of a system of particles will accelerate as if the entire mass of the system was concentrated in it and all external forces were acting on it.

When integrated over time between times $t_0$ and $t_\mathrm{f}$, Newton's Second Law is called the *principle of linear impulse and momentum*:

$$\int_{t_0}^{t_\mathrm{f}} \sum_{i=1}^{N} \boldsymbol{F}_i \mathrm{d}t = \boldsymbol{p}(t_\mathrm{f}) - \boldsymbol{p}(t_0) = \Delta\boldsymbol{p}. \tag{2.7}$$

Hereby, *impulse* is defined as the integral of force over time. So, the principle states that the increment in linear momentum is equal to the impulse of the resultant force that was applied during the same time interval. This form of the relationship is particularly relevant for collisions.

We saw in this section that internal forces cannot effect a change in linear momentum of the system as a whole. Note that this does not mean that internal forces cannot have any effect. On the contrary, if you look *inside* the system, at relative movement of particles, internal forces *can* change motion.

> **Example 2.1** Two persons on skates are standing on a frictionless surface, each holding the end of a connecting rope in their hands. They can pull themselves towards each other by means of this rope, so the internal forces in the rope do have an effect. However, the skaters do not change the position of their combined center of mass.

> **Problem 2.1** A cannon of mass $M$ stands at rest on a frictionless surface. Then, it shoots a ball of mass $m$. Assuming the relative velocity of the ball with respect to the cannon has a magnitude of $v_\mathrm{r}$, what is the resulting rebound speed of the cannon?[a]
>
> [a] *Hint*: After defining a coordinate system, determine the velocity of the center of mass of the combined system consisting of ball and cannon, using Newton's Second Law. Then, re-use your result from Problem 1.11 to relate this velocity to the individual velocities of ball and cannon.

### 2.1.3 Angular Momentum of Particles

The previous subsection showed the relation between the sum of forces on the center of mass of a system of particles and linear momentum of this system. The following subsections will show the relation between the sum of moments acting on a system and a quantity that, intuitively speaking, describes the amount of rotational movement of a mechanical system: *Angular momentum*.

Angular momentum of a particle $i$ about a reference point $Q$ can be seen

First, we consider only the important special case in which the reference point $Q$ is fixed, denoted by $O$. The angular momentum $\boldsymbol{H}_{O,i}$ of a particle about $O$, is defined as the cross product of the particle's position vector $\boldsymbol{r}_i$ with respect to point $O$, and the particle's linear momentum vector $\boldsymbol{p}_i$:

$$\boldsymbol{H}_{O,i} := \boldsymbol{r}_i \times \boldsymbol{p}_i = \boldsymbol{r}_i \times (m_i \dot{\boldsymbol{r}}_i), \tag{2.8}$$

For a general point $Q$, the angular momentum $\boldsymbol{H}_{Q,i}$ of a particle about $Q$, is defined here as:

$$\boldsymbol{H}_{Q,i} := \boldsymbol{r}_{i/Q} \times (m_i \dot{\boldsymbol{r}}_{i/Q}). \tag{2.9}$$

Note that we are using here a relative and not an absolute velocity in the definition. If $Q$ is a fixed point, the definitions are equivalent.

For a system of particles, the particles' individual angular momenta are summed to:

$$\boldsymbol{H}_Q := \sum_{i=1}^{N} \boldsymbol{H}_{Q,i}. \tag{2.10}$$

**Example 2.2** In some machines the power is controlled by a so-called "*centrifugal governor*". The original type, introduced by *James Watt*, is shown in Figure 2.1.

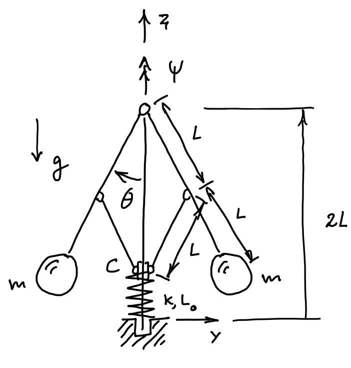



Figure 2.1: Centrifugal governor.

The central spindle, to which the arms carrying the two balls are hinged, rotates at a rate proportional to the speed of the engine. When this rotation rate is constant, the balls, under the action of gravity, a spring force, and centrifugal "forces", will converge to a definite "equilibrium" position that depends on the rotational speed. If the speed increases, the balls diverge outwards, raising the collar $C$ to which some system of levers can be connected, which can turn a valve so as to reduce the power supply. Conversely, when the speed diminishes, the collar descends, and the supply is reinforced.

The inclination of the two identical upper arms to the spindle is denoted by the angle $\theta$, the rotation of the spindle by the angle $\psi$. The rotating coordinate system $xyz$ with associated triad $\mathcal{F}$ is defined in such a way that the balls are always contained in the $yz$-plane and $z$ is pointing upward.

The balls are modeled as particles of mass $m$ and all other parts of the mechanism as massless bodies. The length of the arms carrying the balls is $2L$ and the length of the rods connecting the arms with the collar is $L$. Between the collar $C$ and the fixed base, a linear spring with spring stiffness $k$ and unstressed length $L_0$ is attached. Gravity acts in the downward direction with field strength $g$.

The left ball has the position vector ${}^{\mathcal{F}}\boldsymbol{r}_1 = (0,\, -2L\sin\theta,\, 2L(1-\cos\theta))^{\mathrm{T}}$ and the velocity vector ${}^{\mathcal{F}}\dot{\boldsymbol{r}}_1 = (2L\dot{\psi}\sin\theta,\, -2L\dot{\theta}\cos\theta,\, 2L\dot{\theta}\sin\theta)^{\mathrm{T}}$.

The angular momentum component $H_z$ of the system about the $z$-axis can be written as $H_z = P\dot{\theta} + Q\dot{\psi}$. Determine $P$ and $Q$ as functions of the parameters $m$, $L$, $g$, $k$, $L_0$ and the angle $\theta$.

*Exemplary solution*

The angular momentum component about the $z$-axis is exactly the same for both particles, because of symmetry. So, angular momentum $H_z$ can be calculated from the third component of the left particle's angular momentum vector $\boldsymbol{H}_{O1}$ about the origin $O$ of the $xyz$ coordinate system:

$$H_z = 2\boldsymbol{H}_{O1} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.11E}$$

The angular momentum of the particle with respect to $O$ is the cross product of its position vector $\boldsymbol{r}_1 = \boldsymbol{r}_{1/O}$ and its linear momentum, so:

$$\begin{aligned}
{}^{\mathcal{F}}\boldsymbol{H}_{O1} &= {}^{\mathcal{F}}\boldsymbol{r}_1 \times (m\,{}^{\mathcal{F}}\dot{\boldsymbol{r}}_1) && (2.12\text{E}) \\
&= \begin{pmatrix} 0 \\ -2L\sin\theta \\ 2L\,(1-\cos\theta) \end{pmatrix} \times m \begin{pmatrix} 2L\dot{\psi}\sin\theta \\ -2L\dot{\theta}\cos\theta \\ 2L\dot{\theta}\sin\theta \end{pmatrix} && (2.13\text{E}) \\
\Rightarrow H_z &= 2\cdot 4mL^2\dot{\psi}\sin^2\theta && (2.14\text{E}) \\
&= \underbrace{0}_{P}\cdot\dot{\theta} + \underbrace{8mL^2\sin^2\theta}_{Q}\,\dot{\psi}. && (2.15\text{E})
\end{aligned}$$

Comparing coefficients gives:

$$P = 0, \tag{2.16E}$$

$$Q = 8mL^2\sin^2\theta. \tag{2.17E}$$

◯ **Problem 2.2** A wheel consists of 20 particles of equal mass that are uniformly distributed around the circumference of the otherwise massless wheel. The wheel rotates about its rotational symmetry axis, which is fixed in space. In which direction does the vector of angular momentum of the wheel, with respect to its fixed center, point? Use the cross product in your reasoning. Does your answer change if one of the particles is heavier than the others or if the arrangement is not uniform anymore?

A second important special reference point $Q$ is the system's center of mass $C$. A convenient property of this point is that angular momentum can be calculated either using absolute velocities $\dot{\boldsymbol{r}}_i$ or relative velocities $\dot{\boldsymbol{r}}_{i/C}$, the result is the same (see Problem 2.3):

$$\boldsymbol{H}_C = \sum_{i=1}^{N} (\boldsymbol{r}_{i/C} \times (m_i \dot{\boldsymbol{r}}_{i/C})) = \sum_{i=1}^{N} (\boldsymbol{r}_{i/C} \times (m_i \dot{\boldsymbol{r}}_i)). \tag{2.18}$$

**Problem 2.3** Prove the equivalence of the definitions in (2.18). Make use of vector addition and of the definition of the center of mass of a system of particles. ▶

Knowing a system's angular momentum with respect to its center of mass, it is often still desirable to calculate angular momentum with respect to a different reference point. Helpful reference points are often fixed points. Another frequent case is that the system is part of a larger system, and one wants to calculate the overall system's angular momentum with respect to its overall center of mass.

In such cases, we can make use of the relationship (see Problem 2.5 for a proof) for a (sub)system of mass $m$ and center of mass $C$:

$$\boldsymbol{H}_Q = \boldsymbol{H}_C + \boldsymbol{r}_{C/Q} \times \left(m \dot{\boldsymbol{r}}_{C/Q}\right). \tag{2.19}$$

**Problem 2.4** Check basic plausibility of (2.19) by looking at these two trivial special cases: a) The system's center of mass C coincides with point $Q$, b) The subsystem consists of one single particle[a].

[a] *Hint*: What is $\boldsymbol{H}_C$ for a single particle?

**Problem 2.5** Prove (2.19). Make use of vector addition and of the definition of the center of mass of a system of particles, and of (2.9), (2.10) and (2.18). ▶

### 2.1.4 Euler's Second Law of Motion

We recall that Newton's Second Law for one particle states of mass $m_i$ and position vector $\boldsymbol{r}_i$ states that

$$m_i\ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i + \sum_{i=1}^{N} \boldsymbol{f}_{ij}, \tag{2.20}$$

with external resultant force $\boldsymbol{F}_i$ and internal forces $\boldsymbol{f}_{ij}$. The main step to derive Euler's Second Law is to apply the cross product on both sides:

$$\boldsymbol{r}_{i/Q} \times (m_i\ddot{\boldsymbol{r}}_i) = \boldsymbol{r}_{i/Q} \times \left(\boldsymbol{F}_i + \sum_{i=1}^{N} \boldsymbol{f}_{ij}\right), \tag{2.21}$$

If we sum over all particles, we can simplify the right side, because we saw in (1.21) in Section 1.2.7 that internal forces cause no net moments:

$$\sum_{i=1}^{N} \big(\boldsymbol{r}_{i/Q} \times (m_i\ddot{\boldsymbol{r}}_i)\big) = \sum_{i=1}^{N} \big(\boldsymbol{r}_{i/Q} \times \boldsymbol{F}_i\big) = \boldsymbol{M}_Q, \tag{2.22}$$

We now look for a relationship between the rate of change of angular momentum and the left side of this equation.

To that end, we apply the product rule to obtain the time derivative of $\dot{\boldsymbol{H}}_Q$ in (2.10), using the definition (2.9). We assume constant masses $m_i$, so we obtain

$$\dot{\boldsymbol{H}}_Q = \sum_{i=1}^{N} \dot{\boldsymbol{H}}_{Q,i} = \sum_{i=1}^{N} \big(\dot{\boldsymbol{r}}_{i/Q} \times (m_i\dot{\boldsymbol{r}}_{i/Q}) + \boldsymbol{r}_{i/Q} \times (m_i\ddot{\boldsymbol{r}}_{i/Q})\big). \tag{2.23}$$

The cross product of a vector with itself is zero, so (2.23) simplifies to

$$\dot{\boldsymbol{H}}_Q = \sum_{i=1}^{N} \big(\boldsymbol{r}_{i/Q} \times (m_i\ddot{\boldsymbol{r}}_{i/Q})\big). \tag{2.24}$$

This is not yet the term on the right side of (2.22). However, we can apply vector addition,

$$\ddot{\boldsymbol{r}}_i = \ddot{\boldsymbol{r}}_{i/Q} + \ddot{\boldsymbol{r}}_Q, \tag{2.25}$$

and the definition of the center of mass

$$\sum_{i=1}^{N} m_i\boldsymbol{r}_{i/Q} = m\boldsymbol{r}_{C/Q} \tag{2.26}$$

to obtain Euler's Second Law for a generic point $Q$:

$$\dot{\boldsymbol{H}}_Q + \boldsymbol{r}_{C/Q} \times (m\ddot{\boldsymbol{r}}_Q) = \boldsymbol{M}_Q. \tag{2.27}$$

So, for a generic point of reference, the rate of change of a system's angular momentum is not always equal to the external moments applied. However, there are two important special cases where it is: First, point $Q$ can be chosen as a non-accelerating point $O$, which means $\ddot{\boldsymbol{r}}_Q = \boldsymbol{0}$. In that case:

$$\dot{\boldsymbol{H}}_O = \boldsymbol{M}_O. \tag{2.28}$$

The second special case is point $Q$ being the system's center of mass $C$, because then $\boldsymbol{r}_{C/Q} = \boldsymbol{0}$:

$$\dot{\boldsymbol{H}}_C = \boldsymbol{M}_C. \tag{2.29}$$

Note that a practically less relevant third special case is given if the vectors $\boldsymbol{r}_{C/Q}$ and $\ddot{\boldsymbol{r}}_Q$ are aligned.

This set of three equations, in either of the forms (2.27), (2.28), or (2.29), also called *Euler's second law of motion*, describes rotational movement of a system of particles. Combined with Newton's law for a system of particles (2.6), the resulting six equations form a set of equations of motion for the combined movement of the particles of the system in 3-D: One equation per dimension for linear motion, and one equation per dimension for rotational motion.

Remember that the simple equality between rate of change of angular momentum and external moments, (2.28) or (2.29), which is also called Euler's second law of motion, only holds in two special cases: (1) the reference point $Q$ is a fixed (non-accelerating) point $O$, or (2) the reference point $Q$ is the center of mass $C$.

Also remember that internal forces cannot cause net moments, and therefore they cannot effect a change in angular momentum of the system as a whole. Therefore, when calculating $\boldsymbol{M}_O$ or $\boldsymbol{M}_C$ in (2.28) or (2.29), it suffices to sum only all *external* moments acting on the system.

In analogy to the principle of linear impulse and momentum (see Section 2.1.1), Euler's Second Law when integrated over time is called the *principle of angular impulse and momentum*. Hereby, *angular impulse* is defined as the integral of the moment over time, and it determines angular momentum change during the same time interval:

$$\int_{t_0}^{t_\mathrm{f}} \boldsymbol{M}_C \mathrm{d}t = \boldsymbol{H}_C(t_\mathrm{f}) - \boldsymbol{H}_C(t_0) = \Delta \boldsymbol{H}_C. \tag{2.30}$$

○ **Problem 2.6** For a system of particles where the only external forces are gravitational forces, what can you say about the system's rate of change of angular momentum with respect to its center of mass?

○ **Problem 2.7** A cat is falling off a building, back first. The cat manages to rotate its body and safely lands on its feet. Ignore air drag. Is it correct to state that the cat used its muscle forces to change its angular momentum with respect to its center of mass during the fall? Why/why not? ▷

○ **Problem 2.8** Two particles 1 and 2 are connected by a rigid rod and form one system. In scenario a), an external force of magnitude $F$ is applied to particle 1. In an alternative scenario b), a force of the same direction and magnitude is applied to particle 2 instead.

- Will the acceleration of the system's center of mass differ between the two scenarios?
- Will the system's rate of change of angular momentum with respect to its center of mass differ?
- Will the system's rate of change of angular momentum with respect to a fixed point differ?

Show why for each answer. Can you imagine special cases where the answer to the above three questions is always "No"? Can you imagine special cases where the answer to the above three questions is always "Yes"?

## 2.2 Work and Energy

Another way to describe the kinetics of (a system of) particles is using *work and energy*. This will be introduced in this section and discussed more in Chapter 5.

### 2.2.1 Power and Work of a Force

The instantaneous *power* $P$ of a force $\boldsymbol{F}$ is defined as the scalar product of $\boldsymbol{F}$ and the velocity $\boldsymbol{v}$ of its point of application:

$$P := \boldsymbol{F} \cdot \dot{\boldsymbol{r}} = \boldsymbol{F} \cdot \boldsymbol{v}. \tag{2.31}$$

Power has no direction; it is a scalar, not a vector.

To calculate the *work* done on a particle by the application of a force between times $t_0$ and $t_1$, power is integrated over time:

$$W := \int_{t_0}^{t_1} P \, \mathrm{d}t = \int_{t_0}^{t_1} \boldsymbol{F} \cdot \boldsymbol{v} \, \mathrm{d}t. \tag{2.32}$$

Like power, work is a scalar quantity. As $\mathrm{d}\boldsymbol{r} = \boldsymbol{v}\,\mathrm{d}t$, work $W$ of a force $\boldsymbol{F}$ is the integral of $\boldsymbol{F}$ with respect to the path:

$$W = \int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}, \tag{2.33}$$

where the point of application travels from point $A$ to point $B$, with infinitesimal position vector increments $\mathrm{d}\boldsymbol{r}$.

That means a force $\boldsymbol{F}$ can only do work $W$ on a mechanical system if there is a displacement of the force vector's point of application and when the dot product of $\boldsymbol{F}$ and $\mathrm{d}\boldsymbol{r}$ is not always zero between times $t_0$ and $t_1$.

**Example 2.3** If an apple is held above the ground and then dropped, the work done on the apple as it falls is equal to the weight (a force) of the apple integrated over the path to the ground.

Note that *only* in the special case where the force is constant *and* pointing in direction of the movement (such as in Example 2.3), (2.33) simplifies and one can calculate the work as being equal to the magnitude of the force times distance traveled.

**Example 2.4** A person carries a suitcase along a horizontal path. The upward force the person applies on the suitcase is always perpendicular to application point displacement increment $\mathrm{d}\boldsymbol{r}$, such that the scalar product (2.33) is zero. That means the upward force does *no* work along this path.

**Problem 2.9** A ball rolls down an incline that has an angle of 45° with the vertical. Calculate the distance the ball traveled as a function of height difference covered. Calculate the work done on the ball by its own weight, as a function of this height difference. Show that this work is *not* equal to the magnitude of the weight force times distance traveled. Take care about the signs throughout your calculation.

**Problem 2.10** A force of magnitude $F$ acts on a particle. The force always acts in the direction of the particle's velocity vector, and its magnitude is a function of time $t$, with $F = F_0(1 + \sin(2\pi f t))$, with constant frequency $f$ and amplitude $F_0$. The particle moves with constant speed $v_0$. Calculate the work done by the force $F$ as a function of time $t$ and the given constants. Show that this work is generally *not* equal to the amplitude of the force times distance traveled.

### 2.2.2 Principle of Work and Energy for a Single Particle

Newton's second law for a particle with constant mass $m$, with a resultant force vector $\boldsymbol{F}$ acting on it, states

$$\boldsymbol{F} = \dot{\boldsymbol{p}} = m\ddot{\boldsymbol{r}}. \tag{2.34}$$

If we integrate this over the path between points $A$ and $B$, we obtain work $W$ on the left side of (2.34). If we integrate the right side of (2.34) as well, we obtain (see also Problem B.4)

$$\begin{aligned}\int_A^B \boldsymbol{F}^{\mathrm{T}}\,\mathrm{d}\boldsymbol{r} &= m\int_A^B \ddot{\boldsymbol{r}}^{\mathrm{T}}\,\mathrm{d}\boldsymbol{r}\\ &= m\int_{t_0}^{t_1} \ddot{\boldsymbol{r}}^{\mathrm{T}}\dot{\boldsymbol{r}}\,\mathrm{d}t = m\left[\frac{1}{2}\dot{\boldsymbol{r}}^{\mathrm{T}}\dot{\boldsymbol{r}}\right]_{t_0}^{t_1} = \frac{1}{2}m(v_1^2 - v_0^2),\end{aligned} \tag{2.35}$$

where $v_0$ and $v_1$ are the respective speeds of the particle at time instants $t_0$ and $t_1$, corresponding with the particle being at locations $A$ and $B$.

We define the *kinetic energy* $T$ of the particle with mass $m$ and velocity vector $\boldsymbol{v} = \dot{\boldsymbol{r}}$ with magnitude $v = |\boldsymbol{v}|$ to be the work needed to bring the particle from rest to this velocity. From (2.35), we see (with $v_1 = v$ and $v_0 = 0$) that this quantity is equal to

$$T = \frac{1}{2}m|\boldsymbol{v}|^2 = \frac{1}{2}mv^2. \tag{2.36}$$

Note that in contrast to linear or angular momentum, kinetic energy is a *scalar*, which means that it has no direction.

**Example 2.5** This Example is about the governor of Example 2.2 on page 48. The kinetic energy $T$ of the system can be written in the form $T = \frac{1}{2}A\dot{\theta}^2 + \frac{1}{2}B\dot{\psi}^2$. Determine $A$ and $B$ as functions of the parameters $m$, $L$, $g$, $k$, $L_0$ and the generalized coordinate $\theta$ and explain why the kinetic energy is no function of the generalized coordinate $\psi$.

*Exemplary solution*

Because of symmetry, kinetic energy $T_1$ of particle 1 equals kinetic energy $T_2$ of particle 2, so

$$T = T_1 + T_2 = 2T_1. \tag{2.37E}$$

The balls are modeled as particles, so kinetic energy is

$$T = \sum_{i=1}^{2} \tfrac{1}{2}m\left|\boldsymbol{v}_i\right|^2 = 2\cdot\tfrac{1}{2}m\left|{}^{\mathcal{F}}\dot{\boldsymbol{r}}_1\right|^2 \tag{2.38E}$$

$$= 2\cdot\tfrac{1}{2}m\left(4L^2\dot{\psi}^2\sin^2\theta + 4L^2\dot{\theta}^2\cos^2\theta + 4L^2\dot{\theta}^2\sin^2\theta\right) \tag{2.39E}$$

$$= \tfrac{1}{2}\cdot\underbrace{8mL^2}_{A}\dot{\theta}^2 + \tfrac{1}{2}\cdot\underbrace{8mL^2\sin^2\theta}_{B}\dot{\psi}^2. \tag{2.40E}$$

Comparing coefficients gives $A = 8mL^2$ and $B = 8mL^2\sin^2\theta$.

The kinetic energy is no function of $\psi$, because the magnitude of the velocity vector of either particle does not depend on $\psi$.

When the initial velocity is not zero, we find from the integral of (2.35) that the work $W$ done on any particle by the resultant force $\boldsymbol{F}$ acting on it during an arbitrary time interval equals the *increment* $\Delta T$ of the particle's kinetic energy, which is the difference between the kinetic energy $T_B$ at position $B$ and the kinetic energy $T_A$ at position $A$:

$$W = \int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = T_B - T_A = \Delta T, \tag{2.41}$$

or in short

$$W = \Delta T. \tag{2.42}$$

This is the *principle of work and kinetic energy* for a single particle. Note that we derived it only using Newton's second law.

### 2.2.3 Work of Internal and External Forces

In order to investigate whether internal forces can do work on mechanical systems, we consider a particle $i$ with mass $m_i$ and speed $v_i$, which is part of a system of $N$ particles, as depicted in Figure 1.6 on page 15. The resultant force acting on the $i$-th particle over a position increment $\mathrm{d}\boldsymbol{r}_i$ includes both the resultant external force $\boldsymbol{F}_i$ that acts on this particle *and* the sum of forces $\boldsymbol{f}_{ij}$ exerted on the $i$-th particle by any other $j$-th particle in the system. Therefore, the work $W_i$ done on the particle is

$$W_i = \int_{A_i}^{B_i} \left( \boldsymbol{F}_i + \sum_{j=1}^{N} \boldsymbol{f}_{ij} \right) \cdot \mathrm{d}\boldsymbol{r}_i. \tag{2.43}$$

It is *not* possible to further reduce this equation, because the work done by internal forces does *not* cancel. This is in contrast to what we found in Section 1.2.7 for resultant moments of internal forces. So, internal forces can do work.

◯ **Problem 2.11** The two skaters on a frictionless flat surface from Example 2.1 use the rope they are holding to start pulling towards each other.

Draw individual FBDs of each skater. Identify all external forces for each skater and explain whether these forces do positive, negative, or zero work when the skaters move towards each other, and when they move apart.

Next, draw a FBD of the combined system, consisting of the two skaters, and identify again all external forces. Describe the displacement of the points of application and explain why the work of the *external* forces is zero. Also explain why the work of the *internal* forces is *not* zero.

◯ **Problem 2.12** A wheel is rolling without slip. Explain with a short reasoning whether the friction forces between wheel and ground do work. ▷

### 2.2.4 Principle of Work and Energy for Systems of Particles

In a system of $N$ particles (see Figure 1.6 on page 15), where particle $i$ has the individual kinetic energy

$$T_i = \frac{1}{2} m_i v_i^2, \tag{2.44}$$

the principle of work and energy (2.42) applies to the particle as

$$W_i = \Delta T_i. \tag{2.45}$$

We may apply summation over the left and right side of (2.45). By defining the kinetic energy of the system of particles as the sum of all individual kinetic energies,

$$T := \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2, \tag{2.46}$$

we obtain the principle of work and energy for a system of particles:

$$W = \sum_{i=1}^{N} \int_{A_i}^{B_i} \left( \boldsymbol{F}_i + \sum_{j=1}^{N} \boldsymbol{f}_{ij} \right) \cdot \mathrm{d}\boldsymbol{r}_i = \sum_{i=1}^{N} \Delta T_i = \Delta T. \tag{2.47}$$

This means the change in kinetic energy of a system of particles is given by the work of all external *and* internal forces acting on this system.

◯ **Example 2.6** Consider again the two skaters on a frictionless flat surface from Example 2.1 and Problem 2.11. If the skaters start from rest, their speeds (and thus kinetic energy) will increase. Interpreting the skaters as particles number 1 and 2 within one system, the work done by all external forces acting on this combined system is zero. Still, kinetic energy changes, due to the work of *internal* forces $\boldsymbol{f}_{12}$ and $\boldsymbol{f}_{21}$ transmitted by the rope.

### 2.2.5 Conservative Systems

As an introduction, we approach the topic intuitively. A force is called *conservative* if it is a function of position only and it *conserves* mechanical energy.

◯ **Example 2.7** If a particle of mass $m$ is thrown vertically into the air and rises up to a height of 5 m against the force of gravity, it will lose speed and thereby kinetic energy. However, energy is still *conserved* since one can fully recuperate this mechanical energy by letting the particle fall from the 5 m height back to the ground. In contrast, if a friction force acts on a particle that is sliding over the ground, thereby causing the particle to brake, then mechanical energy is transformed into thermal energy. This energy cannot be recuperated when the particle moves back into the other direction; instead, even more mechanical energy would be transformed into heat.

The work done by conservative forces, such as gravity in Example 2.7, can be expressed as a function of the start and end positions of a system. This saves the integration work since the path the particle takes from start to the end does not affect the work done on the system. In our example, the work done on the system (lifting the particle against gravity) will be the same if we go up and down with the particle two or more times in a row.

Consequently, a conservative force would not do any work along an arbitrary *closed path*, because then the start and end position are identical. Thus, according to the principle of work and energy, the kinetic energy of a system with only conservative forces acting on it will return to its initial value if the initial and final configuration of the system are identical.

◯ **Problem 2.13** A particle is moved from position A to position B along three possible trajectories. A conservative force acts on it. Can you say anything about the work that this force does on the particle along each of these three trajectories?

To check whether or not a force is conservative, the key is the concept of *potential energy*.

Without loss of generality, we first assume there is only a single conservative force acting on a mechanical system. To express the *potential* of this conservative force to do work on the system (in the future), a scalar potential function $V$ is introduced, equivalent to energy. Given that the work of a conservative force purely depends on net displacement of its point of application $\boldsymbol{r}$ between two points, also the associated potential energy $V$ must be a function of $\boldsymbol{r}$, so

$$V = V(\boldsymbol{r}) = V(x, y, z). \tag{2.48}$$

For the force to be conservative, it is not sufficient for it to be only position-dependent. The total energy of the system, consisting of kinetic energy $T$ and potential energy $V$, must equal a constant $c$ at all times:

$$T + V = c. \tag{2.49}$$

In other words, the difference in kinetic energy $\Delta T$ plus the difference in potential energy $\Delta V$ is zero for any two instants in time:

$$\Delta T + \Delta V = 0. \tag{2.50}$$

Note that this implies that along any closed path (where $\Delta V$ will be zero), $\Delta T$ must be zero, such that the system is conservative.

If we substitute the principle of work and energy in (2.42) in (2.50), we obtain

$$W + \Delta V = 0. \tag{2.51}$$

○ **Problem 2.14** What is the change in potential energy $\Delta V$ if a particle starts in one position and ends in the same position? What does your answer mean for the work $W$ that a conservative force does along any path that starts and ends in the same position?[a]

[a] *Hint*: Recall that $V$ is a function of position $\boldsymbol{r}$. Also, use the connection between $W$ and $\Delta V$.

We now consider infinitesimally small increments. This also eliminates the initial value of the potential $V$:

$$\mathrm{d}W + \mathrm{d}V = 0. \tag{2.52}$$

Substituting the definition of work, (2.33), this is equal to

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} + \mathrm{d}V = 0. \tag{2.53}$$

The infinitesimal change of potential energy $\mathrm{d}V$ can be calculated using the partial derivatives of $V$ in respectively $x$, $y$ and $z$ direction:

$$\mathrm{d}V = \frac{\partial V}{\partial x}\,\mathrm{d}x + \frac{\partial V}{\partial y}\,\mathrm{d}y + \frac{\partial V}{\partial z}\,\mathrm{d}z = \begin{pmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial V}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = \nabla V \cdot \mathrm{d}\boldsymbol{r}. \tag{2.54}$$

Here, the nabla operator $\nabla$ produces the gradient of $V(x, y, z)$. See Section B.5 for more explanation on the gradient.

**Example 2.8** For a conservative mechanical system, the elevation in Figure B.1 on page 499 could be interpreted as proportional to a scalar potential $V$, and the arrows could be interpreted as 2-D force vectors $\boldsymbol{F}$ associated with this potential.

If we combine (2.53) with (2.54) we find

$$\boldsymbol{F} = -\nabla V. \tag{2.55}$$

This means that the system is conservative if and only if the force equals the negative gradient of a scalar function $V(\boldsymbol{r})$, the potential energy. So, only if one can find such a scalar function $V$ for a particular force, then this force is a conservative force.

Examples for conservative forces are gravitational force or the force of a spring. An example of a non-conservative force is friction during sliding. Moreover, if all forces acting on a system are conservative, the system itself is conservative.

**Problem 2.15** Draw some more vectors of the gradient field into the hiking map in Figure B.1 on Figure B.1. At which angle do the vectors intersect with the isos lines (representing potential)? How is the length of the arrows related to distance between iso lines? How precisely are the vectors related to gravity?

**Problem 2.16** Try to draw an example of a force field that is position-dependent, but not conservative.[a] ▶

[a] *Hint*: In a conservative force field, the force vector is the negative gradient of the potential field. This means that when one moves in the direction of the force, potential is lost.

**Problem 2.17** Show that friction during sliding motion is not a conservative force, by looking at work done along different paths taken (Note that a single counterexample suffices). ▶

**Problem 2.18** Investigate whether the force vector $\boldsymbol{F} = c(x^2, x^2, 0)^{\mathrm{T}}$, with constant $c$, is a conservative force.[a]

[a] *Hint*: 1. Draw some force vectors of this field in the $xy$ plane. Then look whether you can find any example where the force does work along a closed path. 2. See whether you can find a $V$ via partial integration. What did you conclude from steps 1 and 2, do these findings match?

**Problem 2.19** Use (2.55) to investigate whether gravitational forces, sliding friction, and spring forces are conservative, by trying to find functions

|| $V$. ▷▶

**Example 2.9** This Example is about the governor of Example 2.2 on page 48. The potential energy $V$ of the system can be written in the form $V = Cz^2 + Dz$, with the height $z = 2L(1 - \cos\theta)$ of the collar $C$ and the balls. Determine $C$ and $D$ as functions of the parameters $m$, $L$, $g$, $k$, and $L_0$.

*Exemplary solution*

Each of the two particles has gravitational potential energy $V_g$, which is exactly the same for either, because of symmetry. Moreover, the spring adds potential energy $V_s$ to the system:

$$V = V_s + 2V_g. \tag{2.56E}$$

Substituting the appropriate expressions and given variables into (2.56E) leads to

$$V = \tfrac{1}{2}k\,(z - L_0)^2 + 2mgz, \tag{2.57E}$$

$$= \underbrace{\tfrac{1}{2}k}_{C}\ z^2 + \underbrace{(2mg - kL_0)}_{D}z + \underbrace{\tfrac{1}{2}kL_0^2}_{E(\text{constant})}\ . \tag{2.58E}$$

Comparing coefficients gives $C = \frac{1}{2}k$ and $D = 2mg - kL_0$.

## 2.3 Checking Plausibility and Frequent Mistakes

Some specific ideas for checking plausibility and checking inconsistencies, in addition to Section 1.4, are to:

- do the same problem using work and energy as well as using Newton-Euler.
- solve the equations of motion using MATLAB™ or a numerical program of your choice, and animate the simulation.
- numerically calculate the net energy at each time instant for a simulation. This is particularly useful for conservative systems.

Frequent mistakes specific to this chapter that may be the source of erroneous results are:

- specific, or even numerical values were filled in for positions or velocities at certain time instants *before* taking derivatives, such that the values appear constant and their derivatives zero.
- although point $O$ was not fixed, (2.28) was applied.

- when applying Euler's second law in the forms of (2.28) or (2.29), moments and angular momenta were not calculated about the same point.
- kinetic energy was misinterpreted as a vector quantity.
- the work of a force was calculated incorrectly, in a scalar fashion by just multiplying force with a distance traveled, not taking into account signs nor force or velocity vectors directions, and not taking into account that the force vector might not be constant. To remedy, it is important to always take the scalar product of force and velocity or distance increment vectors, and use integration.

## 2.4 Summary

This chapter covered two main approaches to derive equations of motion for particles and systems of particles: Newton-Euler (or the principles of linear and angular momentum) and work and energy.

Linear and angular momentum of a system of $N$ particles were respectively defined as

$$\boldsymbol{p} := \sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i, \qquad \boldsymbol{H}_Q := \sum_{i=1}^{N} \boldsymbol{\rho}_i \times \boldsymbol{p}_i. \tag{2.59}$$

Angular momentum only has a meaning when a reference point $Q$ is defined. Newton's and Euler's second laws state that

$$\sum_{i=1}^{N} \boldsymbol{F}_i = \dot{\boldsymbol{p}}, \quad \text{and} \quad \sum_{i=1}^{N} \boldsymbol{M}_{O,i} = \dot{\boldsymbol{H}}_O \text{ or } \sum_{i=1}^{N} \boldsymbol{M}_{C,i} = \dot{\boldsymbol{H}}_C, \tag{2.60}$$

respectively, where only the *external* forces and moments acting on the system need to be considered: the effects of the internal ones cancel out. Euler's second law in this form can only be applied for a fixed point $O$ or for the system's center of mass $C$.

Kinetic energy of a system of $N$ particles was defined as

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i |\dot{\boldsymbol{r}}_i|^2, \tag{2.61}$$

and work of a force $\boldsymbol{F}_k$ as

$$W_k = \int_A^B \boldsymbol{F}_k \cdot \mathrm{d}\boldsymbol{r} = \int_{t_0}^{t_1} \boldsymbol{F}_k \cdot \dot{\boldsymbol{r}}\, \mathrm{d}t = \int_{t_0}^{t_1} P\, \mathrm{d}t. \tag{2.62}$$

The principle of work and kinetic energy states that the work of all $n$ internal *and* external forces determines a system's change in kinetic energy:

$$\sum_{k=1}^{n} W_k = \Delta T. \tag{2.63}$$

In case that a potential field $V$ can be found for each internal or external force $\boldsymbol{F}_k$ that does work on a system such that

$$\boldsymbol{F}_k = -\nabla V_k, \text{ with } V_k = V_k(\boldsymbol{r}), \tag{2.64}$$

then the system is conservative, which implies that the sum of the system's kinetic and potential energy is constant:

$$T + \sum_{k=1}^{n} V_k = \text{const.} \tag{2.65}$$

## 2.5 Problems

### 2.5.1 Guided Problems

**Problem 2.20** In a right-handed, orthogonal, inertial coordinate system, the directions of the axes are given by unit vectors $\hat{\boldsymbol{n}}_1$, $\hat{\boldsymbol{n}}_2$, $\hat{\boldsymbol{n}}_3$. With respect to the origin $O$ of the coordinate system, the position vector of a particle of mass $m$ is given as a function of time $t$: $\boldsymbol{r}(t) = a\sin(\omega t)\hat{\boldsymbol{n}}_1 + b\cos(\omega t)\hat{\boldsymbol{n}}_2 + ct\hat{\boldsymbol{n}}_3$, where $a$, $b$, $c$ and $\omega$ are positive constants.

a. State suitable SI-units for $\boldsymbol{r}$, $t$ and parameters $a$, $b$, $c$, and $\omega$. Give your answer in the form: $[a] = ...$, $[b] = ...$, etc.
b. Draw the coordinate system and the particle's motion for time ranging from $t = 0\,\text{s}$ to $t = 4\frac{\pi}{\omega}$. Clearly label the point where the particle is at time $t = 0\,\text{s}$, and also the point where it is at $t = \frac{\pi}{2\omega}$. Use axis ticks involving the constants $a, b, c, \omega$ to clarify the locations. Produce either a clear (e.g. isometric) 3-D projection, or use multiple 2-D projections.
c. Calculate the angular momentum vector $\boldsymbol{H}_O(t)$ of the particle with respect to the origin $O$ as a function of $t$ and the parameters $a$, $b$, $c$, $\omega$, and $m$.
d. Calculate the rate of change of angular momentum $\dot{\boldsymbol{H}}_O(t)$ of the particle with respect to the origin as a function of $t$ and the parameters $a$, $b$, $c$, $\omega$, and $m$.
e. Calculate the acceleration vector $\boldsymbol{a}(t)$ of the particle as a function of $t$ and the parameters $a$, $b$, $c$, and $\omega$.
f. Use inverse dynamics to calculate the resultant force vector acting on the particle. Then calculate the moment this force generates around

the origin, as a function of $t$ and the parameters $a$, $b$, $c$, $\omega$, and $m$. Discuss your result in relationship to part d.

g. Calculate the power transmitted by the resultant force vector acting on the particle, as a function of time $t$ (and of the constants). Also calculate the work done by this force over a finite time interval $\Delta t$.
h. Calculate the kinetic energy $T$ of the particle, as well as the rate of change of kinetic energy $\dot{T}$, as functions of $t$ (and of the constant parameters).
i. Check plausibility and consistency of your answers to the previous question parts. Do this for example by comparing them to each other, and by investigating special cases, such as $a = b$. Conduct at least five independent checks.
j. Under what condition on $a$, $b$ and $c$ are the velocity and the acceleration of the particle perpendicular to each other for all $t$? To verify your answer, make a drawing of the movement in that special case, including velocity and acceleration vectors at one location.
k. Can the resultant force vector acting on the particle be explained by a conservative force field? Clearly answer "yes" or "no", and also provide a mathematical proof and a graphical illustration for your response.
l. To check your drawing, plot the path of the particle. In MATLAB™, you can do this by first creating a linearly spaced time vector and choosing numerical values for all parameters. Then, plot the resulting components of $\boldsymbol{r}$ into 3-D axes using the command `plot3`.

**Problem 2.21** A three-dimensional force vector field is given by

$$\boldsymbol{F}(x, y, z) = -\frac{c}{\sqrt{x^2 + y^2 + z^2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad (2.66P)$$

with a constant $c$ with $\{c\} = 1$, and position coordinates $x$, $y$, $z$ in an inertial Cartesian coordinate system.

a. Determine $[c]$, so the unit of $c$ (see Section 1.1.4).
b. Draw some force vectors of this field in a projection on the $xy$-plane. Then, do the same in projections onto the two other coordinate planes.
c. In your drawing, look whether you can find an example where the force does nonzero net work along a closed path.
d. Try to find a potential function $V$ that can be associated with $\boldsymbol{F}$, meaning that $\boldsymbol{F}$ is the negative gradient of $V$. Compare your findings from this part and parts b and c. Do they match, considering what you know about conservative forces?
e. Give a real-life example of such a force field if possible.
f. Employ MATLAB™ or another software to help with the first part

of the question. To this end, first create matrices **X**, **Y** and **Z** that contain a grid for $x$, $y$ and $z$ (e.g. using the function `ndgrid`), all ranging from $-r$ to $r$, where $r$ is a radius you can choose freely. Then create a matrix **U** that contains the $x$-components of the vector $\boldsymbol{F}$, a matrix **V** that contains the $y$-components and a matrix **W** that contains the $z$-components. Then, (e.g. using the function `quiver3`), plot the vector field in 3-D. Compare to your manual drawing.

g. Repeat parts a-d for the two-dimensional force vector field $\boldsymbol{F}(x, y) = c(x^2, x^2)^{\mathrm{T}}$, with $\{c\} = 1$.

**Problem 2.22** *Kepler's second law* describes the fact that if a planet orbits around a sun under the influence of the sun's gravity, the area covered by lines connecting planet and sun during identical time intervals is always the same. That means that if the path is elliptical, the planet's speed must change.

We will try to derive Kepler's second law on an example system: A planet $P$ orbits a star with an elliptic orbit (see Figure 2.2). The star is located in one of the focal points $O$ of the orbit. The planet and star are modeled as particles with planet mass $m$ and star mass $M$. We assume $M \gg m$, so the center of mass of the star is a fixed point.

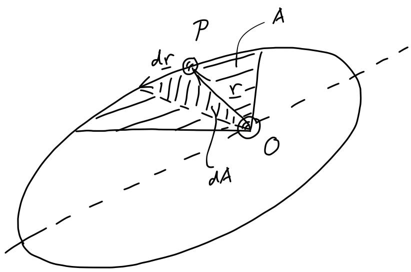



Figure 2.2: Planet in elliptic orbit around a star.

a. Draw a free-body diagram (FBD) of the planet projected onto the plane of the elliptic orbit.
b. Calculate the angular momentum $\boldsymbol{H}_O$ of the planet with respect to $O$ as a function of $\boldsymbol{r}$, $\dot{\boldsymbol{r}}$ and $m$.
c. Calculate the sum of moments $\Sigma \boldsymbol{M}_O$ acting on the planet with respect to $O$ using the FBD from part a.
d. Calculate the rate of change of angular momentum $\dot{\boldsymbol{H}}_O$ using Euler's second law.
e. Calculate the infinitesimally area d$A$ as a function of $\boldsymbol{r}$ and d$r$.
f. Rewrite d$A$ from part e as a function of $\boldsymbol{r}$, $\dot{\boldsymbol{r}}$ and infinitesimal time d$t$.
g. Rewrite d$A$ from part f as a function of $\boldsymbol{H}_O$ and $m$.

h. Calculate the finite area $A$ for a time interval $\Delta t = t_2 - t_1$ by integrating $\mathrm{d}A$ from $t_1$ to $t_2$.
i. Check your result from part h against Kepler's second law.

**Problem 2.23** A particle of mass $m$ slides down a frictionless slide, in a constant gravity field $-g\hat{\boldsymbol{n}}_3$. The surface of the slide is shaped such that its normal vector is always perpendicular to the direction $\hat{\boldsymbol{n}}_1$, see Figure 2.3. The slide has height $H$ between the starting and finish positions. The slide is level (it has zero slope) both at the start and the end positions of the block. At the start, the particle has a velocity vector $\boldsymbol{v}_0 = (v_{X0},\, v_{Y0},\, 0)^{\mathrm{T}}$.

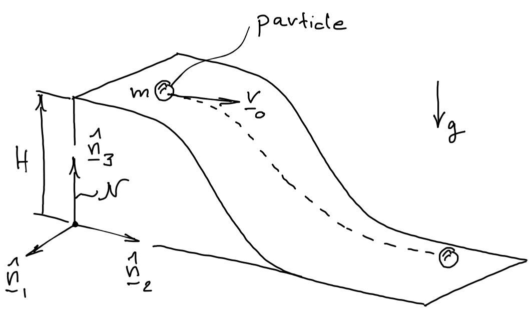



Figure 2.3: A particle sliding down a slide.

a. Draw a FBD of the particle for a general location on the slide.
b. By using conservation of energy, calculate the magnitude of the particle's velocity at the end position, as indicated in the figure.
c. Calculate the angle between the final velocity vector $\boldsymbol{v}_1$ and $\hat{\boldsymbol{n}}_2$. Note that you can re-use your answers to the previous questions.
d. Calculate the work done on the particle by the gravitational force during the movement.

**Problem 2.24** For the suspended camera of mass $m$, as depicted in Example 1.15 on page 35, we investigate the case where the camera moves.

a. The camera accelerates from a standstill, with initial acceleration vector $\boldsymbol{a}_{\mathrm{c}0} = (a_{X0},\, a_{Y0},\, 0)^{\mathrm{T}}$. Calculate the tension in cable r as a function of $l$, $w$, $h$, $m$, $g$, $a_X$, $a_Y$. Start your answer with a free-body diagram.
b. In order to move the camera, we need to change the respective lengths $l_{\mathrm{p}}$, $l_{\mathrm{q}}$, $l_{\mathrm{r}}$ of the three cables. For a generic velocity of the camera of $\boldsymbol{v}_{\mathrm{c}} = (v_X,\, v_Y,\, v_Z)^{\mathrm{T}}$, calculate the speeds $\dot{l}_{\mathrm{p}}$, $\dot{l}_{\mathrm{q}}$, $\dot{l}_{\mathrm{r}}$ with which the three cables need to elongate/shorten as functions of the camera's position and velocity components. Re-write your answer such that the "vector" containing all cable speeds can be calculated by taking the product of

a matrix with the velocity vector $\boldsymbol{v}_\mathrm{c}$.[a]

c. Motorized winches manipulate the lengths of the three cables. The acceleration vector needs to be $\boldsymbol{a}_\mathrm{c} = (a_X,\, a_Y,\, a_Z)^\mathrm{T}$. Calculate the three accelerations $\ddot{l}_\mathrm{p}$, $\ddot{l}_\mathrm{q}$, $\ddot{l}_\mathrm{r}$ with which the cables change length, as functions of the camera location, velocity, and acceleration components.

d. In order to check your above results, implement three functions in MATLAB$^\mathrm{TM}$ or another software: The first calculates the vector of cable lengths as functions of camera location, the second and third respectively provide cable velocities and accelerations as functions of camera velocity and acceleration.
Then, write a script that generates an arbitrary camera trajectory (for example linear or circular motion) and that uses your functions to calculate cable lengths, velocities and accelerations. To check, also calculate the numerical derivatives of cable lengths (for example using the function `gradient`) and check the analytical versus numerical results by plotting them on top of each other.

[a] *Hint*: First, express each cable length as a function of camera position. Then, calculate the derivative of each length with respect to time.

## 2.5.2 Practice Problems

◯ **Problem 2.25** Figure 2.4 shows a ballerina performing a jumping pirouette. A very simple model consists of a massless body and two point-masses, each of mass $m_\mathrm{h}$, representing the hands. The symmetry axis is also the rotation axis, and it remains vertical throughout. The ballerina holds her hands in a symmetric position, each at the same shoulder height and horizontal distance away from the body.

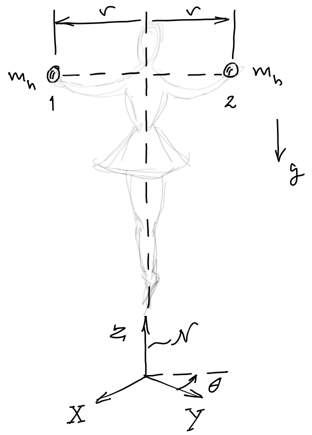



Figure 2.4: Jumping pirouette ballerina.

The ballerina pulls her arms in, by exerting a force on each point mass. The only external force is gravity that acts along the vertical axis of rotation. Explain whether this system conserves all linear momenta, angular momenta about the center of mass, and energy, in one sentence per quantity. ▶

**Problem 2.26** This Problem is about the accelerometer of Problem 1.24 on page 41. We want to calculate the sphere's potential energy due to gravity. We consider a simplified case with only one-dimensional movement, as shown in Figure 2.5. The accelerometer maintains the same orientation throughout.

Now, the positive inertial $Z$-direction is down (not up), and the location of the mass is described by different coordinates.

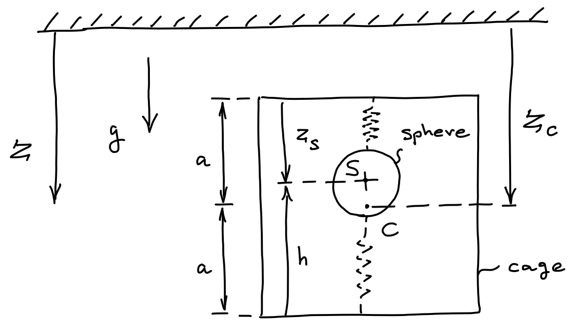



Figure 2.5: Simplified 1-D case of the accelerometer.

With an arbitrary constant energy $V_0$, the potential energy $V$ of the sphere due to gravity can be correctly calculated as (Check all that apply):

A. $V = mgz_S$
B. $V = mg(Z_C - z_S)$
C. $V = mgh$
D. $V = mgZ_C + V_0$
E. $V = -mgz_S + V_0$
F. None of the above

▷

**Problem 2.27** The catapult in Figure 2.6 consists of three springs, a cup, and a ring. Each spring is connected to the cup on one end and to the ring on the other end. The ring is connected to the ground in the $XY$ plane of the inertial $XYZ$ coordinate system.

An object of mass $m$ is to be launched. It is placed in the cup in point $D$ and released with zero initial speed. The initial position vector of $D$ in the inertial coordinate system $XYZ$ is provided as $\boldsymbol{p}_D = (0,\, 0,\, -4l_0)^{\mathrm{T}}$. Neglect the mass of the springs and the cup, and assume the object is a particle.

The connection points of the three springs at the ring are given by position vectors $\boldsymbol{p}_A = r(-\frac{1}{2}\sqrt{3},\, \frac{1}{2},\, 0)^{\mathrm{T}}$, $\boldsymbol{p}_B = r(\frac{1}{2}\sqrt{3},\, \frac{1}{2},\, 0)^{\mathrm{T}}$ and $\boldsymbol{p}_C = r(0,\, -1,\, 0)^{\mathrm{T}}$, with constant parameter $r = 3l_0$. All springs are linear, have stiffness $k$, unstretched/relaxed length $l_0$, and are massless.

Gravity with field strength $g$ acts in $(0,\, 0,\, -1)^{\mathrm{T}}$ direction.

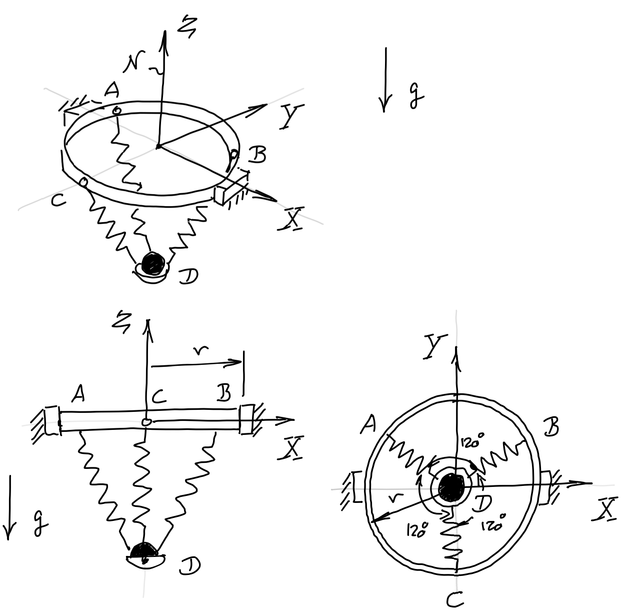



Figure 2.6: Catapult made with a ring, a cup to hold an object, and springs.

a. The object loses contact with the cup when the springs are horizontal. Use energy principles to calculate the height $h$ above the $XY$-plane to which the object flies, when released from the above configuration at rest. Express $h$ as a function of the parameters $m$, $k$, $l_0$ and $g$.
b. Calculate the work done by the springs during the launch.

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# Appendices

# A Vector and Matrix Algebra

## A.1 Vector Addition

Vectors are added (Figure A.1) by adding their individual components. For the addition of vectors, equal rules apply as for scalars summation.

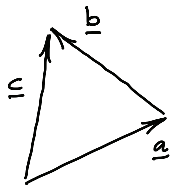


Figure A.1: Vector addition: The vector $\boldsymbol{c}$ is found by adding $\boldsymbol{a}$ and $\boldsymbol{b}$.

**Example A.1** The meaning of vector addition is particularly intuitive for position vectors: Assume for example that the vector $\boldsymbol{r}_{A/O}$ specifies the location of a point $A$ with respect to another point $O$, and that the vector $\boldsymbol{r}_{C/A}$ specifies the location of a point $C$ with respect to $A$. Then, the position vector $\boldsymbol{r}_{C/O}$, which points from $O$ to $C$, is given by $\boldsymbol{r}_{C/O} = \boldsymbol{r}_{A/O} + \boldsymbol{r}_{C/A}$.

**Problem A.1** In Figure A.1, assume the components of $\boldsymbol{c}$ and $\boldsymbol{a}$ are known. How do you calculate $\boldsymbol{b}$?

**Problem A.2** If you know $\boldsymbol{r}_{C/O}$, how do you calculate $\boldsymbol{r}_{O/C}$?

## A.2 Multiplication of a Vector with a Scalar

Multiplication of a vector with a scalar means multiplication of each single vector component by this scalar. This operation changes a vector's magnitude, but *not* its direction. So, scalar multiplication only "scales" a vector.

Accordingly, one can represent any vector by a multiplication of a scalar value (its magnitude) and a unit direction vector. This is particularly helpful when solving problems where the direction of a vector is known, but its magnitude is not. This situation occurs frequently in mechanical systems.

◯ **Problem A.3** Consider again Figure A.1, and assume we are in a 2-dimensional space (so in $\mathbb{R}^2$). Also assume that we know the vector $\boldsymbol{a}$ including its magnitude and direction, but for the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ we only know their directions, given by the unit vectors $\hat{\boldsymbol{e}}_b$ and $\hat{\boldsymbol{e}}_c$, respectively. We are interested in setting up the equations that allow calculating the respective magnitudes $b$ and $c$ of the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$. ▷▶

◯ **Problem A.4** Consider Problem A.3, now assume the vectors are defined in $\mathbb{R}^3$. Will there always be a solution for $b$ and $c$? If not, what is/are the condition(s) on the given variables such that a solution exists?

## A.3 Cross Product and Tilde Matrix

The *cross product* $\boldsymbol{a} \times \boldsymbol{b}$ is a vector perpendicular to both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The cross product of two vectors is the zero vector if both vectors have the same or the exact opposite direction (i.e. the vectors are linearly dependent). The direction of the cross product is determined by the right-hand rule (Figure A.2).

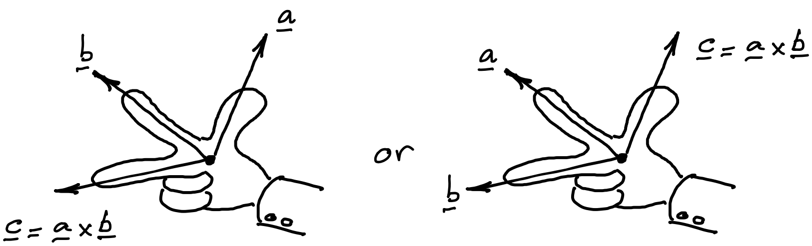


Figure A.2: Right-hand rule.

The magnitude of the cross product equals the area of the parallelogram with the vectors as sides, see Figure A.3. So, the magnitude of a cross product $\boldsymbol{c}$ of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}, \tag{A.1}$$

can be calculated using:

$$|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}| \sin(\theta), \tag{A.2}$$

where $\theta \in [0, \pi]$ is the angle enclosed by the two vectors.

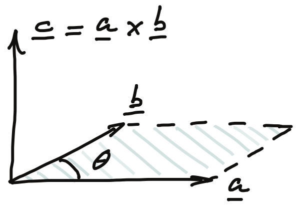


Figure A.3: The magnitude of the cross product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ describes the area of the parallelogram, spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$, which are separated by angle $\theta$.

|| ◯ **Problem A.5** What is the cross product of a vector with itself?

The cross product is used very often in dynamic calculations, for example for moments, angular momenta etc.

When writing the components of the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as

$$\boldsymbol{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}, \tag{A.3}$$

the cross product is defined as:

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}. \tag{A.4}$$

From its definition, it becomes evident that the cross product of two vectors exists only in three-dimensional spaces. The cross product is *anti-commutative*, meaning that $\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{a})$.

|| ◯ **Problem A.6** Use the definition of the cross product to show that $\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{a})$.

Using the *tilde matrix* $\tilde{\mathbf{a}}$, the definition of the cross product, (A.4), can be re-written as a matrix product, namely

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \tilde{\mathbf{a}}\boldsymbol{b}, \tag{A.5}$$

In case we are interested in writing the cross product in terms of a tilde matrix function of vector $\boldsymbol{b}$, we make use of the anti-commutative property $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$, so $\boldsymbol{a} \times \boldsymbol{b} = -\tilde{\mathbf{b}}\boldsymbol{a}$. Also, we can recognize that the tilde matrix is skew-symmetric, so $\tilde{\mathbf{b}}^{\mathrm{T}} = -\tilde{\mathbf{b}}$. This means that alternatively we can write $\boldsymbol{a} \times \boldsymbol{b} = \tilde{\mathbf{b}}^{\mathrm{T}}\boldsymbol{a}$.

◯ **Problem A.7** What is the relationship between the transpose $\tilde{\mathbf{a}}^{\mathrm{T}}$ of the tilde matrix and its negative $-\tilde{\mathbf{a}}$?

## A.4 Dot Product

The *dot product* of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, as illustrated in Figure A.4, is defined as

$$c = \boldsymbol{a} \cdot \boldsymbol{b} := |\boldsymbol{a}||\boldsymbol{b}| \cos(\theta), \tag{A.6}$$

where $\theta$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Note that the dot product results in a scalar value $c$, so it is also called the *scalar product*.

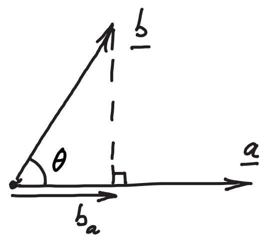


Figure A.4: Dot product.

From this formulation and Figure A.4, a very important property can be seen: The *scalar projection* of $\boldsymbol{b}$ onto $\boldsymbol{a}$, i.e. the component of vector $\boldsymbol{b}$ in direction of $\boldsymbol{a}$, is the signed magnitude $b_{\boldsymbol{a}}$:

$$b_{\boldsymbol{a}} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|}. \tag{A.7}$$

So, to calculate the component of a vector $\boldsymbol{b}$ in a particular direction, one takes the scalar product of $\boldsymbol{b}$ with a unit vector (so a vector having a length equal to 1) in that particular direction. It also implies that if two vectors are orthogonal (perpendicular), their dot product is zero.

In contrast, the *vector projection* of vector $\boldsymbol{b}$ onto $\boldsymbol{a}$ is a vector $\boldsymbol{b_a}$:

$$\boldsymbol{b_a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|} \cdot \frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|^2} \cdot \boldsymbol{a}. \tag{A.8}$$

◯ **Problem A.8** Consider a vector that points in the direction of the $x$-axis. Use the scalar product to calculate the scalar projection of this vector onto the $y$-axis.

When using the same components as in (A.3), the dot product resolves to

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z. \tag{A.9}$$

So, another way to write the dot product is via matrix notation:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^{\mathrm{T}} \boldsymbol{b}. \tag{A.10}$$

Note that the dot product does not only exist in $\mathbb{R}^3$, but also in any other dimension. More broadly in $\mathbb{R}^n$, it is defined as

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{n} a_i b_i. \tag{A.11}$$

In contrast to the cross product, the dot product is commutative (which can be seen directly from (A.9)):

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}. \tag{A.12}$$

◯ **Problem A.9** This Problem is about the top of corks and skewers of Problem 1.8 on page 12. Calculate the distance $h$ from point $Q$ to the ground as a function of the vector $\boldsymbol{r}_{Q/P}$, using a scalar product. ▶

◯ **Problem A.10** Try calculating dot products and cross products by using MATLAB™.

## A.5 Basic Matrix Definitions

Among other interpretations, matrices can be seen as *linear transformations* from one space to another: When a vector $\boldsymbol{x} \in \mathbb{R}^n$ is pre-multiplied by an $m \times n$-matrix $\mathbf{A}$, the result is another vector $\boldsymbol{y} \in \mathbb{R}^m$, mapped to the $n$-dimensional *image* space of $\mathbf{A}$. This space is spanned by the column vectors of $\mathbf{A}$.

The *rank* of $\mathbf{A}$ is the same as the dimension of the column space or image. It is important to note that the rank can be lower than $n$, because it could be that some columns are *linearly dependent*, meaning that they can be constructed by linear combinations of other columns. In that case, the columns cannot span an $n$-dimensional space.

◯ **Problem A.11** What is the rank of an $n \times n$ identity matrix?

**Problem A.12** What is the maximum rank that a $3 \times 4$-matrix can have? Provide a reasoning for your answer.

**Problem A.13** What is the rank of the tilde matrix of an arbitrary nonzero vector? Interpret this result by answering the question: How many possibilities are there for vector $\boldsymbol{b}$ in (A.1) if $\boldsymbol{c}$ and $\boldsymbol{a}$ are given?

There are some matrices with a special structure, for example *square* matrices, where $m = n$. Within this subgroup, there are for example *diagonal* matrices, where all values except for those on the diagonal are zero, or *upper triangular* matrices (also called *right triangular* matrices), where all values below the diagonal are zero.

The *transpose* $\mathbf{A}^{\mathrm{T}}$ of an arbitrary matrix $\mathbf{A}$ is another matrix that contains the columns of $\mathbf{A}$ as rows. For *symmetric* matrices, $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$, which means that all entries are mirror-symmetric with respect to the diagonal.

**Problem A.14** Can non-square matrices be symmetric?

The *trace* of a square matrix is defined as the sum of all diagonal elements of this matrix.

A critical characteristic of a square matrix $\mathbf{A}$ is its *determinant* $|\mathbf{A}|$. The determinant is a scalar value and can be calculated and interpreted as described in Chapter 3 of [40].

**Problem A.15** Calculate the determinant of a diagonal matrix.

**Problem A.16** Calculate the determinant of an upper triangular matrix.

**Problem A.17** If you know the determinant of a matrix $\mathbf{A}$, how can you find the determinant of $\mathbf{A}^{\mathrm{T}}$?

Using its determinant, one can also calculate the *inverse* of a square matrix, as described in Sections 2 and 3 of [40]. When pre- or post-multiplying a matrix $\mathbf{A}$ with its inverse $\mathbf{A}^{-1}$, one obtains the identity matrix. However, not all square matrices can be inverted. Matrices that have an inverse are called *invertible* or *nonsingular* matrices, while those without an inverse are called *singular* matrices.

Consider a square matrix that has *deficient* rank, so where the rank is lower than the number of columns. In that case, mapping a vector $\boldsymbol{x}$ to its counterpart $\boldsymbol{y}$ in the image space, via $\mathbf{A}\boldsymbol{x} = \boldsymbol{y}$ means compressing the original dimension of $\boldsymbol{x}$ to a lower dimensionality.

**Example A.2** Consider a vector $\boldsymbol{x} = (1, 2, 3)^{\mathrm{T}}$ in 3-D that is mapped onto a paper plane via pre-multiplication with a matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We notice that this matrix only has rank 2. So, even though it also has three components, the vector $\boldsymbol{y} = \mathbf{A}\boldsymbol{x} = (1, 2, 0)^{\mathrm{T}}$ is merely a two-dimensional projection of the original vector $\boldsymbol{x}$. If we try to reconstruct the original vector from its shadow, we will not succeed, because there is an infinite number of possible original vectors, for example $(1, 2, 0)^{\mathrm{T}}$ or $(1, 2, 325)^{\mathrm{T}}$. That means there cannot be a unique inverse $\mathbf{A}^{-1}$ such that one can be sure that $\mathbf{A}^{-1}\boldsymbol{y} = \boldsymbol{x}$.

**Problem A.18** What can you say about the inverse of a matrix that has a determinant of 0?

For a matrix that has a rank lower than its number of columns, there is always a set of vectors that must all map to the same image, namely to the zero vector. The set of these vectors is called the *nullspace* or *kernel* of a matrix.

**Problem A.19** What is the relationship for a matrix between its number of columns, the dimension of its kernel, and its rank?

**Problem A.20** Use the MATLAB™ command `null` to calculate the nullspace of the matrix **A** from Example A.2. Does the result match your expectations?

*Definiteness* of a symmetric matrix is an important property that can be employed for example for stability analysis. If a matrix **A** is positive definite, it means that for any nonzero vector $\boldsymbol{x}$, the scalar $\boldsymbol{x}^{\mathrm{T}}\mathbf{A}\boldsymbol{x}$ is positive. If **A** is negative definite, the same scalar will be always negative. Semi-definiteness extends to the case where $\boldsymbol{x}^{\mathrm{T}}\mathbf{A}\boldsymbol{x}$ may also be zero for some vectors $\boldsymbol{x}$.

**Problem A.21** Consider the two-dimensional identity matrix. Can you make a statement about its definiteness?

**Problem A.22** Consider a general two-dimensional diagonal matrix **D** with only negative entries on the diagonal. Can you make a general statement about the definiteness of such a matrix?

## A.6 Eigenvectors and Eigenvalues

A vector $\boldsymbol{a}$ that does not change its direction when it is pre-multiplied by a square matrix $\mathbf{C}$ (which need not be the identity matrix) is called an *eigenvector* of this matrix. The multiplication factor needed to represent the change in length is called the associated *eigenvalue* $\lambda$:

$$\mathbf{C}\boldsymbol{a} = \lambda\boldsymbol{a}. \tag{A.13}$$

If we are interested in finding an eigenvalue for a particular matrix, we can re-write (A.13) as

$$(\mathbf{C} - \lambda\mathbf{E})\boldsymbol{a} = \mathbf{0}, \tag{A.14}$$

where $\mathbf{E}$ is the identity matrix of the same dimension as $\mathbf{C}$.

For example in $\mathbb{R}^3$, this equation system, together with for example the condition that $\boldsymbol{a} = \hat{\boldsymbol{a}}$ has unit length, provides four equations for four unknowns: The components of $\hat{\boldsymbol{a}}$ and the scalar $\lambda$.

If the matrix $\mathbf{C}$ has full rank, we will find as many independent eigenvectors with nonzero eigenvalues as there are dimensions. Remark: It is possible that some or all vectors are complex, in which case also their associated eigenvalues are complex. These solutions always occur in complex conjugate pairs. Algebra of complex numbers is not required in the following; for more information, please consult Appendix H of [70] or Appendix C of [57].

A commonly employed method to solve the equation system is to first establish the determinant of the matrix $\mathbf{C} - \lambda\mathbf{E}$, which delivers the so-called *characteristic polynomial* in $\lambda$:

$$\det(\mathbf{C} - \mathbf{E}\lambda) = 0. \tag{A.15}$$

So, finding the $n$ eigenvalues means finding the $n$ roots of this $n$-degree polynomial. Afterwards, for each of these roots, one finds the associated eigenvector from (A.14).

◯ **Problem A.23** Computing eigenvalues by hand can be much work. However, checking if a vector is an eigenvector is usually quicker. Show that $\begin{pmatrix}1\\1\\1\end{pmatrix}$ is an eigenvector of $\begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\\2 & 3 & 1\end{pmatrix}$.

◯ **Problem A.24** Calculate the eigenvalues of a general diagonal matrix.

It is possible that some eigenvalues have a *multiplicity* larger than 1. If the characteristic polynomial contains factors $(p - \lambda_i)^{k_i}$, then $k_i$ is called the *algebraic multiplicity* of the eigenvalue $\lambda_i$. Note that it could be that there are less than $k_i$ eigenvectors associated with a particular eigenvalue $\lambda_i$. In that case, the *geometric* multiplicity of that eigenvalue is lower, in fact it equals the number of independent eigenvectors for the particular eigenvalue. It will always be at least one.

**Problem A.25** Use the MATLAB™ command `eig` to calculate the eigenvectors and eigenvalues of the matrices given in Example A.2 and in Problem A.23. Do the results match your expectations?

Further useful properties of eigenvalues are that the determinant of a matrix equals the product of all eigenvalues, and the trace of the matrix equals the sum of all eigenvalues.

**Problem A.26** Check plausibility of your answer to Problem A.24 using the determinant and the trace.

**Problem A.27** Calculate the eigenvalues of a general upper triangular matrix. Check plausibility of your answer using the determinant and the trace.

If all its eigenvalues are positive, a symmetric matrix is *positive definite*. If eigenvalues are only non-negative (meaning some could be zero), the matrix is positive *semidefinite*. In analogy, if all eigenvalues are negative (nonpositive), a symmetric matrix is *negative (semi-)definite*.

**Problem A.28** Check your answer to Problem A.21 using eigenvalues.

## A.7 Eigendecomposition of a matrix

Any *diagonalizable* square matrix $\mathbf{A}$ can be decomposed into the form

$$\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1}, \tag{A.16}$$

where the matrix $\mathbf{Q}$ contains the eigenvectors of matrix $\mathbf{A}$, and the matrix $\boldsymbol{\Lambda}$ is a diagonal matrix with the eigenvalues of $\mathbf{A}$ on the diagonal.

This will be shown in the following.

First, we express the $n$-by-$n$ matrix $\mathbf{Q}$ in terms of its column vectors

$$\mathbf{Q} = \begin{pmatrix} \boldsymbol{q}_1 & \dots & \boldsymbol{q}_n \end{pmatrix}, \tag{A.17}$$

If these are all eigenvectors of $\mathbf{A}$, we know from Appendix A.6 that

$$\mathbf{A}\boldsymbol{q}_i = \lambda_i \boldsymbol{q}_i\,, \quad i = 1 \dots n, \tag{A.18}$$

with scalar $\lambda_i$.

We can write this in matrix format:

$$\mathbf{A} \cdot \begin{pmatrix} \boldsymbol{q}_1 & ... & \boldsymbol{q}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 \boldsymbol{q}_1 & ... & \lambda_n \boldsymbol{q}_n . \end{pmatrix} \tag{A.19}$$

By defining the diagonal matrix $\Lambda$ as

$$\boldsymbol{\Lambda} := \begin{pmatrix} \lambda_1 & & \mathbf{0} \\ & ... & \\ \mathbf{0} & & \lambda_n \end{pmatrix}, \tag{A.20}$$

we can re-write (A.19) to

$$\mathbf{A} \cdot \begin{pmatrix} \boldsymbol{q}_1 & ... & \boldsymbol{q}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{q}_1 & ... & \boldsymbol{q}_n \end{pmatrix} \cdot \boldsymbol{\Lambda}, \tag{A.21}$$

or in short

$$\mathbf{AQ} = \boldsymbol{\Lambda}\mathbf{Q}. \tag{A.22}$$

If the matrix $\mathbf{Q}$ is invertible, we obtain the intended relationship (A.16).

A matrix $\mathbf{A}$ that is square but not diagonalizable is a *defective* matrix.

All symmetric matrices are diagonalizable: They possess three real eigenvectors that are mutually orthogonal, such that $\mathbf{Q}$ is invertible.

Diagonalization will be useful for the inertia tensor (see Section 5.2.3).

**Problem A.29** Diagonalize the matrix

$$\mathbf{A} = \begin{pmatrix} 7 & -12 \\ 4 & -7 \end{pmatrix}. \tag{A.23P}$$

▷

**Problem A.30** Show that this matrix is defective:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{A.24P}$$

**Problem A.31** Diagonalize the matrix

$$\mathbf{A} = \begin{pmatrix} 10 & 2 & -2 \\ 2 & 17 & 0 \\ -2 & 0 & 17 \end{pmatrix} \tag{A.25P}$$

For ease of calculation, you may use the MATLAB™ command `eig`. Check your result by re-calculating matrix $\mathbf{A}$ from $\mathbf{Q}$ and $\boldsymbol{\Lambda}$.

# B Differentiation and Integration

## B.1 Derivatives with Respect to Time

Many variables that describe mechanical systems (such as distances or angles) change over time and so are time-dependent.

To denote an infinitesimally small change in a variable, we use the operator d. For example, an infinitesimal increment in the vector $\boldsymbol{r}$ would be denoted as $\mathrm{d}\boldsymbol{r}$.

To represent the rate of change of a quantity with respect to time, one calculates the quotient of this infinitesimal increment and an increment in time $\mathrm{d}t$, which is the derivative with respect to time. To denote the derivative of a vector $\boldsymbol{r}$ or a scalar $r$ with respect to time $t$, we use the abbreviated notations,

$$\dot{\boldsymbol{r}} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)}{\Delta t}, \text{ and } \dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t}, \tag{B.1}$$

respectively. Note that the time derivative of constants, so quantities that do not vary over time, is zero.

The above are all absolute derivatives with respect to time, observed with respect to a system of reference that is not rotating. Section 4.2.2 of Chapter 4, extends this topic to relative derivatives, such as as $(\dot{\boldsymbol{r}})_{\mathcal{F}}$.

## B.2 Product Rule

The product rule of differentiation is applied when an expression contains a multiplication of several functions. The derivative of the scalar function $h(t) = f(t) \cdot g(t)$ with respect to time is:

$$\dot{h} = \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(f(t)g(t)\right) = \frac{\mathrm{d}f(t)}{\mathrm{d}t}g(t) + f(t)\frac{\mathrm{d}g(t)}{\mathrm{d}t}. \tag{B.2}$$

**Problem B.1** Take the time derivative of $h(t) = t^2$ by rewriting $h$ as $h(t) = f(t)g(t)$, where $f(t) = g(t) = t$ and applying the product rule.

**Problem B.2** Take the time derivative of $h(t) = t^3$ by rewriting $h$ as $h(t) = f(t)g(t)$, where $f(t) = t^2$, $g(t) = t$ and applying the product rule.

**Problem B.3** Show (B.2) from the definition of the time derivative in (B.1).

For the scalar product of two vector functions, so $h_1(t) = \boldsymbol{f}(t)^{\mathrm{T}}\boldsymbol{g}(t)$, the rule can equally be applied:

$$\dot{h}_1 = \frac{\mathrm{d}h_1}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{f}(t)^{\mathrm{T}}\boldsymbol{g}(t)\right) = \dot{\boldsymbol{f}}(t)^{\mathrm{T}}\boldsymbol{g}(t) + \boldsymbol{f}(t)^{\mathrm{T}}\dot{\boldsymbol{g}}(t) \tag{B.3}$$

as well as for the cross product $\boldsymbol{h}_2(t) = \boldsymbol{f}(t) \times \boldsymbol{g}(t)$:

$$\dot{\boldsymbol{h}}_2 = \frac{\mathrm{d}\boldsymbol{h}_2}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{f}(t) \times \boldsymbol{g}(t)\right) = \dot{\boldsymbol{f}}(t) \times \boldsymbol{g}(t) + \boldsymbol{f}(t) \times \dot{\boldsymbol{g}}(t) \tag{B.4}$$

**Problem B.4** Use the product rule of differentiation to show that $\int \ddot{\boldsymbol{r}}^{\mathrm{T}}\dot{\boldsymbol{r}}\,\mathrm{d}t = \frac{1}{2}\dot{\boldsymbol{r}}^{\mathrm{T}}\dot{\boldsymbol{r}}$ + constant.

**Problem B.5** Use the scalar formulation (B.2) of the product rule of differentiation, in combination with the definitions of scalar and cross product, to prove (B.3) and (B.4).

## B.3 Chain Rule

The chain rule is used when the expression is the nested composition of two or more functions. To calculate the time derivative of a single nested function $h(g(t))$, the chain rule yields

$$\dot{h} = \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(h(g(t))\right) = \frac{\mathrm{d}h}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}t}. \tag{B.5}$$

**Problem B.6** Calculate the time derivative $\dot{h}$ of $h = \sin(\theta)$ for which you know that $\theta$ is time-dependent, using the chain rule.

**Problem B.7** Calculate the time derivative of $h = 3\theta\sin(\theta)$ for which you know that $\theta$ is time-dependent, using both the product rule and the chain rule.

For a more deeply nested function, the chain rule expands to:

$$\dot{h} = \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(h(g_1(g_2(g_3(t))))\right) = \frac{\mathrm{d}h}{\mathrm{d}g_1}\frac{\mathrm{d}g_1}{\mathrm{d}g_2}\frac{\mathrm{d}g_2}{\mathrm{d}g_3}\frac{\mathrm{d}g_3}{\mathrm{d}t}. \tag{B.6}$$

**Example B.1** To calculate $\dot{h} = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\cos\left(\sqrt{x(t)}\right)}\right)$, we decompose $h$ as a function of $g_1$ first. That is:

1. $h(g_1) = e^{g_1}$. In its turn, $g_1$ is a function of $g_2$. That is
2. $g_1(g_2) = \cos(g_2)$. Then, $g_2$ is a function of $g_3$, namely
3. $g_2(g_3) = \sqrt{g_3}$. Finally, $g_3$ is the time-dependent variable:
4. $g_3(t) = x(t)$.

The separate derivatives are:

1. $\frac{\mathrm{d}h(g_1)}{\mathrm{d}g_1} = e^{g_1}$
2. $\frac{\mathrm{d}g_1(g_2)}{\mathrm{d}g_2} = -\sin(g_2)$
3. $\frac{\mathrm{d}g_2(g_3)}{\mathrm{d}g_3} = \frac{1}{2\sqrt{g_3}}$
4. $\frac{\mathrm{d}g_3(t)}{\mathrm{d}t} = \dot{x}$

This way we find

$$\dot{h} = -\frac{\dot{x}e^{\cos\left(\sqrt{x(t)}\right)}\sin\left(\sqrt{x(t)}\right)}{2\sqrt{x(t)}}. \tag{B.7E}$$

**Problem B.8** Calculate $\dot{h} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\cos\left(x(t)^2\right)\right)$.

**Problem B.9** Show (B.5) from the definition of the time derivative in (B.1).

**Problem B.10** Use the Symbolic Toolbox in MATLAB™ to calculate $\frac{\mathrm{d}h}{\mathrm{d}x}$ from Example B.1 and from Problem B.8 and compare the results with your manual calculations.

## B.4 Partial Derivatives

We consider a function $f$ that depends on multiple variables. A *partial derivative* of $f$ is the derivative of this function with respect to a single variable. Taking a partial derivative, one is only interested in the influence of the variation of one of the variables on $f$, while all the other variables are held constant.

**Example B.2** The elevation at a particular location in a mountain landscape can be given as a function of two variables: latitude and longitude. We could be interested in the influence of the variation of latitude, while the longitude is kept constant. To obtain this information, we calculate the partial derivative of elevation with respect to latitude. This provides the slope of the landscape in the requested direction.

So, a partial derivative represents the dependency of an expression on an isolated variable. The partial derivative of a function $f$ with respect to a variable $x$ is denoted by $\frac{\partial f}{\partial x}$.

**Example B.3**

$$\frac{\partial}{\partial x}\left(ax^2 + 3yx + c\dot{x}^2\right) = 2ax + 3y \tag{B.8E}$$

**Example B.4**

$$\frac{\partial}{\partial \dot{x}}\left(ax^2 + 3yx + c\dot{x}^2\right) = 2c\dot{x} \tag{B.9E}$$

**Example B.5**

$$\frac{\partial}{\partial y}\left(ax^2 + 3yx + c\dot{x}^2\right) = 3x \tag{B.10E}$$

We can use partial derivatives to calculate increments of a function $f$ that depends on multiple variables $x_i$, with $i = 1 \dots N$:

$$\mathrm{d}f = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i. \tag{B.11}$$

**Example B.6** Consider a particle moving in three-dimensional space, within a temperature field. The particle's coordinates $x$, $y$, and $z$ determine the temperature $f(x,\, y,\, z)$. We find the infinitesimal temperature increment $\mathrm{d}f$ that is due to changes in $x$, in $y$, and in $z$ as:

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z. \tag{B.12E}$$

**Example B.7** In Example B.2 about the mountain landscape, the elevation $f$ is only a function of two variables, namely latitude $x_1$ and longitude $x_2$. The relationship (B.11) indicates how much elevation $\mathrm{d}f$ is gained as a function of infinitesimal changes in latitude $\mathrm{d}x_1$ and longitude $\mathrm{d}x_2$.

In case the $x_i$ are all functions of time $t$, we find the *total derivative* of a function $f(x_1(t),\, x_2(t) \dots\, x_N(t))$ with respect to time as:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t}. \tag{B.13}$$

**Problem B.11** Consider a function $f(x(t),\, y(t))$ with

$$f = 3x^2 + 5y, \qquad x = 5t, \qquad y = t^2. \tag{B.14P}$$

Find $\dot{f}$ in two ways: a) by substituting the functions for $x$ and $y$ into $f$, such that it becomes a function of time only, and then taking its derivative, and b) by using (B.13). You should obtain the same result.

## B.5 Gradient

For a function $f$ that depends on multiple variables, for example $x$, $y$, and $z$, the partial derivatives of $f$ in all directions, so in the example respectively in $x$, $y$ and $z$ direction, can be subsumed in a vector, the *gradient* of $f$:

$$\nabla f := \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}. \tag{B.15}$$

The symbol $\nabla$ is called the *nabla* operator; it produces the gradient of a scalar function, like here of $f(x,\, y,\, z)$. At any specific location, this vector points in the direction of the steepest ascent.

A gradient of a function is always orthogonal to the *level curves* (often also denoted *iso lines*) or *level surfaces* of this function, which denotes the manifolds along which the function does not change its value.

**Example B.8** Figure B.1 shows the gradient of a function $f(x,\, y)$ that specifies elevation for a given location on a hiking map. This gradient gives inclination. Each vector points in direction of the greatest change in height at its specific location. The gradient vectors are always perpendicular to the iso lines, which connect all points that have the same elevation.

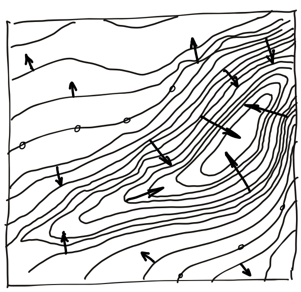

Figure B.1: Visualization of a gradient field. The arrows represent the gradient in elevation on a hiking map with iso lines of elevation.

**Problem B.12** We model Earth as a perfect sphere with center $E$ and radius $r$ (see Figure B.2).

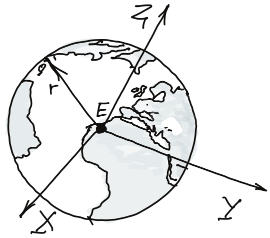


Figure B.2: Earth

a. Describe the location set of points that have equal distance $d$ from the *surface* of the earth in the form $h(X, Y, Z) = d$, where $X, Y, Z$ are the points' coordinates. The unit of the function should be $[h] = \mathrm{m}$.
b. Calculate the gradient of this function using the nabla operator.
c. Draw a few gradient vectors projected on the $XY$-plane, the $YZ$-plane, and the $XZ$-plane. Include at least the gradient at point $(X, Y, Z) = (1.1r, 1.1r, 1.1r)$ and one point on the surface of the Earth.
d. Multiply the gradient by (-1). What might be an interpretation of this (negative) gradient field for the Earth?
e. Check your answer further by plotting the vector field, for example using MATLAB™. You can use the functions `ndgrid` and `quiver3`. You will also need the `./` notation to divide the elements of a matrix by the elements of another.

**Problem B.13** Show that the gradient of a function $f(x, y)$ of two variables $x$ and $y$ at any particular point is orthogonal to the function's level curve through that point. [a]

[a] *Hint*: In order to show this, do the following: Consider an infinitesimal vector that points in a direction that is tangential to the level curve and has the components $(dx, dy)^T$. Express the infinitesimal increment $df$ of $f$ along the level curve, using (B.11), as a function of the individual increments $dx$ and $dy$. This increment should be zero in the direction of the curve. Then, re-write $df$ as a scalar product of two vectors. What can you say about these two vectors?

# B.6 Integration

For single and multiple integrals, please consult for example Chapter 5 of [70], which explains line, surface, and volume integrals in Sections 16.2, 16.7, and 6.2-6.3, respectively. Pay particular attention to the limits of the integrals, they might be functions of other variables.

# C Shorthand Notations

Writing vectors out in components requires additional subscripts. Maintain all subscripts of an original vector can become cumbersome and makes equations less compact. Furthermore, projections require choosing how to typeset several subscripts. Stacked indices are not aesthetic. A comma can be ambiguous, as it sometimes indicates differentiation.

To avoid this choice, one can define new, subscript-free vectors.

◯ **Example C.1** A vector ${}^{\mathcal{B}}\boldsymbol{r}_{A/C}$ that describes the relative location of point $A$ with respect to $C$ already has a lengthy subscript. None of these options for the first component are ideal: $r_{A/C_{b_1}}$, $r_{A/Cb1}$, or $r_{A/C,b1}$.

In this specific problem, one could choose to define a new vector $\boldsymbol{a} := \boldsymbol{r}_{A/C}$ and then write components compactly as ${}^{\mathcal{B}}\boldsymbol{a} = (a_{b1}, a_{b2}, a_{b3})^{\mathrm{T}}$.

This advice does not apply to coding software. On the contrary, variables in code need clear names that maintain all identifying subscripts.

◯ **Example C.2** The first component of vector ${}^{\mathcal{B}}\boldsymbol{r}_{A/C}$ could for example be coded as `r_AwrtC_b1`. There are many equally good options. It is only advised to explicitly choose a type of definition that maintains the original vector name or meaning, the triad, and the number of the component, and to code consistently. Depending on the choices made, also these could be viable variable names: `rCtoA_b1`, or `B_relativeposition_AwrtC_x`, etc. Since all information encoded via typesetting is lost in plain text, this needs to be replaced by a clear definition of sequence of information.

If elements of a point's position vector are also *Cartesian coordinates*, one can for example choose the axis name as main symbol and the point name as index, or vice versa. However, not all position vector components are also Cartesian coordinates. Components of a position vector in triad $\mathcal{F}$ are only coordinates if the point of reference is the origin of a coordinate system that is aligned with triad $\mathcal{F}$. Therefore, it is important to distinguish between vector components that happen to be also Cartesian coordinates, and components that are only projections onto given axis directions, without further meaning.

◯ **Example C.3** In this book (for example in Figure 4.9 on page 123), we frequently use

- an inertial reference point $O$,
- a moving reference point $Q$ (for example the center of mass of a system, a geometric center, or the location of a spherical joint),
- a specific point $P$ (for example a mass element within a rigid body),
- an inertial coordinate system $XYZ$ with origin $O$ and triad $\mathcal{N}$,
- a body-fixed coordinate system $xyz$ with origin $Q$ and triad $\mathcal{B}$.

For the mentioned points, we will often (but not strictly) use these shorthand notations for relative and absolute position vectors:

$$\boldsymbol{P} := \boldsymbol{r}_{P/O}, \quad \boldsymbol{Q} := \boldsymbol{r}_{Q/O}, \quad \boldsymbol{p} := \boldsymbol{r}_{P/Q}. \tag{C.1E}$$

For the above vectors, components are also Cartesian coordinates in the cases of ${}^{\mathcal{N}}\boldsymbol{P}$, ${}^{\mathcal{N}}\boldsymbol{Q}$, and ${}^{\mathcal{B}}\boldsymbol{p}$. The first and the last are both coordinates of point $P$, just in two different coordinate systems: $XYZ$ and $xyz$, respectively. Here, there are multiple alternatives to abbreviate components to variables that carry more meaning:

$${}^{\mathcal{N}}\boldsymbol{P} = {}^{\mathcal{N}}\boldsymbol{r}_{P/O} = \begin{pmatrix} P_{n1} \\ P_{n2} \\ P_{n3} \end{pmatrix} =: \begin{pmatrix} X_P \\ Y_P \\ Z_P \end{pmatrix}, \text{ or } {}^{\mathcal{N}}\boldsymbol{P} =: \begin{pmatrix} P_X \\ P_Y \\ P_Z \end{pmatrix}, \tag{C.2E}$$

$${}^{\mathcal{N}}\boldsymbol{Q} = {}^{\mathcal{N}}\boldsymbol{r}_{Q/O} = \begin{pmatrix} Q_{n1} \\ Q_{n2} \\ Q_{n3} \end{pmatrix} =: \begin{pmatrix} X_Q \\ Y_Q \\ Z_Q \end{pmatrix}, \text{ or } {}^{\mathcal{N}}\boldsymbol{Q} =: \begin{pmatrix} Q_X \\ Q_Y \\ Q_Z \end{pmatrix}, \tag{C.3E}$$

$${}^{\mathcal{B}}\boldsymbol{p} = {}^{\mathcal{B}}\boldsymbol{r}_{P/Q} = \begin{pmatrix} p_{b1} \\ p_{b2} \\ p_{b3} \end{pmatrix} =: \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix}, \text{ or } {}^{\mathcal{B}}\boldsymbol{p} =: \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}. \tag{C.4E}$$

It depends on personal preference which alternative to choose in a given problem. The only requirement is to always make all abbreviated definitions explicit. This book mostly uses the first alternative, so $X_P$ rather than $P_X$.

It is not possible to apply this abbreviated notation in a meaningful way to the three other possible projections, ${}^{\mathcal{B}}\boldsymbol{P}$, ${}^{\mathcal{B}}\boldsymbol{Q}$, and ${}^{\mathcal{N}}\boldsymbol{p}$, because components are *not* Cartesian coordinates. In these cases, we can only use the explicit subscripts that indicate projections:

$${}^{\mathcal{B}}\boldsymbol{P} = {}^{\mathcal{B}}\boldsymbol{r}_{P/O} = \begin{pmatrix} P_{b1} \\ P_{b2} \\ P_{b3} \end{pmatrix}, {}^{\mathcal{B}}\boldsymbol{Q} = {}^{\mathcal{B}}\boldsymbol{r}_{Q/O} = \begin{pmatrix} Q_{b1} \\ Q_{b2} \\ Q_{b3} \end{pmatrix}, {}^{\mathcal{N}}\boldsymbol{p} = {}^{\mathcal{N}}\boldsymbol{r}_{P/Q} = \begin{pmatrix} p_{n1} \\ p_{n2} \\ p_{n3} \end{pmatrix}. \tag{C.5E}$$

Attempting to write something like "$p_X$" as shorthand for $p_{n1}$ would be highly misleading, because it gives the impression of an $X$-coordinate, while the reference point of the position vector is not the origin of $XYZ$.

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# F Answers

**1.8** ${}^{\mathcal{B}}\boldsymbol{r}_{D/P} = \left(-10x_{C^*},\, 0,\, \frac{l}{2} - \Delta\right)^{\mathrm{T}},\, {}^{\mathcal{B}}\boldsymbol{r}_{E/P} = \left(0,\, -10y_{C^*},\, \frac{l}{2} - \Delta\right)^{\mathrm{T}}$

**1.12** $\ddot{\boldsymbol{r}} = \left(-9\sin\frac{3t}{\mathrm{s}} \quad -12\cos\frac{2t}{\mathrm{s}} \quad 8\cos\frac{4t}{\mathrm{s}}\right)^{\mathrm{T}}\ \mathrm{m/s^2}$

**1.14** $\{X_R, Y_L, Y_R, \dot{X}_L\} > 0, \{X_A, \dot{X}_A\} = 0, \{X_L, Y_A, \dot{X}_R, \dot{Y}_L, \dot{Y}_R, \dot{Y}_A\} < 0$

**1.17** $\boldsymbol{r}_{A/P} = (l/\,(2d))\,\boldsymbol{r}_{C/P}$

**1.18** $a = \frac{m_1 L}{m_1+m_2+m_3}, \quad b = \frac{(m_2-m_3)L}{4(m_1+m_2+m_3)}$

**1.19** E.

**1.23** A.

**1.25** $\boldsymbol{F}_{A/D} = -\frac{\boldsymbol{r}_{D/A}}{|\boldsymbol{r}_{D/A}|}\ k\ (|\boldsymbol{r}_{D/A}| - l_n)$

**1.26** A.

**1.27** F.

**2.7** No. The muscle forces are internal forces and therefore cannot change angular momentum of the body as a whole.

**2.12** The friction forces do no work in this case.

**2.19** Gravity and ideal spring forces are conservative forces, sliding friction is not.

**2.26** F.

**3.1** The same: $|{}^{\mathcal{N}}\boldsymbol{r}| = |{}^{\mathcal{B}}\boldsymbol{r}|$

**3.3** ${}^{\mathcal{N}}\hat{\boldsymbol{b}}_1 = (\frac{1}{2}\sqrt{3},\, \frac{1}{2},\, 0)^{\mathrm{T}},\, {}^{\mathcal{N}}\hat{\boldsymbol{b}}_2 = (-\frac{1}{2},\, \frac{1}{2}\sqrt{3},\, 0)^{\mathrm{T}},\, {}^{\mathcal{N}}\hat{\boldsymbol{b}}_3 = (0,\, 0,\, 1)^{\mathrm{T}}$

**3.4** 1

**3.5** 0

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# G Solutions

**1.8** In general, the position vector of the center of mass of a composite body is calculated by (1.11). In this case, with $P$ taking the role of the reference point $A$, two added pieces of cork give the expression

$$ {}^{\mathcal{B}}\boldsymbol{r}_{C/P} = \frac{\frac{1}{10}M^{*}\,{}^{\mathcal{B}}\boldsymbol{r}_{D/P} + \frac{1}{10}M^{*}\,{}^{\mathcal{B}}\boldsymbol{r}_{E/P} + M^{*}\,{}^{\mathcal{B}}\boldsymbol{r}_{C^{*}/P}}{\frac{6}{5}M^{*}}. \tag{G.1} $$

To make the top spin better, the center of mass needs to be on the $z$-axis. Therefore, we desire

$$ {}^{\mathcal{B}}\boldsymbol{r}_{C/P} = \begin{pmatrix} 0 & 0 & z_C \end{pmatrix}^{\mathrm{T}}, \tag{G.2} $$

with arbitrary $z_C$ (although it can be shown that a smaller $z_C$ also makes the top spin better).

Furthermore, we must place one piece of cork on skewer 2 and one piece of cork on skewer 3 to correct for both unbalance in the $x$-direction and unbalance in the $y$-direction. In that case one of the coordinates of the pieces is always zero and its $z$-coordinate is always on the set height of the cork at $Q$. We choose to put cork $D$ on skewer 2 and $E$ on skewer 3:

$$ {}^{\mathcal{B}}\boldsymbol{r}_{D/P} = \begin{pmatrix} x_D \\ 0 \\ \frac{l}{2} - \Delta \end{pmatrix}, \qquad {}^{\mathcal{B}}\boldsymbol{r}_{E/P} = \begin{pmatrix} 0 \\ y_E \\ \frac{l}{2} - \Delta \end{pmatrix} \tag{G.3} $$

By substituting (G.2) and (G.3) in (G.1) a set of equations is obtained:

$$ \begin{pmatrix} \frac{1}{10}M^{*}x_D + 0 + M^{*}x_{C^{*}} \\ 0 + \frac{1}{10}M^{*}y_E + M^{*}y_{C^{*}} \\ 2\frac{1}{10}M^{*}\left(\frac{l}{2} - \Delta\right) + M^{*}r_{C^{*}z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{6}{5}M^{*}z_C \end{pmatrix} \tag{G.4} $$

This set can be solved to find the missing cork coordinates:

$$ x_D = -10x_{C^{*}}, \qquad y_E = -10y_{C^{*}}. \tag{G.5} $$

The position vectors of the two pieces of cork are:

$$ {}^{\mathcal{B}}\boldsymbol{r}_{D/P} = \begin{pmatrix} -10x_{C^{*}} \\ 0 \\ \frac{l}{2} - \Delta \end{pmatrix}, \qquad {}^{\mathcal{B}}\boldsymbol{r}_{E/P} = \begin{pmatrix} 0 \\ -10y_{C^{*}} \\ \frac{l}{2} - \Delta \end{pmatrix}. \tag{G.6} $$

**1.9** Figure G.1 shows a possible FBD. As there is only point contact, no rotations can be prevented by the ground, so no moments are drawn.

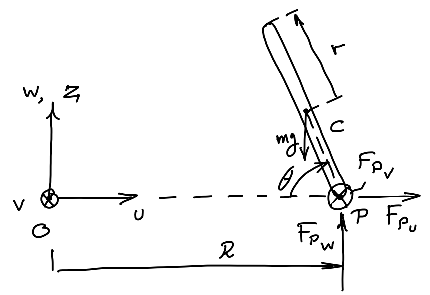


Figure G.1: Disk free-body diagram.

**1.13** The kinematics of all animals are described in Figure G.2.

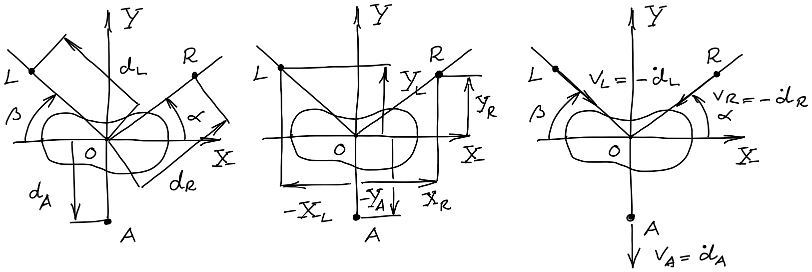


Figure G.2: Describing kinematics of all animals. Left: Location in terms of distance and angle. Center: Cartesian coordinates. Right: Velocity

The correct signs for the rhino's position vector are:

$$\boldsymbol{r}_R = \begin{pmatrix} +X_R \\ +Y_R \\ +Z_R \end{pmatrix} = \begin{pmatrix} +d_R \cos\alpha \\ +d_R \sin\alpha \\ 0 \end{pmatrix} \tag{G.7}$$

The correct signs for its velocity vector are:

$$\boldsymbol{v}_R = \begin{pmatrix} -v_R \cos\alpha \\ -v_R \sin\alpha \\ 0 \end{pmatrix} = +\dot{\boldsymbol{r}}_R = \begin{pmatrix} +\dot{X}_R \\ +\dot{Y}_R \\ +\dot{Z}_R \end{pmatrix} = \begin{pmatrix} \dot{d}_R \cos\alpha - d_R \cancelto{0}{\dot{\alpha}} \sin\alpha \\ \dot{d}_R \sin\alpha + d_R \cancelto{0}{\dot{\alpha}} \cos\alpha \\ 0 \end{pmatrix} \tag{G.8}$$

The correct signs for the antelope's position vector are:

$$\boldsymbol{r}_A = \begin{pmatrix} +X_A \\ +Y_A \\ +Z_A \end{pmatrix} = \begin{pmatrix} 0 \\ -d_A \\ 0 \end{pmatrix} \tag{G.9}$$

The correct signs for its velocity vector are:

$$\boldsymbol{v}_A = \begin{pmatrix} 0 \\ -v_A \\ 0 \end{pmatrix} = +\dot{\boldsymbol{r}}_A = \begin{pmatrix} +\dot{X}_A \\ +\dot{Y}_A \\ +\dot{Z}_A \end{pmatrix} = \begin{pmatrix} 0 \\ -\dot{d}_A \\ 0 \end{pmatrix} \tag{G.10}$$

**1.20** Using the origin of $xyz$ as reference point in (1.11), the center of mass $\boldsymbol{r}_C$ of the combined body is calculated from:

$$\boldsymbol{r}_C = \frac{m\boldsymbol{r}_{\mathrm{D}} + m_a\boldsymbol{r}_A}{m + m_a}, \tag{G.11}$$

with $\boldsymbol{r}_A = (r_x, r_y, r_z)^{\mathrm{T}}$ the position vector of the mass at $A$, with components

$$r_x = d, \qquad r_y = R\cos\varphi, \qquad r_z = R\sin\varphi. \tag{G.12}$$

Substituting these values in (G.11) gives

$$\boldsymbol{r}_C = \frac{1}{m + m_a}\left(mR\begin{pmatrix}\frac{1}{40}\\ \frac{1}{50}\\ \frac{1}{50}\end{pmatrix} + m_a\begin{pmatrix}d\\ R\cos\varphi\\ R\sin\varphi\end{pmatrix}\right). \tag{G.13}$$

For static balance, the center of mass must be on the $x$-axis:

$$\boldsymbol{r}_C \stackrel{!}{=} \begin{pmatrix}r_{Cx} & 0 & 0\end{pmatrix}^{\mathrm{T}}, \tag{G.14}$$

with arbitrary $r_{Cx}$. Combination of (G.13) and (G.14) gives the conditions:

$$\frac{mR}{40} + m_a d \stackrel{!}{=} r_{Cx}, \tag{G.15}$$

$$\frac{mR}{50} + m_a R\cos\varphi \stackrel{!}{=} 0, \tag{G.16}$$

$$\frac{mR}{50} + m_a R\sin\varphi \stackrel{!}{=} 0, \tag{G.17}$$

which means that $d$ is arbitrary, and $\sin\varphi = \cos\varphi$. A possible choice is (avoiding negative mass):

$$\varphi = \frac{5}{4}\pi, \qquad d = 0, \qquad m_a = \frac{\sqrt{2}}{50}m. \tag{G.18}$$

**1.22** The FBD of the skateboarder is shown in Figure G.3. *Remark:*Note that this FBD would not yet be sufficient to determine the equations of motion and the unknown forces and moments. For example, some forces cannot uniquely be determined, and one needs to allow local slip at the wheels for realistic movement.

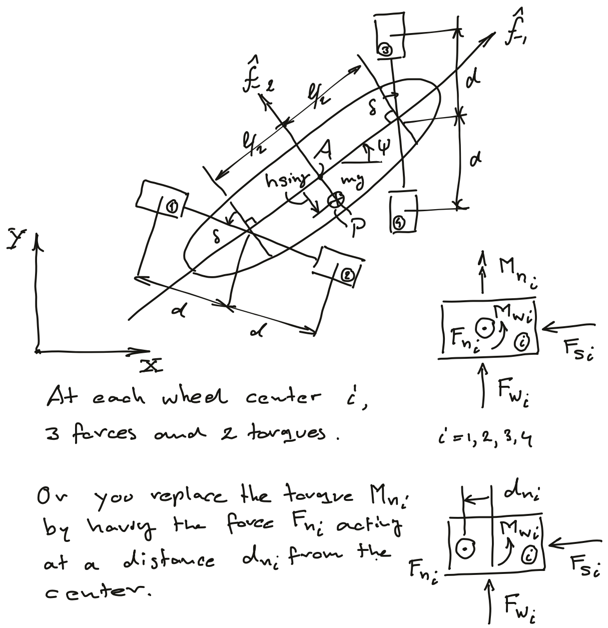


Figure G.3: Free-body diagram of the skateboarder.

**1.24** The FBD for the sphere and the cage are shown in Figure G.4.

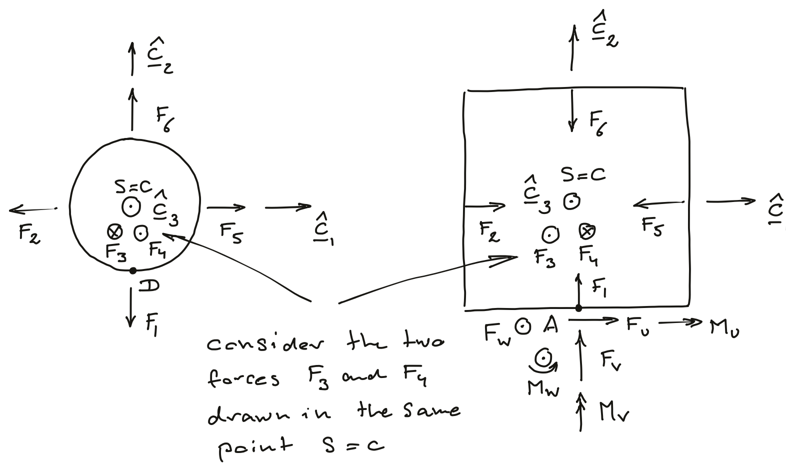


Figure G.4: Free-body diagrams for the accelerometer.

**1.29** Possible FBD's of the scooter's footboard and rear wheel are drawn in Figure G.5.

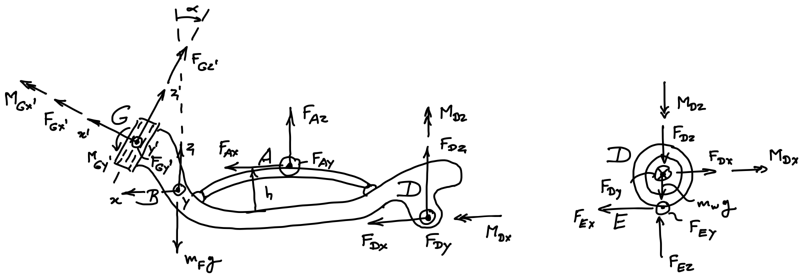

Figure G.5: FBDs of the scooter's footboard and rear wheel.

**2.3** From vector addition, we know the relationship

$$\boldsymbol{r}_{i/C} = \boldsymbol{r}_i - \boldsymbol{r}_C, \tag{G.19}$$

which allows us to re-write the left version of (2.18),

$$\boldsymbol{H}_C = \sum_{i=1}^{N} (\boldsymbol{r}_{i/C} \times (m_i \dot{\boldsymbol{r}}_{i/C})), \tag{G.20}$$

to

$$\begin{aligned} \boldsymbol{H}_C &= \sum_{i=1}^{N} (\boldsymbol{r}_{i/C} \times (m_i(\dot{\boldsymbol{r}}_i - \dot{\boldsymbol{r}}_C))) && \text{(G.21)} \\ &= \sum_{i=1}^{N} (\boldsymbol{r}_{i/C} \times (m_i \dot{\boldsymbol{r}}_i)) - \left( \sum_{i=1}^{N} m_i \boldsymbol{r}_{i/C} \right) \times \dot{\boldsymbol{r}}_C && \text{(G.22)} \end{aligned}$$

From the definition of the center of mass, we know that

$$\sum_{i=1}^{N} m_i \boldsymbol{r}_i = \sum_{i=1}^{N} m_i \boldsymbol{r}_C, \tag{G.23}$$

which implies that

$$\sum_{i=1}^{N} m_i \boldsymbol{r}_{i/C} = \sum_{i=1}^{N} m_i (\boldsymbol{r}_i - \boldsymbol{r}_C) = \boldsymbol{0}, \tag{G.24}$$

so (G.22) simplifies to

$$\boldsymbol{H}_C = \sum_{i=1}^{N} \boldsymbol{r}_{i/C} \times (m_i \dot{\boldsymbol{r}}_i). \tag{G.25}$$

**2.5** For a (sub)system that has its center of mass in point $C$, we find angular momentum with respect to another point $Q$ using summation over each particle $i$ in the (sub)system, following (2.9) and (2.10):

$$\boldsymbol{H}_Q = \sum_{i=1}^{N} \left(\boldsymbol{r}_{i/Q} \times \left(m_i \dot{\boldsymbol{r}}_{i/Q}\right)\right). \tag{G.26}$$

Making use of the vector addition

$$\boldsymbol{r}_{i/Q} = \boldsymbol{r}_{i/C} + \boldsymbol{r}_{C/Q}\,, \quad \dot{\boldsymbol{r}}_{i/Q} = \dot{\boldsymbol{r}}_{i/C} + \dot{\boldsymbol{r}}_{C/Q} \tag{G.27}$$

we can expand (G.26) to

$$\begin{aligned}\boldsymbol{H}_Q &= \sum_{i=1}^{N} \left(\boldsymbol{r}_{i/C} \times \left(m_i \dot{\boldsymbol{r}}_{i/C}\right)\right) + \sum_{i=1}^{N} \left(\boldsymbol{r}_{C/P} \times \left(m_i \dot{\boldsymbol{r}}_{i/C}\right)\right) \\ &\quad + \sum_{i=1}^{N} \left(\boldsymbol{r}_{i/C} \times \left(m_i \dot{\boldsymbol{r}}_{C/Q}\right)\right) + \sum_{i=1}^{N} \left(\boldsymbol{r}_{C/P} \times \left(m_i \dot{\boldsymbol{r}}_{C/Q}\right)\right)\end{aligned} \tag{G.28}$$

We recognize that the first term is nothing else but the (sub)-system's angular momentum with respect to its center of mass. The second and third term can be simplified by recognizing that $\boldsymbol{r}_{C/P}$ and $\dot{\boldsymbol{r}}_{C/P}$ do not depend on $i$, so they can be factored out of the summation. Furthermore, by virtue of the definition of the center of mass, $\sum i = 1 N m_i \boldsymbol{r}_{i/C} = \boldsymbol{0}$ and also $\sum i = 1 N m_i \dot{\boldsymbol{r}}_{i/C} = \boldsymbol{0}$. See a similar and more explicit reasoning for this in the solution of Problem 2.3.

Finally, we retain only the first and fourth summand of (G.28), which can be written as the expected relationship:

$$\boldsymbol{H}_Q = \boldsymbol{H}_C + \boldsymbol{r}_{C/Q} \times \left(m \dot{\boldsymbol{r}}_{C/Q}\right). \tag{G.29}$$

**2.16** One example is drawn in Figure G.6. We can "follow" the force field along a closed, circular path, so net work is being done by the force on that path.

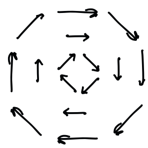

Figure G.6: A non-conservative force field.

**2.17** A sliding friction force is directed against the movement direction, so it is tangent to the path. A specific counterexample showing that such a force is not conservative: A particle moves from point $A$ to point $B$ via two different paths: Path 1 is straight from $A$ to $B$. Path 2 is longer: straight from $A$ to

 

$B$, then straight back to $A$, finally straight back to $B$. The velocity in this example always has constant magnitude, and friction properties are the same in both directions. The work done by the sliding friction force in the second case is three times as large as in the first case, although the two paths' start and end points were the same. Therefore, the work depends on the path taken, and the force is not conservative.

**2.19** (Note that this is also treated as an example in Example 12.7, where illustrations of the conservative cases are shown.)

*Gravitational force*: For the gravitational force, suppose an object is at a positive height $y$ above an arbitrarily defined level of zero potential (which can for example be the ground), and gravity with acceleration $g$ acts in negative $y$-direction (see also Figure 12.11). The object's weight $mg$ is the force acting on the object in negative $y$-direction such that (2.56) becomes

$$F_y = -mg = -\frac{\partial V}{\partial y}, \tag{G.30}$$

which can be integrated to give

$$V(y) = mgy + V_0. \tag{G.31}$$

Only in the special case where $V$ is defined to be zero at $y = 0$, the integration constant is found as $V_0 = 0$.

*Spring force*: For the spring force, we consider an object that is attached to a fixed point by means of a linear spring with stiffness $k$ (see also Figure 12.12). When there is no force acting, the spring has length $x_0$. The object can only move in the $x$-direction, such that (2.56) becomes

$$F_x = -k(x - x_0) = -\frac{\partial V}{\partial x}, \tag{G.32}$$

which can be integrated to give

$$V(x) = \frac{1}{2}k(x - x_0)^2 + V_0. \tag{G.33}$$

In the special case where $V$ is defined to be zero at $x = x_0$, the integration constant is found as $V_0 = 0$.

*Friction force during sliding*: For the friction force, we consider an example where an object is supported by a flat surface and can only slide straight along the $x$-axis with friction. Both Coulomb and viscous friction act on it. The Coulomb friction term is proportional to the normal force of magnitude $N$ that is acting on the object from the surface. With Coulomb friction coefficient $\mu$ and viscous friction coefficient $d$, the force $F_x$ acting on the object in positive $x$-direction is:

$$F_x = -\mu N \mathrm{sgn}(\dot{x}) - d\dot{x}. \tag{G.34}$$

We cannot obtain a potential function $V(x)$ for either of the two damping terms, because the force does not depend on position, but on velocity. So, (2.56) cannot be solved for $V$. This counterexample shows that sliding friction is not a conservative force. Also note that regardless of the direction of motion, the force will always dissipate energy.

**2.25** The system *does not conserve all linear momenta*, because gravity, as an external force, changes the linear momentum component in the vertical direction (so the velocity of the system's center of mass is not constant). The system *does not conserve energy* either, since the internal forces between body and hands do work. However, the system *does conserve angular momenta* about the center of mass, because the only external force (gravity) generates no moment about the system's center of mass.

**3.6** We know from (3.17) that the columns form a triad of orthogonal unit vectors. Therefore, one can calculate the third column by taking the cross product of the first with the second column. Be aware that this order is important, such that the triad is right-handed:

$$ {}^{\mathcal{B}}\hat{\boldsymbol{n}}_3 = {}^{\mathcal{B}}\hat{\boldsymbol{n}}_1 \times {}^{\mathcal{B}}\hat{\boldsymbol{n}}_2. \tag{G.35} $$

**3.7** A rotation of triads about the $Z$-axis is depicted in Figure G.7.

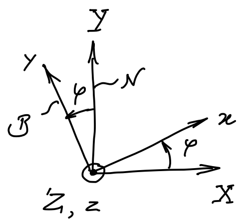


Figure G.7: Rotation of a coordinate triad about the $Z$-axis.

From the drawing, we can write down the components of units vectors of the rotated $x$ and of the $y$-axes ($Z = z$ is unchanged), and we find that these are the rows of the rotation matrix.

**3.8** A rotation of triads about the $Y$-axis is depicted in Figure G.8.

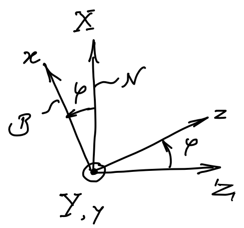


Figure G.8: Rotation of a coordinate triad about the $Y$-axis.

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# Bibliography

[1] S. L. Altmann, *Rotations, quaternions and double groups.* Clarendon Press, New York, Oxford University Press, 1986.

[2] ——, "Hamilton, rodrigues, and the quaternion scandal," *Mathematics Magazine*, vol. 62, no. 5, pp. 291–308, 1989.

[3] H. Baruh, *Analytical dynamics.* McGraw-Hill, 1999.

[4] F. P. Beer, E. R. Johnston, Jr., D. Mazurek, P. Cornwell, and B. Self, *Vector Mechanics for Engineers: Statics and Dynamics*, 11th ed. McGraw Hill, 2015.

[5] Bureau International des Poids et Mesures, "Le Système international d'unités / The International System of Units," SI Brochure, 8th edition, 2006, updated in 2014. [Online]. Available: https://www.bipm.org/en/publications/si-brochure/

[6] P. Bussotti, "On the genesis of the lagrange multipliers," *Journal of Optimization Theory and Applications*, vol. 117, no. 3, pp. 453–459, 2003.

[7] J. C. Butcher, "A history of runge-kutta methods," *Applied numerical mathematics*, vol. 20, no. 3, pp. 247–260, 1996.

[8] ——, *Numerical methods for ordinary differential equations.* John Wiley & Sons, 2016.

[9] M. Ceccarelli, "Screw axis defined by giulio mozzi in 1763 and early studies on helicoidal motion," *Mechanism and Machine Theory*, vol. 35, no. 6, pp. 761–770, 2000.

[10] R. Chandler, C. E. Clauser, J. T. McConville, H. Reynolds, and J. W. Young, "Investigation of inertial properties of the human body," Air Force Aerospace Medical Research Lab Wright-Patterson AFB OH, Tech. Rep., 1975.

[11] M. Chasles, *Note sur les propriétés générales du système de deux corps semblables entr'eux.* Bullettin des Sciences Mathématiques, Physiques et Chimiques, 1830, vol. 14, edited by M. Saigey.

[12] R. J. Cohen, "The tippe top revisited," *American Journal of Physics*, vol. 45, no. 1, pp. 12–17, 1977.

[13] Commission C2 - SUNAMCO, *Symbols, units, nomenclature and fundamental constants in physics (The Red Book).* International Union

of Pure and Applied Physics (IUPAP), 1987 revision (2010 reprint). [Online]. Available: http://iupap.org/wp-content/uploads/2014/05/A4.pdf

[14] E. Eich-Soellner and C. Führer, *Numerical Methods in Multibody Dynamics*, ser. European Consortium for Mathematics in Industry. B.G. Teubner, Stuttgart, 1998.

[15] L. Euler, "Formulae generales pro translatione quacunque corporum rigidorum," *Novi Commentarii academiae scientiarum Petropolitanae*, pp. 189–207, 1775, translation/transcription by John Sten available at www.17centurymaths.com/contents//euler/e478tr.pdf.

[16] C. A. Felippa, "Introduction to finite element methods (asen 5007)," Dec. 13 2017. [Online]. Available: https://www.colorado.edu/engineering/CAS/courses.d/IFEM.d/Home.html

[17] G. Fischer, *Lineare Algebra: Eine Einführung für Studienanfänger*. Springer, 2008.

[18] M. L. Foucault, "Physical demonstration of the rotation of the earth by means of the pendulum," *Journal of the Franklin Institute*, vol. 51, no. 5, pp. 350 – 353, 1851. [Online]. Available: https://doi.org/10.1016/0016-0032(51)90993-3

[19] M. Garcia, A. Chatterjee, A. Ruina, and M. Coleman, "The simplest walking model: Stability, complexity, and scaling," *ASME. J Biomech Eng*, vol. 120, no. 2, pp. 281–288, 1998.

[20] J. Garcia de Jalon and E. Bayo, *Kinematic and dynamic simulation of multibody systems: the real-time challenge*. Springer Verlag, New-York, 1994.

[21] C. W. Gear, *Numerical Initial Value Problems in Ordinary Differential Equations*. Upper Saddle River, NJ, USA: Prentice Hall PTR, 1971.

[22] H. Gelman, "A note on the time dependence of the effective axis and angle of rotation," *Journal of Research of the National Bureau of Standards*, vol. 75, pp. 165–171, 1971.

[23] M. Géradin and D. J. Rixen, *Mechanical vibrations: theory and application to structural dynamics*. Wiley, New York, 1994.

[24] M. Gerardin and A. Cardona, *Flexible multibody dynamics: A finite element approach*. Wiley, New York, 2001.

[25] H. Goldstein, *Classical mechanics*, 2nd ed. Addison-Wesley, Reading, 1980.

[26] M. W. Gomes and A. L. Ruina, "A five-link 2d brachiating ape model with life-like zero-energy-cost motions," *Journal of Theoretical Biology*, vol. 237, no. 3, pp. 265–278, 2005.

[27] D. T. Greenwood, *Advanced Dynamics*. Cambridge Univ. Press, 2003.

[28] G. Hamel, *Theoretische Mechanik*. Springer-Verlag, Berlin, 1965.

[29] E. J. Haug, *Computer aided kinematics and dynamics of mechanical systems, Volume I: Basic methods*. Allyn and Bacon Boston, 1989, vol. 1.

[30] H. Hertz, *Die Prinzipien der Mechanik in neuem Zusammenhange dargestellt*. Johann Ambrosius Barth, 1894, vol. Gesammelte Werke.

[31] J. A. Hrones and G. L. Nelson, *Analysis of the Four-Bar Linkage: Its Application to the Synthesis of Mechanisms*. The Technology Press of MIT and John Wiley & Sons, 1951.

[32] C. Huygens, *Horologium oscillatorium sive de motu pendulorum ad horologia aptato demonstrationes geometricae*. F. Muguet, Paris, 1673, pars quarta, propositio IX, translation by Ian Bruce available at http://17centurymaths.com/contents/huygenscontents.html.

[33] Institute of Electrical and Electronics Engineers (IEEE), *IEEE Conference Template (Overleaf)*. IEEE, accessed July 28, 2020. [Online]. Available: https://www.overleaf.com/latex/templates/ieee-conference-template/grfzhhncsfqn

[34] International Organization for Standardization (ISO), *ISO80000-1:2009 Quantities and Units - Part 1: General*, Std. 1, Nov. 2009. [Online]. Available: https://www.iso.org/obp/ui/#iso:std:iso:80000:-1:ed-1:v1:en

[35] T. R. Kane and D. A. Levinson, *Dynamics: Theory and Applications*. McGraw-Hill Book Company, 1985.

[36] F. Klein and A. Sommerfeld, *Über die Theorie des Kreisels*. Teubner (Leipzig), 1910.

[37] J. B. Kuipers, *Quaternions and rotation sequences*. Princeton University Press, 2002.

[38] J. L. Lagrange, *Analytical Mechanics*. Kluwer, Dordrecht, 1997, original French first edition "Mécanique Analytique," Desaint, Paris, 1788, Translated from the French second edition "Mécanique Analytique," Courcier, Paris, 1811, by A. Boissonnade and V.N. Vagliente. [Online]. Available: https://archive.org/details/mcaniqueanalyt01lagr

[39] C. Lanczos, *The variational principles of mechanics*. University of Toronto Press, Toronto, 1949.

[40] D. C. Lay, S. R. Lay, and J. J. McDonald, *Linear Algebra and Its Applications*, 5th ed. Pearson Education, 2015.

[41] J. le Rond d'Alembert, *Traité de dynamique*. L'Académie Royale des Sciences, Paris, 1743. [Online]. Available: https://archive.org/details/traitdedynamiqu00dalgoog

[42] D. Lemus, A. Berry, S. Jabeen, C. Jayaraman, K. Hohl, F. C. van der Helm, A. Jayaraman, and H. Vallery, “Controller synthesis and clinical exploration of wearable gyroscopic actuators to support human balance,” *Scientific reports*, vol. 10, no. 1, pp. 1–15, 2020.

[43] D. Lemus, J. van Frankenhuyzen, and H. Vallery, “Design and evaluation of a balance assistance control moment gyroscope,” *ASME Journal of Mechanisms and Robotics*, vol. 9, no. 5, pp. 1–9, 2017.

[44] D. Lewis, T. Ratiu, J. Simo, and J. E. Marsden, “The heavy top: a geometric treatment,” *Nonlinearity*, vol. 5, no. 1, pp. 1–48, 1992.

[45] J. P. Meijaard, J. M. Papadopoulos, A. Ruina, and A. L. Schwab, “Linearized dynamics equations for the balance and steer of a bicycle: a benchmark and review,” *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 463, no. 2084, pp. 1955–1982, 2007.

[46] F. C. Moon, *Applied dynamics.* Wiley, New York, 1998.

[47] G. Mozzi, *Discorso matematico sopra il rotamento momentaneo dei corpi.* Stamperia di Donato Campo, 1763.

[48] J. I. Neĭmark and N. A. Fufaev, *Dynamics of nonholonomic systems.* Providence, RI: American Mathematical Society., 1972, (Transl. from the Russian edition, Nauka, Moscow, 1967.).

[49] I. Newton, *The Mathematical Principles of Natural Philisophy.* Benjamin Motte, 1729, vol. 1, translated from Latin to English by Andrew Motte.

[50] P. E. Nikravesh, *Computer-aided analysis of mechanical systems.* Prentice-Hall, Inc., 1988.

[51] ——, *Planar multibody dynamics: formulation, programming and applications.* CRC press, 2007.

[52] J. J. O’Connor and E. F. Robertson, “Martin Wilhelm Kutta,” Aug. 2006. [Online]. Available: http://www-history.mcs.st-andrews.ac.uk/Biographies/Kutta.html

[53] ——, “Carl David Tolmé Runge,” Dec. 2008. [Online]. Available: http://www-history.mcs.st-andrews.ac.uk/Biographies/Runge.html

[54] L. A. Pars, *A treatise on analytical dynamics.* Heinemann, London, 1965.

[55] A.-M. Pendrill, “Rollercoaster loop shapes,” *Physics Education*, vol. 40, no. 6, p. 517, 2005.

[56] Physical Review Journals, *Axis Labels and Scales on Graphs.* American Physical Society, 2012. [Online]. Available: https://journals.aps.org/authors/axis-labels-and-scales-on-graphs-h18

[57] D. Poole, *Linear Algebra: A Modern Introduction*, 4th ed. Cengage Learning, 2015.

[58] R. E. Roberson and R. Schwertassek, *Dynamics of multibody systems.* Springer, 1988.

[59] E. J. Routh, *The advanced part of a treatise on the dynamics of a system of rigid bodies*, 4th ed. Macmillan and Co. (London), 1884, art. 211 ff.

[60] H. Schaub and J. L. Junkins, *Analytical Mechanics of Space Systems*, 3rd ed. Reston, VA: AIAA Education Series, 2014.

[61] W. Schiehlen *et al.*, *Multibody systems handbook.* Springer, 1990.

[62] A. L. Schwab and J. P. Meijaard, "How to draw Euler Angles and utilize Euler Parameters," in *Proceedings of IDETC/CIE - ASME International Design Engineering Technical Conferences & Computers and Information in Engineering Conference*, Philadelphia, USA, Sep. 2006, pp. 259–265.

[63] A. L. Schwab and M. Wisse, "Basin of attraction of the simplest walking model," in *Proceedings of the ASME design engineering technical conference*, Pittsburgh, USA, Sep. 2001.

[64] A. Shabana, *Dynamics of multibody systems.* Wiley, New York, 1998.

[65] L. F. Shampine and M. K. Gordon, *Computer solution of ordinary differential equations: the initial value problem.* San Francisco, CA: W. H. Freeman and Company, 1975.

[66] L. F. Shampine and M. W. Reichelt, "The MATLAB ODE Suite," *SIAM Journal on Scientific Computing*, vol. 18, no. 1, pp. 1–22, Jan. 1997.

[67] A. Sommerfeld, *Vorlesungen über theoretische Physik, Band 1: Mechanik.* Klemm, Wiesbaden, 1949.

[68] H. Sperl, "Wendekreisel," German Patent 63 261, Oct. 7, 1891.

[69] J. Steiner, "Von dem Krümmungs-Schwerpuncte ebener Curven," *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, vol. 21, pp. 33–63, 1840, §VII.

[70] J. Stewart, *Calculus: Early Transcendentals*, 8th ed. Cengage Learning, 2015.

[71] B. Storey, "Numerical methods for ordinary differential equations," Reader, University of Maryland, 2014. [Online]. Available: http://www.atmos.umd.edu/~rjs/class/fall2014/readings/storey_ode_matlab.pdf

[72] G. J. Talbourdet, "Mathematical solution of four-bar linkages," *Machine Design*, vol. 13, no. 5-7, 1941.

[73] United States Federal Aviation Administration, "A hazard in aerobatics: effects of g-forces on pilots," US Department of Transportation, Tech. Rep., 1984.

[74] WagonWorldCoLtd, "EzyRoller - The Ultimate Riding Machine," Dec. 2011. [Online]. Available: https://youtu.be/GMnJ9q1D4hU

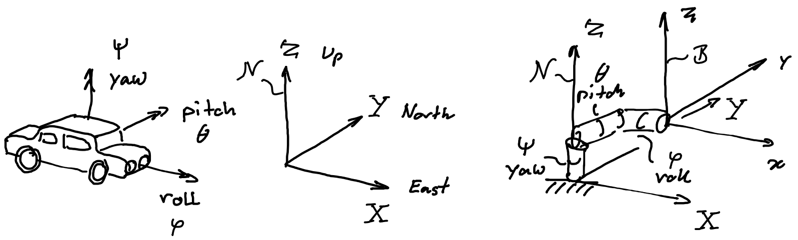
[75] E. T. Whittaker, *A treatise on the analytical dynamics of particles and rigid bodies*, 4th ed. Cambridge University Press, Cambridge, 1937.

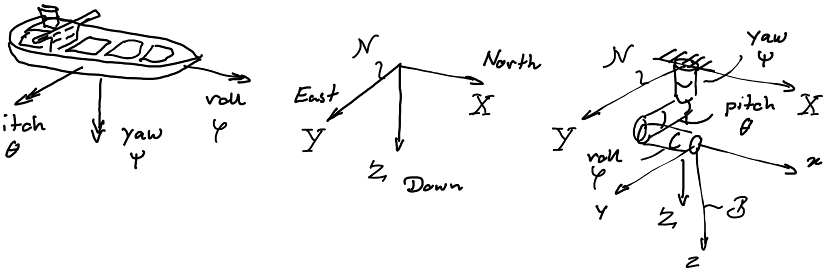
[76] windy54m, "Gibbon Brachiating," Nov. 26 2006. [Online]. Available: https://youtu.be/eV-gOL4t9Vk

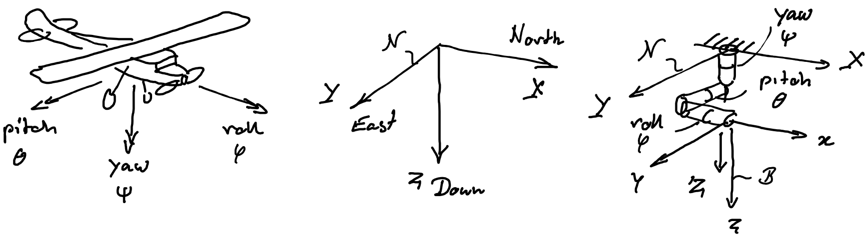
[77] J. Wittenburg, *Dynamics of Systems of Rigid Bodies.* Vieweg+Teubner Verlag, 1977.

# Index

**Dynamics of rigid bodies and multibody systems in three-dimensional spaces represents a fundamental pillar for many engineering sciences. This book is aimed at undergraduate and graduate students who already have some background in statics and dynamics of particles and planar systems, but who wish to extend their knowledge towards three-dimensional problems involving one or multiple rigid bodies, and towards variational methods. To facilitate application of this knowledge to real-world problems, the book also treats several methods and examples for the numerical simulation of dynamic systems. For all introduced concepts, practical examples and problems for self-study are included.**

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